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NUMERICAL METHODS FOR COUPLED SYSTEMS OF QUASILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract This paper is concerned with numerical solutions of a coupled system of arbitrary number of quasilinear elliptic equations under combined Dirichlet and nonlinear boundary conditions. A finite difference system for a transformed system of the quasilinear equations is formulated, and three monotone iterative schemes for the computation of numerical solutions are given using the method of upper and lower solutions. It is shown that each of the three monotone iterations converges to a minimal solution or a maximal solution depending on whether the initial iteration is a lower solution or an upper solution. A comparison result among the three iterative schemes is given. Also shown is the convergence of the minimal and maximal discrete solutions to the corresponding minimal and maximal solutions of the continuous system as the mesh size tends to zero. These results are applied to a heat transfer problem with temperature dependent thermal conductivity and a Lotka-Volterra cooperation system with degenerate diffusion. This degenerate property leads to some interesting distinct property of the system when compared with the non-degenerate semilinear systems. Numerical results are given to the above problems, and in each problem an explicit continuous solution is constructed and is used to compare with the computed solution.

Keywords Quasilinear elliptic equations, degenerate diffusion, monotone iterative schemes, upper and lower solutions, convergence of discrete solution, Lotka-Volterra cooperation system.

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1. Introduction

In the theory of heat transfer if the thermal conductivity is temperature dependent and the boundary is subjected to a Boltzmann fourth-power radiation law then for certain participating media such as glass, fibers, and powders, the steady–state temperature $u(x)$ is governed by the quasilinear boundary-value problem

$$\begin{aligned} -\nabla\cdot(D(u)\nabla u) &= f(x,u), (x\in\Omega),\\ D(u)\partial u/\partial\nu &= \sigma_0(x)(a_0^4-u^4), (x\in\partial\Omega), \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^p$ with boundary $\partial\Omega$ $(p=1,2,\ldots)$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$, $\sigma_0(x)\geq 0$ on $\partial\Omega$ is the so-called Stefan-Boltzmann function, and $a_0>0$ is the surrounding temperature. The functions

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$D(u)$ and $f(x,u)$ are given by

$$D(u) = k_c + k_r u^3, \ \ f(x,u) = p(x) - c(x)u^\alpha \tag{1.1a}$$

for some $\alpha \geq 1$, where k_c and k_r are the thermal conductivity constants based on conduction and radiation, respectively, while $p(x)$ and $c(x)$ are positive functions representing the internal source and evaporation coefficients(cf [15, p249]). The function $D(u)$ in (1.1a) is due to the effect of simultaneous conduction and radiation in the participating medium.

On the other hand, in the Lotka–Volterra cooperation system of two-cooperating species the steady-state density functions of the species $u(x)$, $v(x)$ are governed by the coupled system

$$\begin{aligned} &-\Delta u^m = u(a^{(1)} - b^{(1)}u + c^{(1)}v), && (x \in \Omega),\\ &-\Delta v^n = v(a^{(2)} - b^{(2)}v + c^{(2)}u), && (x \in \Omega),\\ &u(x) = v(x) = 0, && (x \in \partial\Omega), \end{aligned} \tag{1.2}$$

where $m > 1$, $n > 1$ and $(a^{(l)}, b^{(l)}, c^{(l)})$, $l = 1, 2$, are positive constants(cf [17, 19]). The terms Δu^m, Δv^n with $m > 1$, $n > 1$ implies that the diffusion rate of movement from high density region to low density region is slow, and it models a tendency to avoid crowding(cf. [12, 13, 23]).

Problem (1.2) and its corresponding time-dependent system have been extensively investigated in the field of ecology, but most of the investigations are devoted to the semilinear case $m = n = 1$ (cf [6–8, 13, 27, 28]). This system has also been used as a mathematical model in economics where u and v represent the wealth of two cooperating nations or regions (cf. [1]).

Motivated by the above model problems and many others we consider a general class of coupled system of quasilinear elliptic boundary-value problems in the form

$$\begin{aligned} &-\nabla\cdot(D^{(l)}(u^{(l)})) + \mathbf{b}^{(l)}(x)(D^{(l)}(u^{(l)})\nabla u^{(l)}) = f^{(l)}(x,\mathbf{u}), \ l = 1,\ldots,N, && (x \in \Omega),\\ &u^{(l)}(x) = \xi^{(l)}(x), \ l = 1,\ldots,n_0 - 1, && (x \in \partial\Omega),\\ &D^{(l)}(u^{(l)})\partial u^{(l)}/\partial\nu = g^{(l)}(x,\mathbf{u}), \ l = n_0,\ldots,N, && (x \in \partial\Omega), \end{aligned} \tag{1.3}$$

where $\mathbf{u} = (u^{(1)},\ldots,u^{(N)})$, $1 \leq n_0 \leq N+1$, and for each $l = 1,\ldots,N$, $D^{(l)}(u^{(l)})$, $\mathbf{b}^{(l)}(x) = (b_1^{(l)}(x),\ldots,b_N^{(l)}(x))$, $\xi^{(l)}(x)$, $f^{(l)}(x,\mathbf{u})$ and $g^{(l)}(x,\mathbf{u})$ are continuous functions satisfying the conditions in Hypothesis (H_1) in Section 2. It is assumed that $D^{(l)}(u^{(l)}) > 0$ for $u^{(l)} > 0$ and $D^{(l)}(0) \geq 0$, including the degenerate case $D^{(l)}(0) = 0$ and $\xi^{(l)} = 0$. The condition on n_0 implies that the system (1.3) may consists of Dirichlet boundary condition for some i and nonlinear boundary condition for the remaining i, including the linear Neumann-Robin boundary condition

$$\partial u^{(l)}/\partial\nu + \beta^{(l)}u^{(l)} = g^{(l)}(x), \ \ (x \in \partial\Omega).$$

In particular, the boundary condition in (1.3) is of Dirichlet type for all l if $n_0 = N+1$, and it is nonlinear (or linear Neumann-Robin type) if $n_0 = 1$. We also allow $D^{(l)}(u^{(l)}) = d^{(l)}$ to be independent of $u^{(l)}$ for some or all l. In the later case, problem (1.3) becomes a coupled system of semilinear elliptic boundary problems.

The purpose of this paper is the following: (a) To present three monotone iterative schemes for the computation of positive minimal and maximal solutions of a discrete system of (1.3), including the existence of these solutions. (b) To show the convergence of the discrete solutions to the corresponding solutions of the continuous problem as the mesh size decreases to zero. (c) To apply the results in (a) and (b) to the heat transfer problem (1.1) and the cooperation system (1.2), including some numerical results for these two problems. In the computation of numerical solutions we first construct a source function so that a continuous solution of the problem is explicitly known and is used to compare with the computed solution by the monotone iterative schemes. Our numerical results demonstrate excellent agreement between the computed solution and the true continuous solution.

Literature dealing with numerical solutions of quasilinear elliptic equations is extensive and various topics of the problem, such as method of computation, error estimate, and convergence of the discrete solution have been discussed (cf. [2,9,11,21, 26]). Most of the discussions in the above works are for scalar equations with linear boundary condition. The work in [21] treated a scalar quasilinear equation with either Dirichlet or nonlinear boundary condition. Similar discussions for semilinear elliptic equations, including coupled system of two or more equation have been treated by either the finite difference method or the finite element method (cf. [10, 16, 18, 19]). For numerical solutions of quasilinear elliptic equations, it is often assumed that the equation is non-degenerate and the problem has a unique solution. An example of this situation is the heat transfer problem (1.1) where the thermal conductivity is temperature dependent. This equation with $D(u) = k_c + k_r u^3$ and the physical meaning of the constants k_c and k_r have been derived in [3, 15]. Other models and many relevant references in this area can be found in chapter 9 of [15]. On the other hand, in many reaction-diffusion problems with density dependent diffusion coefficients the governing equations are either degenerate or it is non-degenerate but the solution is not unique. This type of equations, including the system (1.3) have been treated recently in [22]. The emphasis in this paper is on the degenerate property of the system. This property and the non-uniqueness of the solution is especially true for coupled system of equations, including semilinear systems. For example, in the semilinear system (1.2) where $m = n = 1$, the problem has only the trivial solution $(0, 0)$ if $a^{(l)} \leq \lambda_0$, $l = 1, 2$, where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$\Delta\phi + \lambda\phi = 0 \ \text{ in } \ \Omega, \quad \phi = 0 \ \text{ on } \ \partial\Omega. \tag{1.4}$$

In the case $a^{(l)} > \lambda_0$ the problem has two semitrivial solutions in the form $(u_s, 0)$, $(0, v_s)$, where $u_s > 0$ and $v_s > 0$ in Ω. In addition, it has at least one positive solution if $b^{(1)}b^{(2)} > c^{(1)}c^{(2)}$, and it has no positive solution if $b^{(1)}b^{(2)} < c^{(1)}c^{(2)}$ (cf. [17, p678]). However, the situation is quite different for the quasilinear system (1.2) and its corresponding finite difference system when $m > 1$, $n > 1$. In fact, we show that for any positive constants $(a^{(l)}, b^{(l)}, c^{(l)})$, $l = 1, 2$, the corresponding finite difference system of (1.2) has two semitrivial solutions $(u(x_i), 0)$, $(0, v(x_i))$, where x_i denotes a mesh point in $\bar{\Omega}$. If, in addition, $1/m + 1/n < 1$ then the problem has a positive minimal solution $(\underline{u}(x_i), \underline{v}(x_i))$ and a positive maximal solution $(\overline{u}(x_i), \overline{v}(x_i))$. If, in addition, $(\underline{u}(x_i), \underline{v}(x_i)) = (\overline{u}(x_i), \overline{v}(x_i))$ then their common value is the unique positive finite difference solution (see Theorem 5.3). The above conclusions hold true for every positive constants $(a^{(l)}, b^{(l)}, c^{(l)})$, including the case $a^{(l)} \leq \lambda_0$ or $b^{(1)}b^{(2)} \leq c^{(1)}c^{(2)}$. These results are in sharp contrast to the semilinear case $m = n = 1$.

To compute numerical solutions of the above problems, including positive minimal and maximal solutions we use the method of upper and lower solutions and its associated monotone iterations for a finite difference system of the coupled system (1.3). This approach leads to some general results for the system which can be used not only for the problems in (1.1) and (1.2) but also for many other degenerate and non-generate reaction-convection-diffusion system with density-dependent diffusions. An important feature of this method is that it can be used to compute positive minimal and maximal solutions without the uniqueness assumption. In particular, if the solution is unique then the monotone convergence of the minimal and maximal sequences gives a simple and reliable error estimate of the solution without any explicit knowledge of the solution.

The plan of the paper is as follows: In section 2, we transform the system (1.3) into a system of semilinear and algebraic equations which are discretised into a finite difference system. A monotone iterative scheme is given for computing positive solutions using the method of upper and lower solutions. Proofs of the monotone convergence of the sequence of iterations together with two other monotone iterative schemes are given in Section 3. In Section 4, we show the convergence of the finite difference solution to the continuous solution as the mesh size tends to zero. The above results are applied to problem (1.1) and (1.2) in Section 5 where explicit lower and upper solutions are constructed. Finally, in Section 6 we give some numerical results for these two problems, including the construction of known continuous solutions which are used to compare with the computed solutions.

2. Finite Difference Systems

To develop computational schemes for numerical solutions of (1.3) we make a transformation by letting

$$w^{(l)} = I^{(l)}[u^{(l)}] \equiv \int_0^{u^{(l)}} D^{(l)}(s)ds, \quad \text{for} \quad u^{(l)} \geq 0, \quad l = 1, \ldots, N. \tag{2.1}$$

In view of $dw^{(l)}/du^{(l)} = D^{(l)}(u^{(l)}) > 0$ for $u^{(l)} > 0$ the inverse function of $w^{(l)} = I^{(l)}[u^{(l)}]$, denoted by $u^{(l)} = q^{(l)}(w^{(l)})$, exists and $dq^{(l)}/dw^{(l)} = 1/D^{(l)}(u^{(l)})$. Since for each $l = 1, \ldots, N$,

$$\nabla w^{(l)} = D^{(l)}(u^{(l)})\nabla u^{(l)}, \quad \partial w^{(l)}/\partial\nu = D^{(l)}(u^{(l)})\partial u^{(l)}/\partial\nu,$$

we may write (1.3) as

$$\begin{aligned}
&-\Delta w^{(l)} + \mathbf{b}^{(l)} \cdot \nabla w^{(l)} = f^{(l)}(x, \mathbf{u}), \quad l = 1, \ldots, N, (x \in \Omega), \\
&w^{(l)} = \eta^{(l)}, \quad l = 1, \ldots, n_0 - 1, (x \in \partial\Omega), \\
&\partial w^{(l)}/\partial\nu = g^{(l)}(x, \mathbf{u}), \quad l = n_0, \ldots, N, (x \in \partial\Omega), \\
&u^{(l)} = q^{(l)}(w^{(l)}), \quad l = 1, \ldots, N, (x \in \bar{\Omega}),
\end{aligned} \tag{2.2}$$

where $\eta^{(l)} = I^{(l)}[\xi^{(l)}]$. Although the above system may be written in a semilinear form in terms of $w^{(l)}$ by using $\mathbf{u} = \mathbf{q}(\mathbf{w}) = (q^{(1)}(w^{(1)}), \ldots, q^{(N)}(w^{(N)}))$ we prefer to use its present form as a coupled system of $\mathbf{w}$ and $\mathbf{u}$ where $\mathbf{w} = (w^{(1)}, \ldots, w^{(N)})$. One reason for this is that in the degenerate case $D^{(l)}(0) = 0$ the functions $f^{(l)}(x, \mathbf{q}(\mathbf{w}))$, $g^{(l)}(x, \mathbf{q}(\mathbf{w}))$ are not Lipschitz continuous in $w^{(l)}$ at $w^{(l)} = 0$

because $dw^{(l)}/du^{(l)} = 1/D^{(l)}(u^{(l)})$ which is not defined at $u^{(l)} = 0$. It is obvious that problem (2.2) can be discretised into a discrete system in the same fashion as for semilinear elliptic boundary-value problems. In this paper, we use the finite difference method for the discrete system although it can also be formulated by the finite element method with a suitable choice of the basis functions (see Remark 3.1).

Let $i = (i_1, \dots, i_p)$ be a multiple index with $i_\nu = 1, \dots, M_\nu$, and let $x_i = (x_{i_1}, \dots, x_{i_p})$ be a mesh point in $\bar{\Omega} \equiv \Omega \cup \partial\Omega$, where $\nu = 1, \dots, p$ and M_ν is the total number of intervals in the x_ν − direction. Denote by Λ, $\partial\Lambda$ and $\bar{\Lambda}$ the sets of mesh points in Ω, $\partial\Omega$ and $\bar{\Omega}$, respectively. When no confusion arises we write $i \in \Lambda'$ for $x_i \in \Lambda'$ where Λ' stands for Λ, $\partial\Lambda$ or $\bar{\Lambda}$. Let h_ν be the spatial increment in the x_ν–direction, and let $u_i^{(l)} = u^{(l)}(x_i)$, $w_i^{(l)} = w^{(l)}(x_i)$, and $\mathbf{u}_i = (u^{(1)}(x_i), \dots, u^{(N)}(x_i))$. Define

$$
\begin{aligned}
D^{(l)}(u_i^{(l)}) &= D^{(l)}(u^{(l)}(x_i)), & q^{(l)}(w_i^{(l)}) &= q^{(l)}(w^{(l)}(x_i)),\\
f_i^{(l)}(\mathbf{u}_i) &= f^{(l)}(x_i, \mathbf{u}(x_i)), & g_i^{(l)}(\mathbf{u}_i) &= g^{(l)}(x_i, \mathbf{u}(x_i)),
\end{aligned}
$$

where $l = 1, \dots, N$. Then by the central difference approximations

$$
\begin{aligned}
\Delta_p[w_i^{(l)}] &= \sum_{\nu=1}^{p} \Delta^{(\nu)} w_i^{(l)} \equiv \sum_{\nu=1}^{p} [w^{(l)}(x_i + h_\nu e_\nu) - 2w^{(l)}(x_i) + w^{(l)}(x_i - h_\nu e_\nu)],\\
\mathbf{b}^{(l)} \cdot \delta_p[w_i^{(l)}] &= \sum_{\nu=1}^{p} (b_\nu^{(l)}(x_i)/2h_\nu)[w^{(l)}(x_i + h_\nu e_\nu) - w^{(l)}(x_i - h_\nu e_\nu)]
\end{aligned}
\tag{2.3}
$$

and the boundary approximation

$$
B^{(l)}[w_i^{(l)}] = [w^{(l)}(x_i^{(b)}) - w^{(l)}(\hat{x}_i)]/(x_i^{(b)} - \hat{x}_i), \quad (x_i^{(b)} \in \partial\Lambda, \hat{x}_i \in \Lambda), \tag{2.4}
$$

where e_ν is the unit vector in $\mathbb{R}^p$ with the ν-th component one and zero elsewhere and $\hat{x}_i$ is a neighboring point of $x^{(b)}$ in Λ, we approximate the transformed system (2.2) by the finite difference system

$$
\begin{aligned}
&-\Delta_p[w_i^{(l)}] + \mathbf{b}_i^{(l)} \cdot \delta_p[w_i^{(l)}] = f_i^{(l)}(\mathbf{u}_i), \quad l = 1, \dots, N, (i \in \Lambda),\\
&w_i^{(l)} = \eta_i^{(l)}, \quad l = 1, \dots, n_0 - 1, (i \in \partial\Lambda),\\
&B^{(l)}[w_i^{(l)}] = g_i^{(l)}(\mathbf{u}_i), \quad l = n_0, \dots, N, (i \in \partial\Lambda),\\
&u_i^{(l)} = q^{(l)}(w_i^{(l)}), \quad l = 1, \dots, N, (i \in \bar{\Lambda}),
\end{aligned}
\tag{2.5}
$$

(cf. [5, 24]).

To develop monotone iterative schemes for the computation of solutions of (2.5) we use the method of lower and upper solutions. The definition of these solutions, denoted by $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) = ((\hat{u}_i^{(1)}, \dots, \hat{u}_i^{(N)}), (\hat{w}_i^{(1)}, \dots, \hat{w}_i^{(N)}))$ and $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i) = ((\tilde{u}_i^{(1)}, \dots, \tilde{u}_i^{(N)}), (\tilde{w}_i^{(1)}, \dots, \tilde{w}_i^{(N)}))$ respectively, are defined in the following.

Definition 2.1. A function $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) \in \mathbb{R}^N \times \mathbb{R}^N$ is called a lower solution of (2.5) if its components $(\hat{u}_i^{(l)}, \hat{w}_i^{(l)})$ satisfy the relation

$$\begin{aligned}
&-\Delta_p[\hat{w}_i^{(l)}] + \mathbf{b}_i^{(l)} \cdot \delta_p[\hat{w}_i^{(l)}] \le f_i^{(l)}(\hat{\mathbf{u}}_i), && l = 1, \ldots, N, && (i \in \Lambda),\\
&\hat{w}_i^{(l)} \le \eta_i^{(l)}, && l = 1, \ldots, n_0 - 1, && (i \in \partial\Lambda),\\
&B^{(l)}[\hat{w}_i^{(l)}] \le g_i^{(l)}(\hat{\mathbf{u}}_i), && l = n_0, \ldots, N, && (i \in \partial\Lambda),\\
&\hat{u}_i^{(l)} \le q^{(l)}(\hat{w}_i^{(l)}), && l = 1, \ldots, N, && (i \in \bar{\Lambda}).
\end{aligned} \tag{2.6}$$

Similarly, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ is called an upper solution of (2.5) if its components satisfy all the inequalities in (2.6) in reversed order.

The pair $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ are said to be ordered if $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) \le (\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$. For a given pair of ordered lower and upper solutions $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ we set

$$\begin{aligned}
&S^{(1)} \equiv \{\mathbf{u}_i \in \mathbb{R}^N;\ \hat{\mathbf{u}}_i \le \mathbf{u}_i \le \tilde{\mathbf{u}}_i\},\\
&S^{(2)} \equiv \{(\mathbf{u}_i, \mathbf{w}_i) \in \mathbb{R}^N \times \mathbb{R}^N;\ (\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) \le (\mathbf{u}_i, \mathbf{w}_i) \le (\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)\}.
\end{aligned} \tag{2.7}$$

In the following discussion we assume that a pair of ordered lower and upper solutions $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ exist and $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) \ge (0, 0)$. In addition, we make the following basic hypothesis for each $l = 1, \ldots, N$:

(H_1) (i) $D^{(l)}(u^{(l)}) > 0$ for $u^{(l)} > 0$, $D^{(l)}(0) \ge 0$, and $h_\nu < (|b_\nu^{(l)}(x)|)^{-1}$ for $\nu = 1, \ldots, p$.

(ii) $f^{(l)}(x, \cdot)$, $g^{(l)}(x, \cdot)$, $\hat{\mathbf{b}}^{(l)}(x)$ and $\xi^{(l)}(x)$ are continuous functions of x, and $f^{(l)}(\cdot, \mathbf{u})$, $g^{(l)}(\cdot, \mathbf{u})$ are C^1–functions of $\mathbf{u}$ for $\mathbf{u} \in S^{(1)}$.

(iii) $f^{(l)}(\cdot, \mathbf{u})$ and $g^{(l)}(\cdot, \mathbf{u})$ are quasi-monotone nondecreasing in $\mathbf{u}$ for $\mathbf{u} \in S^{(1)}$ and there exist nonnegative functions $\gamma^{(l)}(x)$ and $\overline{\gamma}^{(l)}(x)$, not both identical zero, such that

$$\begin{aligned}
&\gamma^{(l)}(x) D^{(l)}(u^{(l)}) + \frac{\partial f^{(l)}}{\partial u^{(l)}}(x, \mathbf{u}) \ge 0 \text{ for } \mathbf{u} \in S^{(1)}, x \in \Omega,\\
&\overline{\gamma}^{(l)}(x') D^{(l)}(u^{(l)}) + \frac{\partial g^{(l)}}{\partial u^{(l)}}(x', \mathbf{u}) \ge 0 \text{ for } \mathbf{u} \in S^{(1)}, x' \in \partial\Omega.
\end{aligned} \tag{2.8}$$

In the hypothesis $(H_1) - (i)$ we allow $D^{(l)}(0) = 0$ for some or all l. In this situation the system (2.5) is degenerate on the boundary if $\xi^{(l)}(x) = 0$ for some or all $x \in \Omega$. Recall that $f^{(l)}(\mathbf{u})$ is quasi-monotone nondecreasing in $\mathbf{u}$ for $\mathbf{u} \in S^{(1)}$ if

$$\frac{\partial f^{(l)}}{\partial u^{(j)}}(\cdot, \mathbf{u}) \ge 0 \quad \text{for all} \quad \mathbf{u} \in S^{(1)} \quad \text{with} \quad j \ne l.$$

It is easy to see from the hypothesis $D^{(l)}(u^{(l)}) > 0$ for $u^{(l)} > 0$ that condition (2.8) holds by any nonnegative functions $\gamma^{(l)}(x)$, $\overline{\gamma}^{(l)}(x)$ if $\partial f^{(l)}(\cdot, \mathbf{u})/\partial u^{(l)} \ge 0$ and $\partial g^{(l)}(\cdot, \mathbf{u})/\partial u^{(l)} \ge 0$. It is also satisfied by some positive functions $\gamma^{(l)}(x)$, $\overline{\gamma}^{(l)}(x)$ if $D^{(l)}(0) > 0$. Hence this condition is needed only for the degenerate case $D^{(l)}(0) = 0$.

Define

$$\begin{aligned}
&L^{(l)}[w_i^{(l)}] = -\Delta_p[w_i^{(l)}] + \mathbf{b}_i^{(l)} \cdot \delta_p[w_i^{(l)}] + \gamma_i^{(l)} w_i^{(l)},\\
&F_i^{(l)}(\mathbf{u}_i) = \gamma_i^{(l)} I^{(l)}[u_i^{(l)}] + f_i^{(l)}(\mathbf{u}_i),\\
&G_i^{(l)}(\mathbf{u}_i) = \overline{\gamma}_i^{(l)} I^{(l)}[u_i^{(l)}] + g_i^{(l)}(\mathbf{u}_i), \qquad l = 1, \ldots, N,
\end{aligned} \tag{2.9}$$

where $w_i^{(l)} = I^{(l)}[u_i^{(l)}]$ is given by (2.1). Then we may write problem (2.5) in the equivalent form

$$
\begin{aligned}
&L^{(l)}[w_i^{(l)}] = F_i^{(l)}(\mathbf{u}_i), && l = 1, \dots, N, && (i \in \Lambda),\\
&w_i^{(l)} = \eta_i^{(l)}, && l = 1, \dots, n_0 - 1, && (i \in \partial\Lambda),\\
&B^{(l)}[w_i^{(l)}] + \overline{\gamma}_i^{(l)} w_i^{(l)} = G_i^{(l)}(\mathbf{u}_i), && l = n_0, \dots, N, && (i \in \partial\Lambda),\\
&u_i^{(l)} = q^{(l)}(w_i^{(l)}), && l = 1, \dots, N, && (i \in \bar{\Lambda}).
\end{aligned} \tag{2.10}
$$

It is obvious from (2.8), (2.9) and $(d/du^{(l)})I[u_i^{(l)}] = D^{(l)}(u_i^{(l)})$ that

$$
\begin{aligned}
&\frac{\partial F^{(l)}}{\partial u^{(l)}}(\cdot, \mathbf{u}_i) = \gamma_i^{(l)} D^{(l)}(u_i^{(l)}) + \frac{\partial f^{(l)}}{\partial u^{(l)}}(\cdot, \mathbf{u}_i) \geq 0,\\
&\frac{\partial G^{(l)}}{\partial u_i^{(l)}}(\cdot, \mathbf{u}_i) = \overline{\gamma}_i^{(l)} D^{(l)}(u_i^{(l)}) + \frac{\partial g^{(l)}}{\partial u^{(l)}}(\cdot, \mathbf{u}_i) \geq 0 \qquad \text{for } \hat{\mathbf{u}}_i \leq \mathbf{u}_i \leq \tilde{\mathbf{u}}_i.
\end{aligned}
$$

By the quasi-monotone nondecreasing property of $f^{(l)}(\cdot, \mathbf{u})$, $g^{(l)}(\cdot, \mathbf{u})$ in the hypothesis $(H_1) - (iii)$ we see that for every $l = 1, \dots, N$,

$$
\begin{aligned}
&F_i^{(l)}(\mathbf{u}_i) \geq F_i^{(l)}(\mathbf{v}_i),\\
&G_i^{(l)}(\mathbf{u}_i) \geq G_i^{(l)}(\mathbf{v}_i) \ \text{ when } \ \hat{\mathbf{u}}_i \leq \mathbf{v}_i \leq \mathbf{u}_i \leq \tilde{\mathbf{u}}_i.
\end{aligned} \tag{2.11}
$$

Using any $\mathbf{u}_i^{(0)}$ as the initial iteration we can construct a sequence $\{\mathbf{u}_i^{(k)}, \mathbf{w}_i^{(k)}\}$ from the linear iteration process:

$$
\begin{aligned}
&L^{(l)}[(w_i^{(l)})^{(k)}] = F_i^{(l)}(\mathbf{u}_i^{(k-1)}), \quad l = 1, \dots, N, (i \in \Lambda),\\
&(w_i^{(l)})^{(k)} = \eta_i^{(l)}, \quad l = 1, \dots, n_0 - 1, (i \in \partial\Lambda),\\
&B^{(l)}[(w_i^{(l)})^{(k)}] + \overline{\gamma}_i^{(l)} (w_i^{(l)})^{(k)} = G_i^{(l)}(\mathbf{u}_i^{(k-1)}), \quad l = n_0, \dots, N, (i \in \partial\Lambda),\\
&(u_i^{(l)})^{(k)} = q^{(l)}((w_i^{(l)})^{(k)}), \quad l = 1, \dots, N, (i \in \bar{\Lambda}),
\end{aligned} \tag{2.12}
$$

where $((u_i^{(l)})^{(k)}, (w_i^{(l)})^{(k)})$, $l = 1, \dots, N$, are the components of $(\mathbf{u}_i^{(k)}, \mathbf{w}_i^{(k)})$. It is obvious that the sequence $\{\mathbf{u}_i^{(k)}, \mathbf{w}_i^{(k)}\}$ is well-defined and can be easily computed. Specifically, starting from any $\mathbf{u}_i^{(0)}$ we can compute the solution $(w_i^{(l)})^{(1)}$ of (2.12) for $k = 1$ by solving a linear finite difference system under Dirichlet boundary condition for $l = 1, \dots, n_0 - 1$ and under Robin boundary condition for $l = n_0, \dots, N$. This is because for $k = 1$ the functions at the right-hand side of the first three equations are known. Knowing the value of $(w_i^{(l)})^{(1)}$ we then compute $(u_i^{(l)})^{(1)}$ from the last equation in (2.12). This gives the first iteration $((u_i^{(l)})^{(1)}, (w_i^{(l)})^{(1)})$. Using $\mathbf{u}_i^{(1)} \equiv ((u_i^{(1)})^{(1)}, \dots, (u_i^{(N)})^{(1)})$ in (2.12) instead of $\mathbf{u}^{(0)}$ the same computational procedure for $k = 2$ yields the second iteration $((u_i^{(l)})^{(2)}, (w_i^{(l)})^{(2)})$. A continuation of this process leads to the k-th iteration $((u_i^{(l)})^{(k)}, (w_i^{(l)})^{(k)})$ for every k. It is clear that this sequence of iterations depends on the initial iteration $\mathbf{u}_i^{(0)}$. To obtain the monotone convergence of this sequence we use either $\mathbf{u}_i^{(0)} = \hat{\mathbf{u}}_i$ or $\mathbf{u}_i^{(0)} = \tilde{\mathbf{u}}_i$ and denote the corresponding sequence by $\{\underline{\mathbf{u}}_i^{(k)}, \underline{\mathbf{w}}_i^{(k)}\}$ and $\{\overline{\mathbf{u}}_i^{(k)}, \overline{\mathbf{w}}_i^{(k)}\}$, respectively. The following theorem gives the monotone convergence of these sequences.

Theorem 2.1. *Let* $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ *be a pair of ordered lower and upper solutions of (2.5), and let hypothesis* (H_1) *be satisfied. Then the following statements hold true:*
(a) The sequence $\{\underline{\mathbf{u}}_i^{(k)}, \underline{\mathbf{w}}_i^{(k)}\}$ *converges to a minimal solution* $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$ *of (2.5), the sequence* $\{\overline{\mathbf{u}}_i^{(k)}, \overline{\mathbf{w}}_i^{(k)}\}$ *converges to a maximal solution* $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$*, and they satisfy the relation*

$$\begin{aligned}(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) &\le (\underline{\mathbf{u}}_i^{(k)}, \underline{\mathbf{w}}_i^{(k)}) \le (\underline{\mathbf{u}}_i^{(k+1)}, \underline{\mathbf{w}}_i^{(k+1)}) \le (\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i) \le (\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i) \\ &\le (\overline{\mathbf{u}}_i^{(k+1)}, \overline{\mathbf{w}}_i^{(k+1)}) \le (\overline{\mathbf{u}}_i^{(k)}, \overline{\mathbf{w}}_i^{(k)}) \le (\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i), \quad k = 1, 2, \ldots.\end{aligned}$$

(b) Every other solution $(\mathbf{u}_i, \mathbf{w}_i)$ *of (2.5) in* $S^{(2)}$*, if any, satisfies the relation*

$$(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i) \le (\mathbf{u}_i, \mathbf{w}_i) \le (\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i). \tag{2.13}$$

(c) If either $\underline{\mathbf{u}}_i = \overline{\mathbf{u}}_i$ *or* $\underline{\mathbf{w}}_i = \overline{\mathbf{w}}_i$ *then* $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i) = (\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)(\equiv (\mathbf{u}_i^*, \mathbf{w}_i^*))$ *and* $(\mathbf{u}_i^*, \mathbf{w}_i^*)$ *is the unique solution of (2.5) in* $S^{(2)}$.

Although the above theorem can be proved directly from (2.12) we postpone its proof to the next section (along with two other monotone iterative schemes) using vector form of the finite difference system.

Remark 2.1. (a). In view of the relation (2.13), we call $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$ and $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$ the respective minimal and maximal solutions of (2.5) in $S^{(2)}$. This minimal and maximal property is with respect to the pair of lower and upper solutions in $S^{(2)}$, and different pair of lower and upper solutions may yield different minimal and maximal solutions.
(b). The condition $h_\nu < |b_\nu^{(l)}|^{-1}$ in Hypothesis $(H_1) - (i)$ can be removed by using an upwind difference scheme if $|b_\nu^{(l)}|$ is extremely large.

3. Monotone Sequences

To show the monotone convergence of the sequence given by (2.12) and the sequences governed by two other iterative schemes, called Gauss-Seidel and Jacobi iterations, we formulate the system (2.5) in vector form. Let

$$\begin{aligned}M^{(l)} &= (M_1 - 1)(M_2 - 1) \cdots (M_p - 1), \, l = 1, \ldots, n_0 - 1, \\ \overline{M}^{(l)} &= (M_1 + 1)(M_2 + 1) \cdots (M_p + 1), \, l = n_0, \ldots, N,\end{aligned}$$

where M_ν, $\nu = 1, \ldots, p$, are the number of intervals in x_ν-direction in $\overline{\Lambda}$, $M^{(l)}$ is the total number of mesh points in Λ, and $\overline{M}^{(l)}$ is the total number of mesh points in $\overline{\Lambda}$. For each $l = 1, \ldots, N$, we define vectors

$$\begin{aligned}&U^{(l)} = (u_1^{(l)}, \ldots, u_{M'}^{(l)})^T, \; W^{(l)} = (w_1^{(l)}, \ldots, w_{M'}^{(l)})^T, \\ &Q^{(l)}(W^{(l)}) = (q^{(1)}(w_1^{(l)}), \ldots, q^{(N)}(w_{M'}^{(l)}))^T, \; \boldsymbol{\eta} = (\eta_1^{(l)}/h_1^2, \ldots, \eta_{M'}^{(l)}/h_{M'}^2)^T, \\ &\mathbf{U} = (U^{(1)}, \ldots, U^{(N)})^T, \\ &\mathbf{F}^{(l)}(\mathbf{U}) = (F_1^{(l)}(U^{(1)}), \ldots, F_{M'}^{(l)}(U^{(N)}))^T, \mathbf{G}^{(l)}(\mathbf{U}) = (G_1^{(l)}(U^{(1)}), \ldots, G_{M'}^{(l)}(U^{(N)}))^T\end{aligned} \tag{3.1}$$

and matrices

$$\begin{aligned} &\mathcal{A}^{(l)} = A^{(l)} + \Gamma^{(l)}, \Gamma^{(l)} = \mathrm{diag}(\gamma_1^{(l)}, \cdots, \gamma_{M'}^{(l)}), \; l = 1, \ldots, n_0 - 1, \\ &\overline{\mathcal{A}}^{(l)} = \overline{A}^{(l)} + \overline{\Gamma}^{(l)}, \overline{\Gamma}^{(l)} = \mathrm{diag}(\overline{\gamma}_1^{(l)}, \cdots, \overline{\gamma}_{M'}^{(l)}), \; l = n_0, \ldots, N, \end{aligned} \tag{3.2}$$

where $(\cdot)^T$ denotes the transpose of a row vector, M' stands for either $M^{(l)}$ or $\overline{M}^{(l)}$, and $\gamma_i^{(l)}$ and $\overline{\gamma}_i^{(l)}$ are the functions in (2.8). The matrices $A^{(l)}$, $\overline{A}^{(l)}$ are the respective $M^{(l)}$ by $M^{(l)}$ and $\overline{M}^{(l)}$ by $\overline{M}^{(l)}$ block matrices associated with the finite difference approximation in (2.3)-(2.4). Then we may write the system (2.10), which is equivalent to (2.5), in the vector form

$$\begin{aligned} &\mathcal{A}^{(l)} W^{(l)} = \mathbf{F}^{(l)}(\mathbf{U}) + \boldsymbol{\eta}^{(l)}, \quad l = 1, \ldots, n_0 - 1, \\ &\overline{\mathcal{A}}^{(l)} W^{(l)} = \mathbf{F}^{(l)}(\mathbf{U}) + \mathbf{G}^{(l)}(\mathbf{U}), \quad l = n_0, \ldots, N, \\ &U^{(l)} = Q^{(l)}(W^{(l)}), \quad l = 1, \ldots, N. \end{aligned} \tag{3.3}$$

Since our main concern in the proof of monotone convergence of the sequences is the mathematical structure of the discrete system (2.10), detailed formulation of (3.3) is omitted (see [18] for some detailed discussion). However, it is to be noted from (2.3)-(2.4) that for $l = 1, \ldots, n_0 - 1$ (Dirichlet boundary condition) the size of the matrix $A^{(l)}$ is $M^{(l)}$ by $M^{(l)}$, while for $l = n_0, \ldots, N$ (Robin boundary condition) the size of $\overline{A}^{(l)}$ is $\overline{M}^{(l)}$ by $\overline{M}^{(l)}$. Hence the number of components of the vector $\mathbf{U}$ is $M^* = M^{(1)} + \cdots + M^{(n_0-1)} + \overline{M}^{(n_0)} + \cdots + \overline{M}^{(N)}$.

In relation to the vector form (3.3) we have the following definition of lower and upper solutions:

Definition 3.1. A Vector $(\hat{\mathbf{U}}, \hat{\mathbf{W}}) \equiv ((\hat{U}^{(1)}, \cdots, \hat{U}^{(N)}), (\hat{W}^{(1)}, \cdots, \hat{W}^{(N)}))$ is called a lower solution of (3.3) if

$$\begin{aligned} &\mathcal{A}^{(l)} \hat{W}^{(l)} \le \mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \boldsymbol{\eta}^{(l)}, \quad \text{for } l = 1, \ldots, n_0 - 1, \\ &\overline{\mathcal{A}}^{(l)} \hat{W}^{(l)} \le \mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \mathbf{G}^{(l)}(\hat{\mathbf{U}}), \quad \text{for } l = n_0, \ldots, N, \\ &\hat{\mathbf{U}} \le Q^{(l)}(\hat{W}^{(l)}), \quad \text{for } l = 1, \ldots, N. \end{aligned} \tag{3.4}$$

Similarly, $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}}) \equiv ((\tilde{U}^{(1)}, \cdots, \tilde{U}^{(N)}), (\tilde{W}^{(1)}, \cdots, \tilde{W}^{(N)}))$ is called an upper solution of (3.3) if it satisfies (3.4) in reversed order.

The pair of lower and upper solutions $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$, $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$ are said to be ordered if $(\hat{\mathbf{U}}, \hat{\mathbf{W}}) \le (\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$. For a given pair of ordered lower and upper solutions $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$, $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$ we set

$$\begin{aligned} &\mathcal{S}^{(l)} \equiv \{U^{(l)} \in \mathbb{R}^{m^{(l)}}; \hat{U}^{(l)} \le U^{(l)} \le \tilde{U}^{(l)}\}, \quad (l = 1, \ldots, N), \\ &\mathcal{S} \equiv \{(\mathbf{U}, \mathbf{W}) \in \mathbb{R}^{M^*}; (\hat{\mathbf{U}}, \hat{\mathbf{W}}) \le (\mathbf{U}, \mathbf{W}) \le (\tilde{\mathbf{U}}, \tilde{\mathbf{W}})\}, \end{aligned}$$

where $m^{(l)} = M^{(l)}$ for $l = 1, \ldots, n_0 - 1$ and $m^{(l)} = \overline{M}^{(l)}$ for $l = n_0, \ldots, N$. It can be easily shown that if $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$ is a lower solution of (2.5), where $\hat{\mathbf{u}}_i = (\hat{u}_i^{(1)}, \cdots, \hat{u}_i^{(N)})$, $\hat{\mathbf{w}}_i = (\hat{w}_i^{(1)}, \cdots, \hat{w}_i^{(N)})$, then its vector representation $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$ with $\hat{\mathbf{U}} = (\hat{U}^{(1)}, \cdots, \hat{U}^{(N)})$, $\hat{\mathbf{W}} = (\hat{W}^{(1)}, \cdots, \hat{W}^{(N)})$ is a lower solution of (3.3). The same is true for an upper solution. Since many of the techniques for the construction of lower and upper solutions of the differential system (2.2) can be used for the finite difference system (2.5) it is often convenient to use Definition 2.1 for the search of explicit lower and upper solutions of specific problems.

To ensure the existence of a positive solution of (3.3) and to obtain a computational algorithm for its numerical values we assume that a pair of ordered lower and upper solutions $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$, $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$ exist and impose the following hypothesis for each $l = 1, \ldots, N$.

(H_2) (i) The matrix $A^{(l)} = (a_{jk}^{(l)})$ is irreducible, and $a_{jj}^{(l)} > 0$, $a_{jk}^{(l)} \le 0$ for $k \ne j$, and

$$\sum_{k=1}^{m^{(l)}} a_{jk}^{(l)} \ge 0 \quad \text{for all} \quad j = 1, \ldots, m^{(l)}. \tag{3.5}$$

(ii) The functions $\mathbf{F}^{(l)}(\mathbf{U})$, $\mathbf{G}^{(l)}(\mathbf{U})$ are nondecreasing in $\mathbf{U}$ for $\hat{\mathbf{U}} \le \mathbf{U} \le \tilde{\mathbf{U}}$.

It is well-known that under the Hypothesis $(H_2) - (i)$ the inverse matrix $(\mathcal{A}^{(l)})^{-1}$ exists and is a positive matrix if strict inequality in (3.5) holds for at least one j (cf. [25,29]). This implies that for any nonnegative matrix $\Gamma^{(l)}$, the inverse $(\mathcal{A}^{(l)})^{-1}$ exists and is positive for $l = 1, \ldots, n_0 - 1$ (In fact, $\mathcal{A}^{(l)}$ is an M-matrix). The same is true for $l = n_0, \ldots, N$ if $\Gamma^{(l)} \ge 0$ and $\Gamma^{(l)} \ne 0$. This property ensures that $U \ge V$ whenever $\mathcal{A}^{(l)} U \ge \mathcal{A}^{(l)} V$ and $\mathcal{A}^{(l)}$ is sometimes called a monotone matrix (cf. [14]). This type of matrices has been widely used in matrix theory and in linear and semilinear elliptic equations(cf. [14,16,18,19,21]). It is easy to verify from the approximation (2.3)-(2.4) and the assumption $h_\nu < |b_\nu^{(l)}(x)|^{-1}$ that the condition in $(H_2) - (i)$ on $a_{ij}^{(l)}$ are satisfied. The same approximation and the connectedness assumption on Ω implies that $A^{(l)}$ is irreducible. Moreover, condition (2.11) ensures that the condition in $(H_2) - (ii)$ on $\mathbf{F}(\mathbf{U})$ and $\mathbf{G}(\mathbf{U})$ are satisfied. Hence, under the hypothesis (H_1) all the requirements in (H_2) are fulfilled. We first consider some computational algorithms for numerical solutions.

3.1. Three monotone iterative schemes.

In this subsection we present three monotone iterations called Picard, Gauss-Seidel, and Jacobi iterations, using either $\hat{\mathbf{U}}$ or $\tilde{\mathbf{U}}$ as the initial iteration.

(A) Picard iteration: For the Picard iteration the sequence is governed by

$$\begin{aligned} &A^{(l)}(W^{(l)})^{(k)} = \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \boldsymbol{\eta}^{(l)}, \quad l = 1, \ldots, n_0 - 1, \\ &\overline{A}^{(l)}(W^{(l)})^{(k)} = \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \mathbf{G}^{(l)}(\mathbf{U}^{(k-1)}), \quad l = n_0, \ldots, N, \\ &(U^{(l)})^{(k)} = Q^{(l)}((W^{(l)})^{(k)}), \quad l = 1, \ldots, N, \end{aligned} \tag{3.6}$$

where $k = 1, 2, \ldots$. It is easily seen that the sequence given by (3.6) is simply a vector representation of the sequence governed by (2.12). Since starting from $k = 1$ the function $\mathbf{U}^{(0)}$ is either $\hat{\mathbf{U}}$ or $\tilde{\mathbf{U}}$ the right-hand side of the first two equations in (3.6) is known. This implies that the first iteration $(W^{(l)})^{(1)}$ exists and can be computed by solving a linear algebraic equation for each $l = 1, \ldots, N$. Knowing the values of $(W^{(l)})^{(1)}$ we can compute the value of $(U^{(l)})^{(1)}$ from the last equation in

(3.6). Using the value of $\mathbf{U}^{(1)} \equiv ((U^{(1)})^{(1)}, \cdots, (U^{(N)})^{(1)})$ instead of $\mathbf{U}^{(0)}$ we can compute the solution $(W^{(l)})^{(2)}$ and then $(U^{(l)})^{(2)}$ for each l. Repeating this process leads to the sequence $\{\mathbf{U}^{(k)}, \mathbf{W}^{(k)}\}$ for every $k = 1, 2, \ldots$. Denote the sequence by $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$ if $\mathbf{U}^{(0)} = \hat{\mathbf{U}}$ and by $\{\overline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)}\}$ if $\mathbf{U}^{(0)} = \tilde{\mathbf{U}}$, and refer to them as minimal and maximal sequence, respectively. The following theorem gives the monotone convergence of these sequences.

Theorem 3.1. *Let $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$, $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$ be a pair of ordered lower and upper solutions of (3.3), and let Hypothesis (H_2) be satisfied. Then the following statements hold:*
(a). The minimal sequence $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$ converges to a minimal solution $(\underline{\mathbf{U}}, \underline{\mathbf{W}})$ of (3.3) in $\mathcal{S}$, and the maximal sequence $\{\overline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)}\}$ converges to a maximal solution $(\overline{\mathbf{U}}, \overline{\mathbf{W}})$. Moreover, they satisfy the relation

$$\begin{aligned}(\hat{\mathbf{U}}, \hat{\mathbf{W}}) &\le (\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}) \le (\underline{\mathbf{U}}^{(k+1)}, \underline{\mathbf{W}}^{(k+1)}) \le (\underline{\mathbf{U}}, \underline{\mathbf{W}}) \le (\overline{\mathbf{U}}, \overline{\mathbf{W}}) \\ &\le (\overline{\mathbf{U}}^{(k+1)}, \overline{\mathbf{W}}^{(k+1)}) \le (\overline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)}) \le (\tilde{\mathbf{U}}, \tilde{\mathbf{W}}), \quad k = 1, 2, \ldots. \end{aligned} \tag{3.7}$$

(b). Any solution $(\mathbf{U}, \mathbf{W})$ of (3.3) in $\mathcal{S}$ satisfies the relation

$$(\underline{\mathbf{U}}, \underline{\mathbf{W}}) \le (\mathbf{U}, \mathbf{W}) \le (\overline{\mathbf{U}}, \overline{\mathbf{W}}). \tag{3.8}$$

(c). If either $\underline{\mathbf{U}} = \overline{\mathbf{U}}$ or $\underline{\mathbf{W}} = \overline{\mathbf{W}}$ then $(\underline{\mathbf{U}}, \underline{\mathbf{W}}) = (\overline{\mathbf{U}}, \overline{\mathbf{W}}) (\equiv (\mathbf{U}^*, \mathbf{W}^*))$ and $(\mathbf{U}^*, \mathbf{W}^*)$ is the unique solution in $\mathcal{S}$.

Proof. (a) Consider the minimal sequence $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$ where $\underline{\mathbf{U}}^{(k)} = ((\underline{U}^{(1)})^{(k)}, \cdots, (\underline{U}^{(N)})^{(k)})$ and $\underline{\mathbf{W}}^{(k)} = ((\underline{W}^{(1)})^{(k)}, \cdots, (\underline{W}^{(N)})^{(k)})$. By (3.6), (3.4) and $\underline{\mathbf{U}}^{(0)} = \hat{\mathbf{U}}$ we have

$$\begin{aligned} A^{(l)}((\underline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(0)}) &= \mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \boldsymbol{\eta}^{(l)} - A^{(l)}(\underline{W}^{(l)})^{(0)} \\ &= \mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \boldsymbol{\eta}^{(l)} - A^{(l)}(\hat{W}^{(l)}) \\ &\ge 0 \quad \text{for} \quad l = 1, \ldots, n_0 - 1, \\ \overline{A}^{(l)}((\underline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(0)}) &= \mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(0)}) - \overline{A}^{(l)}(\underline{W}^{(l)})^{(0)} \\ &= \mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \mathbf{G}^{(l)}(\hat{\mathbf{U}}) - \overline{A}^{(l)}(\hat{W}^{(l)}) \\ &\ge 0 \quad \text{for } l = n_0, \ldots, N. \end{aligned}$$

The positivity of $(A^{(l)})^{-1}$ and $(\overline{A}^{(l)})^{-1}$ ensures that $(\underline{W}^{(l)})^{(1)} \ge (\underline{W}^{(l)})^{(0)}$ for every $l = 1, \ldots, N$. Since $(\underline{U}^{(l)})^{(1)} - (\underline{U}^{(l)})^{(0)} = Q^{(l)}((\underline{W}^{(l)})^{(1)}) - Q^{(l)}((\underline{W}^{(l)})^{(0)})$ the non-decreasing property of $Q^{(l)}$ implies that $(\underline{U}^{(l)})^{(1)} \ge (\underline{U}^{(l)})^{(0)}$ for $l = 1, \ldots, N$. This shows that $((\underline{U}^{(l)})^{(1)}, (\underline{W}^{(l)})^{(1)}) \ge ((\underline{U}^{(l)})^{(0)}, (\underline{W}^{(l)})^{(0)})$. A similar argument using the property of an upper solution gives $((\overline{U}^{(l)})^{(1)}, (\overline{W}^{(l)})^{(1)}) \le ((\overline{U}^{(l)})^{(0)}, (\overline{W}^{(l)})^{(0)})$. Moreover, by (3.6) and the condition in $(H_2) - (ii)$, we have

$$\begin{aligned} A^{(l)}((\overline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(1)}) &= (\mathbf{F}^{(l)}(\overline{\mathbf{U}}^{(0)}) + \boldsymbol{\eta}^{(l)}) - (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \boldsymbol{\eta}^{(l)}) \\ &\ge 0, \quad l = 1, \ldots, n_0 - 1, \\ \overline{A}((\overline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(1)}) &= (\mathbf{F}^{(l)}(\overline{\mathbf{U}}^{(0)}) + \mathbf{G}^{(l)}(\overline{\mathbf{U}}^{(0)})) \\ &\quad -(\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(0)})) \\ &\ge 0, \quad l = n_0, \ldots, N. \end{aligned}$$

This leads to $(\overline{W}^{(l)})^{(1)} \ge (\underline{W}^{(l)})^{(1)}$ for every $l = 1, \dots, N$. This result and $(\overline{U}^{(l)})^{(1)} - (\underline{U}^{(l)})^{(1)} = Q^{(l)}((\overline{W}^{(l)})^{(1)}) - Q^{(l)}((\underline{W}^{(l)})^{(1)})$ yield $(\overline{U}^{(l)})^{(1)} \ge (\underline{U}^{(l)})^{(1)}$. The above conclusions show that

$$\begin{aligned}((\underline{U}^{(l)})^{(0)}, (\underline{W}^{(l)})^{(0)}) &\le ((\underline{U}^{(l)})^{(1)}, (\underline{W}^{(l)})^{(1)})\\ &\le ((\overline{U}^{(l)})^{(1)}, (\overline{W}^{(l)})^{(1)})\\ &\le ((\overline{U}^{(l)})^{(0)}, (\overline{W}^{(l)})^{(0)}).\end{aligned}$$

Assume, by induction, that

$$\begin{aligned}((\underline{U}^{(l)})^{(k-1)}, (\underline{W}^{(l)})^{(k-1)}) &\le ((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)})\\ &\le ((\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)})\\ &\le ((\overline{U}^{(l)})^{(k-1)}, (\overline{W}^{(l)})^{(k-1)})\end{aligned}$$

for some $k > 1$. Then by (3.6) and $(H_2) - (ii)$,

$$\begin{aligned}A^{(l)}((\underline{W}^{(l)})^{(k+1)} - (\underline{W}^{(l)})^{(k)}) &= (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k)}) + \boldsymbol{\eta}^{(l)}) - (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k-1)}) + \boldsymbol{\eta}^{(l)})\\ &\ge 0, \quad l = 1, \dots, n_0 - 1.\\ \overline{A}^{(l)}((\underline{W}^{(l)})^{(k+1)} - (\underline{W}^{(l)})^{(k)}) &= (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k)}) + \mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(k)})) - (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k-1)})\\ &\quad +\mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(k-1)}))\\ &\ge 0, \quad l = n_0, \dots, N.\end{aligned}$$

The positivity of $(A^{(l)})^{-1}$ and $(\overline{A}^{(l)})^{-1}$ implies that $(\underline{W}^{(l)})^{(k+1)} \ge (\underline{W}^{(l)})^{(k)}$ for every $l = 1, \dots, N$. This leads to $(\underline{U}^{(l)})^{(k+1)} \ge (\underline{U}^{(l)})^{(k)}$ which shows that $((\underline{U}^{(l)})^{(k+1)}, (\underline{W}^{(l)})^{(k+1)}) \ge ((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)})$, $l = 1, \dots, N$. A similar argument gives $((\overline{U}^{(l)})^{(k+1)}, (\overline{W}^{(l)})^{(k+1)}) \le ((\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)})$ and $((\underline{U}^{(l)})^{(k+1)}, (\underline{W}^{(l)})^{(k+1)}) \le ((\overline{U}^{(l)})^{(k+1)}, (\overline{W}^{(l)})^{(k+1)})$. The monotone property of the minimal and maximal property in (3.7) follows from the principle of induction. This monotone property implies that for every $l = 1, \dots, N$, the limits

$$\begin{aligned}\lim_{k\to\infty}((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)}) &= (\underline{U}^{(l)}, \underline{W}^{(l)}),\\ \lim_{k\to\infty}((\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)}) &= (\overline{U}^{(l)}, \overline{W}^{(l)}),\end{aligned} \tag{3.9}$$

exist and the vectors $(\underline{\mathbf{U}}, \underline{\mathbf{W}})$, $(\overline{\mathbf{U}}, \overline{\mathbf{W}})$, where

$$\begin{aligned}\underline{\mathbf{U}} &= (\underline{U}^{(1)}, \cdots, \underline{U}^{(N)}),\ \underline{\mathbf{W}} = (\underline{W}^{(1)}, \cdots, \underline{W}^{(N)}),\\ \overline{\mathbf{U}} &= (\overline{U}^{(1)}, \cdots, \overline{U}^{(N)}),\ \overline{\mathbf{W}} = (\overline{W}^{(1)}, \cdots, \overline{W}^{(N)}),\end{aligned}$$

satisfy the relation (3.7). Letting $k \to \infty$ in (3.6) shows that both $(\underline{\mathbf{U}}, \underline{\mathbf{W}})$ and $(\overline{\mathbf{U}}, \overline{\mathbf{W}})$ are solutions of (3.3). (The minimal and maximal property of these solutions is a consequence of the result in (b) below). This proves the conclusion in (a).

(b). To show the relation (3.8) for any solution $(\mathbf{U}, \mathbf{W}) \in \mathcal{S}$ we observe from $(H_2)-(ii)$ and $\hat{\mathbf{U}} \le \mathbf{U} \le \tilde{\mathbf{U}}$ that the components $(U^{(l)}, W^{(l)})$ of $(\mathbf{U}, \mathbf{W})$ satisfy the relation

$$A^{(l)}(W^{(l)} - (\underline{W}^{(l)})^{(1)}) = (\mathbf{F}^{(l)}(\mathbf{U}) + \boldsymbol{\eta}^{(l)}) - (\mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \boldsymbol{\eta}^{(l)}) \ge 0,$$
$$l = 1, \ldots, n_0 - 1,$$
$$\overline{A}^{(l)}(W^{(l)} - (\underline{W}^{(l)})^{(1)}) = (\mathbf{F}^{(l)}(\mathbf{U}) + \mathbf{G}^{(l)}(\mathbf{U})) - (\mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \mathbf{G}^{(l)}(\hat{\mathbf{U}})) \ge 0,$$
$$l = n_0, \ldots, N.$$

This yields $W^{(l)} \ge (\underline{W}^{(l)})^{(1)}$. In view of $U^{(l)} - (\underline{U}^{(l)})^{(1)} = Q^{(l)}(W^{(l)}) - Q^{(l)}((\underline{W}^{(l)})^{(1)}) \ge 0$ we have $(U^{(l)}, W^{(l)}) \ge ((\underline{U}^{(l)})^{(1)}, (\underline{W}^{(l)})^{(1)})$. It follows by an induction argument that $(U^{(l)}, W^{(l)}) \ge ((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)})$ for every $k = 1, 2, \ldots$. A similar argument gives $(U^{(l)}, W^{(l)}) \le ((\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)})$ for every k. Letting $k \to \infty$ and using (3.9) leads to the relation $(\underline{U}^{(l)}, \underline{W}^{(l)}) \le (U^{(l)}, W^{(l)}) \le (\overline{U}^{(l)}, \overline{W}^{(l)})$ for every $l = 1, \ldots, N$. This proves the relation (3.8), and therefore the minimal and maximal property of $(\underline{\mathbf{U}}, \underline{\mathbf{W}})$ and $(\overline{\mathbf{U}}, \overline{\mathbf{W}})$.

(c). If either $\underline{\mathbf{U}} = \overline{\mathbf{U}}$ or $\underline{\mathbf{W}} = \overline{\mathbf{W}}$ then the relation $\underline{\mathbf{U}} = Q(\underline{\mathbf{W}})$ and $\overline{\mathbf{U}} = Q(\overline{\mathbf{W}})$, where $Q(\mathbf{W}) = (Q^{(1)}(W^{(1)}), \cdots, Q^{(N)}(W^{(N)}))$, ensures that $(\underline{\mathbf{U}}, \underline{\mathbf{W}}) = (\overline{\mathbf{U}}, \overline{\mathbf{W}})$. The uniqueness of the solution $(\mathbf{U}^*, \mathbf{W}^*)$ in $\mathcal{S}$ follows from (3.8). This proves the theorem. □

To treat the system (3.3) by the Gauss-Seidel and Jacobi iterations we write $A^{(l)}$ in the split form $A^{(l)} = \mathcal{D}^{(l)} - \mathcal{U}^{(l)} - \mathcal{L}^{(l)}$ for each l, where $\mathcal{D}^{(l)}$, $(-\mathcal{U}^{(l)})$ and $(-\mathcal{L}^{(l)})$ are the diagonal, upper-off-diagonal, and lower-off-diagonal sub-matrices of $A^{(l)}$, respectively. Similarly, we write $\overline{A}^{(l)} = \overline{\mathcal{D}}^{(l)} - \overline{\mathcal{U}}^{(l)} - \overline{\mathcal{L}}^{(l)}$. It is obvious from the Hypothesis $(H_2) - (i)$ that the diagonal elements of $\mathcal{D}^{(l)}$ and $\overline{\mathcal{D}}^{(l)}$ are positive and all the elements of $\mathcal{U}^{(l)}$, $\mathcal{L}^{(l)}$, $\overline{\mathcal{U}}^{(l)}$ and $\overline{\mathcal{L}}^{(l)}$ are nonnegative. Define

$$\mathcal{G}^{(l)} = \mathcal{D}^{(l)} + \Gamma^{(l)} - \mathcal{L}^{(l)}, \; \mathcal{J}^{(l)} = \mathcal{D}^{(l)} + \Gamma^{(l)} \quad \text{for } l = 1, \ldots, n_0 - 1,$$
$$\overline{\mathcal{G}}^{(l)} = \overline{\mathcal{D}}^{(l)} + \overline{\Gamma}^{(l)} - \overline{\mathcal{L}}^{(l)} \;\; \overline{\mathcal{J}}^{(l)} = \overline{\mathcal{D}}^{(l)} + \overline{\Gamma}^{(l)} \quad \text{for } l = n_0, \ldots, N.$$

Then $\mathcal{G}^{(l)} = \mathcal{A}^{(l)} - \mathcal{U}^{(l)}$, $\overline{\mathcal{G}}^{(l)} = \overline{\mathcal{A}}^{(l)} - \overline{\mathcal{U}}^{(l)}$ and $\mathcal{J}^{(l)} = \mathcal{G}^{(l)} - \mathcal{L}^{(l)}$, $\overline{\mathcal{J}}^{(l)} = \overline{\mathcal{G}}^{(l)} - \overline{\mathcal{L}}^{(l)}$.

Using the above notation we have the following additional iterative schemes:

(B) Gauss-Seidel Iteration.

$$\mathcal{G}(W^{(l)})^{(k)} = \mathcal{U}^{(l)}(W^{(l)})^{(k-1)} + \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \boldsymbol{\eta}^{(l)}, \quad l = 1, \ldots, n_0 - 1,$$
$$\overline{\mathcal{G}}(W^{(l)})^{(k)} = \overline{\mathcal{U}}^{(l)}(W^{(l)})^{(k-1)} + \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \mathbf{G}^{(l)}(\mathbf{U}^{(k-1)}), \quad l = n_0, \ldots, N,$$
$$(U^{(l)})^{(k)} = Q^{(l)}((W^{(l)})^{(k)}), \quad l = 1, \ldots, N. \tag{3.10}$$

(C) Jacobi iteration.

$$\mathcal{J}^{(l)}(W^{(l)})^{(k)} = (\mathcal{U}^{(l)} + \mathcal{L}^{(l)})(W^{(l)})^{(k-1)} + \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \boldsymbol{\eta}^{(l)}, \quad l = 1, \ldots, n_0 - 1,$$
$$\overline{\mathcal{J}}^{(l)}(W^{(l)})^{(k)} = (\overline{\mathcal{U}}^{(l)} + \overline{\mathcal{L}}^{(l)})(W^{(l)})^{(k-1)} + \mathbf{F}^{(l)}(\mathbf{U}^{(k-1)}) + \mathbf{G}^{(l)}(\mathbf{U}^{(k-1)}),$$
$$l = n_0, \ldots, N,$$
$$(U^{(l)})^{(k)} = Q^{(l)}((W^{(l)})^{(k)}), \quad l = 1, \ldots, N. \tag{3.11}$$

where $(\mathbf{U}^{(0)}, \mathbf{W}^{(0)})$ is either $(\hat{\mathbf{U}}, \hat{\mathbf{W}})$ or $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}})$, and $k = 1, 2, \ldots$. Since by the Hypothesis $(H_2) - (i)$ the inverses $(\mathcal{G}^{(l)})^{-1}$, $(\mathcal{J}^{(l)})^{-1}$ and $(\overline{\mathcal{G}}^{(l)})^{-1}$, $(\overline{\mathcal{J}}^{(l)})^{-1}$ exist and are nonnegative, the sequences governed by (3.10) and (3.11) are well-defined and can be computed by solving a linear algebraic system with triangular and diagonal coefficient matrices, respectively. Denote the sequence again by $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$ if $(\mathbf{U}^{(0)}, \mathbf{W}^{(0)}) = (\hat{\mathbf{U}}, \hat{\mathbf{W}})$ and by $\{\overline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)}\}$ if $(\mathbf{U}^{(0)}, \mathbf{W}^{(0)}) = (\hat{\mathbf{U}}, \hat{\mathbf{W}})$, and refer to them as minimal and maximal sequences, respectively. The following theorem gives the monotone convergence of these sequences.

Theorem 3.2. *Let the conditions in Theorem 3.1 be satisfied. Then all the conclusions in (a), (b) and (c) of Theorem 3.1 hold true for the minimal and maximal sequences* $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$, $\{\overline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)}\}$ *governed either by the Gauss-Seidel iteration (3.10) or by the Jacobi iteration (3.11).*

Proof. **Gauss-Seidel iteration** We first show the monotone property of the minimal and maximal sequences governed by (3.10). Let $((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)})$, $l = 1, \ldots, N$, be the components of $\{\underline{\mathbf{U}}^{(k)}, \underline{\mathbf{W}}^{(k)}\}$. By (3.4), (3.10), $(\underline{\mathbf{U}}^{(0)}, \underline{\mathbf{W}}^{(0)}) = (\hat{\mathbf{U}}, \hat{\mathbf{W}})$, $\mathcal{A}^{(l)} = \mathcal{G}^{(l)} - \mathcal{U}^{(l)}$ for $l = 1, \ldots, n_0 - 1$ and $\overline{\mathcal{A}}^{(l)} = \overline{\mathcal{G}}^{(l)} - \overline{\mathcal{U}}^{(l)}$ for $l = n_0, \ldots, N$, we have

$$\begin{aligned}
\mathcal{G}^{(l)}((\underline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(0)}) &= [\mathcal{U}^{(l)}(\underline{W}^{(l)})^{(0)} + \mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \boldsymbol{\eta}^{(l)}] - \mathcal{G}^{(l)}(\underline{W}^{(l)})^{(0)} \\
&= (\mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \boldsymbol{\eta}^{(l)}) - \mathcal{A}^{(l)}\hat{W}^{(l)} \\
&\geq 0, \quad l = 1, \ldots, n_0 - 1, \\
\overline{\mathcal{G}}^{(l)}((\underline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(0)}) &= [\overline{\mathcal{U}}^{(l)}(\underline{W}^{(l)})^{(0)} + \mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(0)})] - \overline{\mathcal{G}}^{(l)}(\underline{W}^{(l)})^{(0)} \\
&= \mathbf{F}^{(l)}(\hat{\mathbf{U}}) + \mathbf{G}^{(l)}(\hat{\mathbf{U}}) - \overline{\mathcal{A}}^{(l)}(\hat{W}^{(l)}) \\
&\geq 0, \quad l = n_0, \ldots, N.
\end{aligned}$$

The positivity of $(\mathcal{G}^{(l)})^{-1}$ and $(\overline{\mathcal{G}}^{(l)})^{-1}$ implies that $(\underline{W}^{(l)})^{(1)} \geq (\underline{W}^{(l)})^{(0)}$ for every $l = 1, \ldots, N$. By the relation $(\underline{U}^{(l)})^{(1)} - (\underline{U}^{(l)})^{(0)} = Q^{(l)}((\underline{W}^{(l)})^{(1)}) - Q^{(l)}((\underline{W}^{(l)})^{(0)}) \geq 0$ we obtain $((\underline{U}^{(l)})^{(1)}, (\underline{W}^{(l)})^{(1)}) \geq ((\underline{U}^{(l)})^{(0)}, (\underline{W}^{(l)})^{(0)})$ for $l = 1, \ldots, N$. A similar argument for the maximal sequence $\{(\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)}\}$ gives $((\overline{U}^{(l)})^{(0)}, (\overline{W}^{(l)})^{(0)}) \geq ((\overline{U}^{(l)})^{(1)}, (\overline{W}^{(l)})^{(1)})$ for $l = 1, \ldots, N$. Moreover, by (3.10), $\mathcal{U} \geq 0$ and $(H_2) - (ii)$,

$$\begin{aligned}
\mathcal{G}^{(l)}((\overline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(1)}) &= \mathcal{U}^{(l)}((\overline{W}^{(l)})^{(0)} - (\underline{W}^{(l)})^{(0)}) + (\mathbf{F}^{(l)}(\overline{\mathbf{U}}^{(0)}) \\
&\quad + \boldsymbol{\eta}^{(l)}) - (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \boldsymbol{\eta}^{(l)}) \\
&\geq 0, \quad l = 1, \ldots, n_0 - 1, \\
\overline{\mathcal{G}}^{(l)}((\overline{W}^{(l)})^{(1)} - (\underline{W}^{(l)})^{(1)}) &= \mathcal{U}^{(l)}((\overline{W}^{(l)})^{(0)} - (\underline{W}^{(l)})^{(0)}) + (\mathbf{F}^{(l)}(\overline{\mathbf{U}}^{(0)}) \\
&\quad + \mathbf{G}^{(l)}(\overline{\mathbf{U}}^{(0)})) - (\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)}) + \mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(0)})) \\
&\geq 0, \quad l = n_0, \ldots, N.
\end{aligned}$$

This yields $(\overline{W}^{(l)})^{(1)} \geq (\underline{W}^{(l)})^{(1)}$ which ensures $(\overline{U}^{(l)})^{(1)} \geq (\underline{U}^{(l)})^{(1)}$ for $l = 1, \ldots, N$.

The above conclusions show that

$$\begin{aligned}((\underline{U}^{(l)})^{(0)},(\underline{W}^{(l)})^{(0)}) &\le ((\underline{U}^{(l)})^{(1)},(\underline{W}^{(l)})^{(1)})\\ &\le ((\overline{U}^{(l)})^{(1)},(\overline{W}^{(l)})^{(1)})\\ &\le ((\overline{U}^{(l)})^{(0)},(\overline{W}^{(l)})^{(0)})\end{aligned}$$

for $l=1,\dots,N$. Assume, by induction, that

$$\begin{aligned}((\underline{U}^{(l)})^{(k-1)},(\underline{W}^{(l)})^{(k-1)}) &\le ((\underline{U}^{(l)})^{(k)},(\underline{W}^{(l)})^{(k)})\\ &\le ((\overline{U}^{(l)})^{(k)},(\overline{W}^{(l)})^{(k)})\\ &\le ((\overline{U}^{(l)})^{(k-1)},(\overline{W}^{(l)})^{(k-1)})\end{aligned}$$

for some $k>1$. Then

$$\begin{aligned}\mathcal{G}^{(l)}((\underline{W}^{(l)})^{(k+1)}-(\underline{W}^{(l)})^{(k)}) &= \mathcal{U}^{(l)}((\underline{W}^{(l)})^{(k)}-(\underline{W}^{(l)})^{(k-1)})+(\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k)})\\ &\quad +\boldsymbol{\eta}^{(l)})-(\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k-1)})+\boldsymbol{\eta}^{(l)})\\ &\ge 0 \quad \text{for} \quad l=1,\dots,n_0-1,\\ \overline{\mathcal{G}}^{(l)}((\underline{W}^{(l)})^{(k+1)}-(\underline{W}^{(l)})^{(k)}) &= \mathcal{U}^{(l)}((\underline{W}^{(l)})^{(k)}-(\underline{W}^{(l)})^{(k-1)})+(\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k)})\\ &\quad +\mathbf{G}^{(l)}(\underline{U}^{(k)}))-(\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(k-1)})+\mathbf{G}^{(l)}(\underline{\mathbf{U}}^{(k-1)}))\\ &\ge 0 \quad \text{for} \quad l=n_0,\dots,N.\end{aligned}$$

This leads to $(\underline{W}^{(l)})^{(k+1)}\ge(\underline{W}^{(l)})^{(k)}$ which yields $(\underline{U}^{(l)})^{(k+1)}\ge(\underline{U}^{(l)})^{(k)}$. A similar argument gives $((\overline{U}^{(l)})^{(k+1)},(\overline{W}^{(l)})^{(k+1)})\le((\overline{U}^{(l)})^{(k)},(\overline{W}^{(l)})^{(k)})$ and $((\underline{U}^{(l)})^{(k+1)},(\underline{W}^{(l)})^{(k+1)})\le((\overline{U}^{(l)})^{(k+1)},(\overline{W}^{(l)})^{(k+1)})$. It follows from the principle of induction that the above sequences possess the monotone property in (3.7). This implies that the limits $(\underline{U}^{(l)},\underline{W}^{(l)})$, $(\overline{U}^{(l)},\overline{W}^{(l)})$ in (3.9) exist and satisfy relation (3.7). Letting $k\to\infty$ in (3.10) shows that both $(\underline{U}^{(l)},\underline{W}^{(l)})$ and $(\overline{U}^{(l)},\overline{W}^{(l)})$ satisfy the equation

$$\begin{aligned}&\mathcal{G}^{(l)}W^{(l)}=\mathcal{U}^{(l)}W^{(l)}+\mathbf{F}^{(l)}(\mathbf{U})+\boldsymbol{\eta}^{(l)}, \quad l=1,\dots,n_0-1,\\ &\overline{\mathcal{G}}^{(l)}W^{(l)}=\overline{\mathcal{U}}^{(l)}W^{(l)}+\mathbf{F}^{(l)}(\mathbf{U})+\mathbf{G}^{(l)}(\mathbf{U}), \quad l=n_0,\dots,N,\\ &U^{(l)}=Q^{(l)}(W^{(l)}), \quad l=1,\dots,N.\end{aligned}$$

Since $\mathcal{G}^{(l)}-\mathcal{U}^{(l)}=\mathcal{A}^{(l)}$ and $\overline{\mathcal{G}}^{(l)}-\overline{\mathcal{U}}^{(l)}=\overline{\mathcal{A}}^{(l)}$ we conclude that both $(\underline{\mathbf{U}},\underline{\mathbf{W}})$ and $(\overline{\mathbf{U}},\overline{\mathbf{W}})$ are solutions of (3.3). This proves the result in (a).

(b). Let $(\mathbf{U},\mathbf{W})$ be any solution of (3.3) in $\mathcal{S}$. Then its components $(U^{(l)},W^{(l)})$ satisfy the relation

$$\begin{aligned}\mathcal{G}^{(l)}(W^{(l)}-(\underline{W}^{(l)})^{(1)}) &= (\mathcal{U}^{(l)}W^{(l)}+\mathbf{F}^{(l)}(\mathbf{U})+\boldsymbol{\eta}^{(l)})-(\mathcal{U}^{(l)}(\underline{W}^{(l)})^{(0)}\\ &\quad +\mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(0)})+\boldsymbol{\eta}^{(l)})\\ &= \mathcal{U}^{(l)}(W^{(l)}-\hat{W}^{(l)})+\mathbf{F}^{(l)}(\mathbf{U})-\mathbf{F}^{(l)}(\hat{\mathbf{U}})\\ &\ge 0, \quad l=1,\dots,n_0-1,\\ \mathcal{G}^{(l)}(W^{(l)}-(\underline{W}^{(l)})^{(1)}) &= (\overline{\mathcal{U}}^{(l)}W^{(l)}+\mathbf{F}^{(l)}(\mathbf{U})+\mathbf{G}^{(l)}(\mathbf{U}))\end{aligned}$$

$$\begin{aligned}
&-(\overline{\mathcal{U}}^{(l)}(\underline{W}^{(l)})^{(0)} + \mathbf{F}^{(l)}(\underline{\mathbf{U}}^{(l)})^{(0)} + \mathcal{G}^{(l)}(\underline{\mathbf{U}}^{(l)})^{(0)})\\
&= \overline{\mathcal{U}}^{(l)}(W^{(l)} - \hat{W}^{(l)}) + (\mathbf{F}^{(l)}(\mathbf{U}) - \mathbf{F}^{(l)}(\hat{\mathbf{U}}))\\
&\quad +(\mathbf{G}^{(l)}(\mathbf{U}) - \mathbf{G}^{(l)}(\hat{\mathbf{U}}))\\
&\geq 0, \quad l = n_0, \ldots, N.
\end{aligned}$$

The positivity of $(\mathcal{G}^{(l)})^{-1}$ and $(\overline{\mathcal{G}}^{(l)})^{-1}$ ensures that $W^{(l)} \geq (\underline{W}^{(l)})^{(1)}$ for every $l = 1, \ldots, N$. This leads to $U^{(l)} \geq (\underline{U}^{(l)})^{(1)}$. A similar argument using the property of an upper solution gives $(U^{(l)}, W^{(l)}) \leq ((\overline{U}^{(l)})^{(1)}, (\overline{W}^{(l)})^{(1)})$. This proves $((\underline{U}^{(l)})^{(1)}, (\underline{W}^{(l)})^{(1)}) \leq (U^{(l)}, W^{(l)}) \leq ((\overline{U}^{(l)})^{(1)}, (\overline{W}^{(l)})^{(1)})$. It follows by an induction argument that

$$((\underline{U}^{(l)})^{(k)}, (\underline{W}^{(l)})^{(k)}) \leq (U^{(l)}, W^{(l)}) \leq ((\overline{U}^{(l)})^{(k)}, (\overline{W}^{(l)})^{(k)})$$

for every $k = 1, 2, \ldots$. Letting $k \to \infty$ shows that $(\underline{U}^{(l)}, \underline{W}^{(l)}) \leq (U^{(l)}, W^{(l)}) \leq (\underline{U}^{(l)}, \underline{W}^{(l)})$, $l = 1, \ldots, N$. This proves the relation (3.8).
(c). The proof for the result in (c) is the same as for Picard iteration.

Jacobi Iteration For Jacobi iteration, we replace the matrices $(\mathcal{G}^{(l)}, \overline{\mathcal{G}}^{(l)})$ by $(\mathcal{J}^{(l)}, \overline{\mathcal{J}}^{(l)})$ and $(\mathcal{U}, \overline{\mathcal{U}})$ by $(\mathcal{U}+\mathcal{L}, \overline{\mathcal{U}}+\overline{\mathcal{L}})$, respectively, in the proof for Gauss-Seidel iteration. Then by the positive property of $((\mathcal{J}^{(l)})^{-1}, (\overline{\mathcal{J}}^{(l)})^{-1})$ and the nonnegative property of $((\mathcal{U}+\mathcal{L}), (\overline{\mathcal{U}}+\overline{\mathcal{L}}))$ the same reasoning leads to the conclusions in (a), (b) and (c) of Theorem 3.1. Details are omitted. □

3.2. Comparison of monotone sequences.

It is seen from Theorem 3.1 and Theorem 3.2 that the three monotone iterations in (3.6), (3.10) and (3.11) lead to the same minimal and maximal solutions $(\underline{\mathbf{U}}, \underline{\mathbf{W}})$, $(\overline{\mathbf{U}}, \overline{\mathbf{W}})$ of (3.3). Denote the corresponding minimal and maximal sequences by

$$\begin{aligned}
&(\{\underline{\mathbf{U}}_P^{(k)}, \underline{\mathbf{W}}_P^{(k)}\}, \{\overline{\mathbf{U}}_P^{(k)}, \overline{\mathbf{W}}_P^{(k)}\}),\\
&(\{\underline{\mathbf{U}}_G^{(k)}, \underline{\mathbf{W}}_G^{(k)}\}, \{\overline{\mathbf{U}}_G^{(k)}, \overline{\mathbf{W}}_G^{(k)}\}),\\
&(\{\underline{\mathbf{U}}_J^{(k)}, \underline{\mathbf{W}}_J^{(k)}\}, \{\overline{\mathbf{U}}_J^{(k)}, \overline{\mathbf{W}}_J^{(k)}\}),
\end{aligned} \tag{3.12}$$

respectively, where

$$\begin{aligned}
(\underline{\mathbf{U}}_P^{(0)}, \underline{\mathbf{W}}_P^{(0)}) = (\underline{\mathbf{U}}_G^{(0)}, \underline{\mathbf{W}}_G^{(0)}) = (\underline{\mathbf{U}}_J^{(0)}, \underline{\mathbf{W}}_J^{(0)}) = (\hat{\mathbf{U}}, \hat{\mathbf{W}}),\\
(\overline{\mathbf{U}}_P^{(0)}, \overline{\mathbf{W}}_P^{(0)}) = (\overline{\mathbf{U}}_G^{(0)}, \overline{\mathbf{W}}_G^{(0)}) = (\overline{\mathbf{U}}_J^{(0)}, \overline{\mathbf{W}}_J^{(0)}) = (\tilde{\mathbf{U}}, \tilde{\mathbf{W}}).
\end{aligned}$$

Then we have the following comparison results among these sequences:

Theorem 3.3. *Let the conditions in Theorem 3.1 be satisfied. Then the three minimal and maximal sequences in (3.12) possess the following comparison property:*

$$\begin{aligned}
&(\underline{\mathbf{U}}_P^{(k)}, \underline{\mathbf{W}}_P^{(k)}) \leq (\underline{\mathbf{U}}_G^{(k)}, \underline{\mathbf{W}}_G^{(k)}) \leq (\underline{\mathbf{U}}_J^{(k)}, \underline{\mathbf{W}}_J^{(k)}),\\
&(\overline{\mathbf{U}}_P^{(k)}, \overline{\mathbf{W}}_P^{(k)}) \geq (\overline{\mathbf{U}}_G^{(k)}, \overline{\mathbf{W}}_G^{(k)}) \geq (\overline{\mathbf{U}}_J^{(k)}, \overline{\mathbf{W}}_J^{(k)}),\ k = 1, 2, \ldots.
\end{aligned} \tag{3.13}$$

Proof. We prove the theorem for the minimal sequence in (3.13). The proof for the maximal sequence is similar. Let $(Z^{(l)})^{(k)} = (\underline{W}_P^{(l)})^{(k)} - (\underline{W}_G^{(l)})^{(k)}$. Since $\mathcal{A}^{(l)} = \mathcal{D}^{(l)} - \mathcal{U}^{(l)} - \mathcal{L}^{(l)} + \Gamma^{(l)} = \mathcal{G}^{(l)} - \mathcal{U}^{(l)}$ and $\overline{\mathcal{A}}^{(l)} = \overline{\mathcal{G}}^{(l)} - \overline{\mathcal{U}}^{(l)}$ we see from (3.6), (3.10) and $(\underline{W}^{(l)})_G^{(k)} \geq (\underline{W}_G^{(l)})^{(k-1)}$ that

$$\begin{aligned} \mathcal{A}^{(l)}(Z^{(l)})^{(k)} &= [\mathbf{F}^{(l)}(\underline{\mathbf{U}}_P^{(k-1)}) + \boldsymbol{\eta}^{(l)}] - [\mathcal{G}^{(l)}(\underline{W}_G^{(l)})^{(k)} - \mathcal{U}^{(l)}(\underline{W}_G^{(l)})^{(k)}] \\ &= [\mathbf{F}^{(l)}(\underline{\mathbf{U}}_P^{(k-1)}) + \boldsymbol{\eta}^{(l)}] - [\mathcal{U}^{(l)}((\underline{W}_G^{(l)})^{(k-1)} - (\underline{W}_G^{(l)})^{(k)}) \\ &\quad + (\mathbf{F}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)}) + \boldsymbol{\eta}^{(l)})] \\ &\geq \mathbf{F}^{(l)}(\underline{\mathbf{U}}_P^{(k-1)}) - \mathbf{F}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)}) \quad \text{for} \quad l = 1, \ldots, n_0 - 1. \end{aligned} \tag{3.14}$$

The same reasoning gives

$$\begin{aligned} \overline{\mathcal{A}}^{(l)}(Z^{(l)})^{(k)} &\geq (\mathbf{F}^{(l)}(\underline{\mathbf{U}}_P^{(k-1)}) - \mathbf{F}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)})) \\ &\quad + (\mathbf{G}^{(l)}(\underline{\mathbf{U}}_P^{(k-1)}) - \mathbf{G}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)})) \quad \text{for} \quad l = n_0, \ldots, N. \end{aligned} \tag{3.15}$$

Consider the case $k = 1$. Since $(\underline{U}_P^{(l)})^{(0)} = (\underline{U}_G^{(l)})^{(0)} = \hat{U}^{(l)}$ the inequalities in (3.14), (3.15) imply that

$$\mathcal{A}^{(l)}(Z^{(l)})^{(1)} \geq 0 \quad \text{for} \quad l = 1, \ldots, n_0 - 1 \quad \text{and} \quad \overline{\mathcal{A}}^{(l)}(Z^{(l)})^{(1)} \geq 0 \quad \text{for} \quad l = n_0, \ldots, N.$$

This gives $(Z^{(l)})^{(1)} \geq 0$ for every $l = 1, \ldots, N$, or equivalently, $(\underline{W}_P^{(l)})^{(1)} \geq (\underline{W}_G^{(l)})^{(1)}$. It follows from $(\underline{U}_P^{(l)})^{(1)} - (\underline{U}_G^{(l)})^{(1)} = Q^{(l)}((\underline{W}_P^{(l)})^{(1)}) - Q^{(l)}((\underline{W}_G^{(l)})^{(1)}) \geq 0$ that

$$((\underline{U}_P^{(l)})^{(1)}, (\underline{W}_P^{(l)})^{(1)}) \geq ((\underline{U}_G^{(l)})^{(1)}, (\underline{W}_G^{(l)})^{(1)}) \quad \text{for every} \quad l = 1, \ldots, N.$$

Assume by induction that

$$((\underline{U}_P^{(l)})^{(k)}, (\underline{W}_P^{(l)})^{(k)}) \geq ((\underline{U}_G^{(l)})^{(k)}, (\underline{W}_G^{(l)})^{(k)}) \quad (l = 1, \ldots, N)$$

for some $k > 1$. Then by (3.14), (3.15) and $(H_2) - (ii)$ with k replaced by $(k+1)$ we obtain

$$\mathcal{A}^{(l)}(Z^{(l)})^{(k+1)} \geq 0 \quad \text{for} \quad l = 1, \ldots, n_0 - 1 \quad \text{and} \quad \overline{\mathcal{A}}^{(l)}(Z^{(l)})^{(k+1)} \geq 0 \quad \text{for} \quad l = n_0, \ldots, N.$$

This gives $(Z^{(l)})^{(k+1)} \geq 0$ which yields $(\underline{W}_P^{(l)})^{(k+1)} \geq (\underline{W}_G^{(l)})^{(k+1)}$ for every $l = 1, \ldots, N$. The nondecreasing property of $Q^{(l)}$ ensures that $(\underline{U}_P^{(l)})^{(k+1)} \geq (\underline{U}_G^{(l)})^{(k+1)}$. The first inequality in (3.13) for the minimal sequence follows from the principle of induction.

To show the second inequality of (3.13) for the minimal sequences we let $(Z^{(l)})^{(k)} = (W_G^{(l)})^{(k)} - (W_J^{(l)})^{(k)}$ for $l = 1, \ldots, N$. Then by (3.6), $\mathcal{G}^{(l)} = \mathcal{J}^{(l)} - \mathcal{L}^{(l)}$ and the nonnegative property of $\mathcal{L}^{(l)}$ we have

$$\begin{aligned} \mathcal{G}^{(l)}(Z^{(l)})^{(k)} &= \mathcal{G}^{(l)}(\underline{W}_G^{(l)})^{(k)} - [\mathcal{J}^{(l)}(\underline{W}_J^{(l)})^{(k)} - \mathcal{L}^{(l)}(\underline{W}_J^{(l)})^{(k)}] \\ &= [\mathcal{U}^{(l)}(\underline{W}_G^{(l)})^{(k-1)} + (\mathbf{F}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)}) + \boldsymbol{\eta}^{(l)})] - [\mathcal{U}^{(l)}(\underline{W}_J^{(l)})^{(k-1)} \\ &\quad - \mathcal{L}^{(l)}((\underline{W}_J^{(l)})^{(k)} - (\underline{W}_J^{(l)})^{(k-1)}) + (\mathbf{F}^{(l)}(\underline{\mathbf{U}}_J^{(k-1)}) + \boldsymbol{\eta}^{(l)})] \\ &\geq \mathcal{U}^{(l)}((\underline{W}_G^{(l)})^{(k-1)} - (\underline{W}_J^{(l)})^{(k-1)}) + (\mathbf{F}^{(l)}(\mathbf{U}_G^{(k-1)}) - \mathbf{F}^{(l)}(\underline{\mathbf{U}}_J^{(k-1)})) \\ &\quad \text{for} \quad l = 1, \ldots, n_0 - 1. \end{aligned}$$

Similarly

$$\overline{\mathcal{G}}^{(l)}(Z^{(l)})^{(k)} \geq \overline{\mathcal{U}}^{(l)}((\underline{W}_G^{(l)})^{(k-1)} - (\underline{W}_J^{(l)})^{(k-1)}) + (\mathbf{F}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)}) - \mathbf{F}^{(l)}(\underline{\mathbf{U}}_J^{(k-1)})) \\ +(\mathbf{G}^{(l)}(\underline{\mathbf{U}}_G^{(k-1)}) - \mathbf{G}^{(l)}(\underline{\mathbf{U}}_J^{(k-1)})) \;\text{ for }\; l = n_0, \ldots, N.$$

Using the relation $(\underline{\mathbf{U}}_G^{(0)}, \underline{\mathbf{W}}_G^{(0)}) = (\underline{\mathbf{U}}_J^{(0)}, \underline{\mathbf{W}}_J^{(0)}) = (\hat{\mathbf{U}}, \hat{\mathbf{W}})$ for $k = 1$ in the above inequalities gives

$$\mathcal{G}^{(l)}(Z^{(l)})^{(1)} \geq 0 \;\text{ for }\; l = 1, \ldots, n_0 - 1 \;\text{ and }\; \overline{\mathcal{G}}^{(l)}(Z^{(l)})^{(1)} \geq 0 \;\text{ for }\; l = n_0, \ldots, N.$$

This yields $(\underline{W}_G^{(l)})^{(1)} \geq (\underline{W}_J^{(l)})^{(1)}$ and therefore $(\underline{U}_G^{(l)})^{(1)} \geq (\underline{U}_J^{(l)})^{(1)}$ for every $l = 1, \ldots, N$. It follows by an induction argument that

$$(\underline{W}_G^{(l)})^{(k)} \geq (\underline{W}_J^{(l)})^{(k)} \;\text{ and }\; (\underline{U}_G^{(l)})^{(k)} \geq (\underline{U}_J^{(l)})^{(k)} \;\text{ for every }\; k = 1, 2, \ldots.$$

This proves the second inequality of (3.13) for the minimal sequences. □

Remark 3.1. (a). Theorem 3.2 implies that with the same initial iteration $\hat{\mathbf{U}}$ or $\tilde{\mathbf{U}}$ the sequence governed by the Picard iteration converges faster than the Gauss-Seidel iteration which, in turn, converges faster than the Jacobi iteration. However, the Jacobi iteration is the simplest to use in practical computation while the Picard iteration may require additional iterations for each k if the spatial domain Ω is of two or higher dimension. On the other hand, since $\mathcal{G}^{(l)}$ and $\overline{\mathcal{G}}^{(l)}$ are triangular matrices it is more suitable for practical computation when Ω is of higher dimension. (b). In Theorem 3.1 and Theorem 3.2 if $\underline{\mathbf{U}} = \overline{\mathbf{U}}$ or $\underline{\mathbf{W}} = \overline{\mathbf{W}}$ then the solution $(\mathbf{U}^*, \mathbf{W}^*)$ is unique in $\mathcal{S}$ and satisfies the relation (3.7). This implies that an error estimate for $(\mathbf{U}^*, \mathbf{W}^*)$ is given by

$$(\overline{\mathbf{U}}^{(k)} - \underline{\mathbf{U}}^{(k)}, \overline{\mathbf{W}}^{(k)} - \underline{\mathbf{W}}^{(k)}), \quad k = 1, 2, \ldots.$$

and this error decreases to zero as $k \to \infty$. This result and the monotone convergence of the minimal and maximal sequences to $(\mathbf{U}^*, \mathbf{W}^*)$ is very useful in practical computation since it does not require any explicit knowledge of the solution.

4. Convergence of finite difference solutions

Using the method of lower and upper solutions for both continuous and finite difference problems we can prove the convergence of the finite difference solution $(\mathbf{u}_i, \mathbf{w}_i)$ of (2.5) to the continuous solution $(\mathbf{u}(x_i), \mathbf{w}(x_i))$ of (2.2) at every mesh point x_i of a given partition $\overline{\Lambda}^*$. Recall that a smooth function $(\hat{\mathbf{u}}(x), \hat{\mathbf{w}}(x))$ is called a lower solution of (2.2) if it satisfies (2.2) with all the equality sign "=" replaced by the inequality sign "≤". Similarly, $(\tilde{\mathbf{u}}(x), \tilde{\mathbf{w}}(x))$ is called an upper solution if it satisfies (2.2) with "=" replace by "≥" (cf [22]). This pair of lower and upper solutions are said to be ordered if $(\hat{\mathbf{u}}(x), \hat{\mathbf{w}}(x)) \leq (\tilde{\mathbf{u}}(x), \tilde{\mathbf{w}}(x))$ in $\overline{\Omega}$. Assume that problem (2.2) has a pair of ordered lower and upper solutions, and the functions $c(x), \eta(x), f(x, \mathbf{u})$ and $g(x, \mathbf{u})$ are smooth functions in their respective domains. Then by using either $(\hat{\mathbf{u}}(x), \hat{\mathbf{w}}(x))$ or $(\tilde{\mathbf{u}}(x), \tilde{\mathbf{w}}(x))$ as the initial iteration $(\mathbf{u}^{(0)}, \mathbf{w}^{(0)})$ we can construct a sequence $\{\mathbf{u}^{(k)}, \mathbf{w}^{(k)}\}$ from the linear iteration process

$$\begin{aligned}
&L^*[(w^{(l)})^{(k)}] = F^{(l)}(x, \mathbf{u}^{(k-1)}), && l = 1, \ldots, N, && (x \in \Omega),\\
&(w^{(l)})^{(k)} = \eta^{(l)}, && l = 1, \ldots, n_0 - 1, && (x \in \partial\Omega),\\
&\frac{\partial}{\partial \nu}(w^{(l)})^{(k)} + \overline{\gamma}^{(l)}(w^{(l)})^{(k)} = G^{(l)}(x, \mathbf{u}^{(k-1)}), && l = n_0, \ldots, N, && (x \in \partial\Omega),\\
&(u^{(l)})^{(k)} = q^{(l)}((w^{(l)})^{(k)}), && l = 1, \ldots, N, && (x \in \overline{\Omega})
\end{aligned} \tag{4.1}$$

for $k = 1, 2, \ldots$, where $((u^{(l)})^{(k)}, (w^{(l)})^{(k)})$, $l = 1, \ldots, N$, are the components of $(\mathbf{u}^{(k)}, \mathbf{w}^{(k)})$ and

$$L^*[w^{(l)}] \equiv -\Delta w^{(l)} + \mathbf{b} \cdot \nabla w^{(l)} + \gamma^{(l)} w^{(l)}.$$

It is obvious that the sequence $\{\mathbf{u}^{(k)}, \mathbf{w}^{(k)}\}$ governed by (4.1) is well-defined and can be obtained by solving a linear Dirichlet boundary problem for $l = 1, \ldots, n_0 - 1$ and a Robin boundary problem for $l = n_0, \ldots, N$. We denote the sequence by $\{\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}\}$ if $(\mathbf{u}^{(0)}, \mathbf{w}^{(0)}) = (\hat{\mathbf{u}}, \hat{\mathbf{w}})$ and by $\{\overline{\mathbf{u}}^{(k)}, \overline{\mathbf{w}}^{(k)}\}$ if $(\mathbf{u}^{(0)}, \mathbf{w}^{(0)}) = (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$, and refer to them as minimal and maximal sequence, respectively. In the following theorem we state the monotone convergence of these sequences from [22].

Theorem 4.1. *Let $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$, $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ be a pair of ordered lower and upper solutions of (2.2), and let Hypothesis (H_1) be satisfied. Then as $k \to \infty$, the sequence $\{\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}\}$ converges to a minimal solution $(\underline{\mathbf{u}}, \underline{\mathbf{w}})$ of (2.2), and $\{\overline{\mathbf{u}}^{(k)}, \overline{\mathbf{w}}^{(k)}\}$ converges to a maximal solution $(\overline{\mathbf{u}}, \overline{\mathbf{w}})$. Moreover,*

$$\begin{aligned}
(\hat{\mathbf{u}}, \hat{\mathbf{w}}) &\le (\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}) \le (\underline{\mathbf{u}}^{(k+1)}, \underline{\mathbf{w}}^{(k+1)}) \le (\underline{\mathbf{u}}, \underline{\mathbf{w}})\\
&\le (\overline{\mathbf{u}}, \overline{\mathbf{w}}) \le (\overline{\mathbf{u}}^{(k+1)}, \overline{\mathbf{w}}^{(k+1)}) \le (\overline{\mathbf{u}}^{(k)}, \overline{\mathbf{w}}^{(k)})\\
&\le (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \quad \textit{for every} \quad k = 1, 2, \ldots.
\end{aligned} \tag{4.2}$$

and if $(\underline{\mathbf{u}}, \underline{\mathbf{w}}) = (\overline{\mathbf{u}}, \overline{\mathbf{w}}) (\equiv (\mathbf{u}^*, \mathbf{w}^*))$ then $(\mathbf{u}^*, \mathbf{w}^*)$ is the unique solution of (2.2) between $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$ and $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$.

A proof of the above theorem can be found in a recent article [22]. By using the above monotone convergence theorem and the results in Theorem 2.1 we show the convergence of the minimal and maximal finite difference solutions to the corresponding continuous solutions at every mesh point $x_i \in \overline{\Lambda}^*$, where $\overline{\Lambda}^*$ is a fixed partition of $\overline{\Omega}$. It is assumed that every refinement of $\overline{\Lambda}^*$ contains $\overline{\Lambda}^*$, and there exist a pair of ordered lower and upper solutions $(\hat{\mathbf{u}}(x), \hat{\mathbf{w}}(x))$, $(\tilde{\mathbf{u}}(x), \tilde{\mathbf{w}}(x))$ of (2.2) and $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ of (2.5). It is also assumed that given any $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for $|h| < \delta_0$ and $l = 1, \ldots, N$,

$$\begin{aligned}
&|\hat{u}^{(l)}(x_i) - \hat{u}_i^{(l)}| + (\hat{w}^{(l)}(x_i) - \hat{w}_i^{(l)}| < \epsilon_0,\\
&|\tilde{u}^{(l)}(x_i) - \tilde{u}_i^{(l)}| + |\tilde{w}^{(l)}(x_i) - \tilde{w}_i^{(l)}| < \epsilon_0,
\end{aligned} \tag{4.3}$$

where $(\hat{u}^{(l)}(x_i), \hat{w}^{(l)}(x_i))$ and $(\tilde{u}^{(l)}(x_i), \tilde{w}^{(l)}(x_i))$ are the respective components of $(\hat{\mathbf{u}}(x_i), \hat{\mathbf{w}}(x_i))$ and $(\tilde{\mathbf{u}}(x_i), \tilde{\mathbf{w}}(x_i))$ and $|h| = h_1 + \ldots + h_p$.

Theorem 4.2. *Let Hypothesis (H_1) and condition (4.3) be satisfied, and let $((\hat{\mathbf{u}}(x), \hat{\mathbf{w}}(x)), (\tilde{\mathbf{u}}(x), \tilde{\mathbf{w}}(x)))$ and $((\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i), (\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i))$ be pairs of lower and upper solutions of (2.2) and (2.5), respectively. Then at every point $x_i \in \overline{\Lambda}^*$, $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$ converges to $(\underline{\mathbf{u}}(x_i), \underline{\mathbf{w}}(x_i))$ and $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$ converges to $(\overline{\mathbf{u}}(x_i), \overline{\mathbf{w}}(x_i))$ as $|h| \to 0$.*

Proof. We prove the convergence of the minimal solution $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$ to $(\hat{\mathbf{u}}(x_i), \hat{\mathbf{w}}(x_i))$. For this purpose, it suffices to show that given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\underline{\mathbf{u}}(x_i) - \underline{\mathbf{u}}_i| + |\underline{\mathbf{w}}(x_i) - \underline{\mathbf{w}}_i| < \epsilon \quad \text{when} \quad |h| < \delta, \tag{4.4}$$

where $|\mathbf{z}| = |z^{(1)}| + \ldots + |z^{(N)}|$ for any $\mathbf{z} = (z^{(1)}, \cdots, z^{(N)}) \in \mathbb{R}^N$. Let $\{\underline{\mathbf{u}}^{(k)}(x_i), \underline{\mathbf{w}}^{(k)}(x_i)\}$ be the minimal sequence governed by (4.1). By Theorem 4.1 and Theorem 2.1 there exists $k^* \geq 1$ such that for all $i \in \overline{\Lambda}^*$,

$$\begin{aligned}
&|\underline{\mathbf{u}}^{(k)}(x_i) - \underline{\mathbf{u}}(x_i)| + |\mathbf{w}^{(k)}(x_i) - \underline{\mathbf{w}}(x_i)| < \epsilon/3, \\
&|\underline{\mathbf{u}}_i^{(k)} - \underline{\mathbf{u}}_i| + |\underline{\mathbf{w}}_i^{(k)} - \underline{\mathbf{w}}_i| < \epsilon/3 \quad \text{when} \quad k \geq k^*.
\end{aligned}$$

Since

$$\begin{aligned}
|\underline{\mathbf{u}}(x_i) - \underline{\mathbf{u}}_i| \; &\leq |\underline{\mathbf{u}}(x_i) - \underline{\mathbf{u}}^{(k)}(x_i)| + |\underline{\mathbf{u}}^{(k)}(x_i) - \underline{\mathbf{u}}_i^{(k)}| + |\underline{\mathbf{u}}_i^{(k)} - \underline{\mathbf{u}}_i|, \\
|\underline{\mathbf{w}}(x_i) - \underline{\mathbf{w}}_i| &\leq |\underline{\mathbf{w}}(x_i) - \underline{\mathbf{w}}^{(k)}(x_i)| + |\underline{\mathbf{w}}^{(k)}(x_i) - \underline{\mathbf{w}}_i^{(k)}| + |\underline{\mathbf{w}}_i^{(k)} - \underline{\mathbf{w}}_i|
\end{aligned}$$

for every k and i, condition (4.4) is satisfied if there exists $k \geq k^*$ such that

$$|\underline{\mathbf{u}}^{(k)}(x_i) - \underline{\mathbf{u}}_i^{(k)}| + |\underline{\mathbf{w}}^{(k)}(x_i) - \underline{\mathbf{w}}_i^{(k)}| < \epsilon/3 \quad \text{when} \quad |h| < \delta. \tag{4.5}$$

To show this we observe from (4.1) and (2.3)-(2.4) that the minimal sequence $\{\underline{\mathbf{u}}^{(k)}(x_i), \underline{\mathbf{w}}^{(k)}(x_i)\}$ satisfies the relation

$$\begin{aligned}
&L^{(l)}[(\underline{w}^{(l)}(x_i))^{(k)}] = F_i^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) + o^{(k)}(|h|^2), \quad l = 1, \ldots, N, (x_i \in \Lambda^*), \\
&(\underline{w}^{(l)}(x_i))^{(k)} = \eta^{(l)}(x_i), \quad l = 1, \ldots, n_0 - 1, (x_i \in \partial\Lambda^*), \\
&B[(\underline{w}^{(l)}(x_i))^{(k)}] = G^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) + o^{(k)}(|h|), \quad l = n_0, \ldots, N, (x_i \in \partial\Lambda^*), \\
&(\underline{u}^{(l)}(x_i))^{(k)} = q^{(l)}((\underline{w}^{(l)}(x_i))^{(k)}), \quad l = 1, \ldots, N, (x_i \in \overline{\Lambda}^*),
\end{aligned} \tag{4.6}$$

where $((\underline{u}^{(l)}(x_i), \underline{w}^{(l)}(x_i))$, $l = 1, \ldots, N$, are the components of $(\underline{\mathbf{u}}(x_i), \underline{\mathbf{w}}(x_i))$ and $o^{(k)}(|h|) \to 0$ as $|h| \to 0$. Let

$$(v_i^{(l)})^{(k)} = (\underline{u}^{(l)}(x_i))^{(k)} - (\underline{u}_i^{(l)})^{(k)}, \quad (z_i^{(l)})^{(k)} = (\underline{w}^{(l)}(x_i))^{(k)} - (\underline{w}_i^{(l)})^{(k)}.$$

Then a subtraction of (2.12) from (4.6) gives

$$\begin{aligned}
&L^{(l)}[(z_i^{(l)})^{(k)}] = F_i^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) - F_i^{(l)}(\underline{\mathbf{u}}_i^{(k-1)}) + o^{(k)}(|h|^2), \quad l = 1, \ldots, N, \\
&(z_i^{(l)})^{(k)} = 0, \quad l = 1, \ldots, n_0 - 1, \\
&B^{(l)}[(z_i^{(l)})^{(k)}] + \overline{\gamma}_i^{(l)}(z_i^{(l)})^{(k)} = G_i^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) - G_i^{(l)}(\underline{\mathbf{u}}_i^{(k-1)}) + o^{(k)}(|h|), \\
&\qquad\qquad l = n_0, \ldots, N, \\
&(v_i^{(l)})^{(k)} = q^{(l)}((\underline{w}^{(l)}(x_i))^{(k)}) - q^{(l)}((\underline{w}_i^{(l)})^{(k)}), \quad l = 1, \ldots, N.
\end{aligned}$$

Since by the mean-value theorem,

$$\begin{aligned}&F_i^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) - F_i^{(l)}(\underline{\mathbf{u}}_i^{(k-1)})\\&=\sum_{j=1}^{N}(\frac{\partial F_i^{(l)}(\xi)}{\partial u^{(j)}})((\underline{u}^{(j)}(x_i))^{(k-1)} - (\underline{u}_i^{(j)})^{(k-1)})\\&\equiv\sum_{j=1}^{N} K_{ij}^{(l)}(v_i^{(j)})^{(k-1)}\end{aligned}$$

and similarly,

$$\begin{aligned}&G_i^{(l)}(\underline{\mathbf{u}}^{(k-1)}(x_i)) - G_i^{(l)}(\underline{\mathbf{u}}_i^{(k-1)}) = \sum_{j=1}^{N}\overline{K}_{ij}^{(l)}(v_i^{(j)})^{(k-1)},\\&q^{(l)}((\underline{w}^{(l)}(x_i))^{(k)}) - q^{(l)}((\underline{w}_i^{(l)})^{(k)}) = \hat{K}_i^{(l)}(z_i^{(l)})^{(k)},\end{aligned}$$

where

$$K_{ij}^{(l)} = \frac{\partial F_i^{(l)}}{\partial u^{(j)}}(\xi),\quad \overline{K}_{ij}^{(l)} = \frac{\partial G_i^{(l)}}{\partial u^{(j)}}(\overline{\xi}),\quad \hat{K}_i^{(l)} = \frac{\partial q^{(l)}}{\partial w^{(l)}}(\hat{\xi})$$

and ξ, $\overline{\xi}$ and $\hat{\xi}$ are some intermediate values between $(\underline{u}^{(j)}(x_i))^{(k-1)}$ and $(\underline{u}_i^{(j)})^{(k-1)}$ and between $(\underline{w}^{(j)}(x_i))^{(k)}$ and $(\underline{w}_i^{(j)})^{(k)}$, respectively, we see that the above system is equivalent to

$$\begin{aligned}&L^{(l)}[(z_i^{(l)})^{(k)}] = \sum_{j=1}^{N} K_{ij}^{(l)}(v_i^{(j)})^{(k-1)} + o^{(k)}(|h|^2), && l = 1,\ldots,N,\\&(z_i^{(l)})^{(k)} = 0, && l = 1,\ldots,n_0-1,\\&B^{(l)}[(z_i^{(l)})^{(k)}] + \overline{\gamma}_i(z^{(l)})^{(k)} = \sum_{j=1}^{N}\overline{K}_{ij}^{(l)}(v^{(j)})^{(k-1)} + o^{(k)}(|h|), && l = n_0,\ldots,N,\\&(v_i^{(l)})^{(k)} = \hat{K}_i^{(l)}(z_i^{(l)})^{(k)}, && l = 1,\ldots,N.\end{aligned}\tag{4.7}$$

Notice from the nondecreasing property of $F_i^{(l)}(\mathbf{u})$, $G_i^{(l)}(\mathbf{u})$ and $q^{(l)}(w_i)$ that the values of $K_{ij}^{(l)}$, $\overline{K}_{ij}^{(l)}$ and $\hat{K}_i^{(l)}$ are nonnegative and are bounded for all (i,j,l).

Let $(V^{(l)})^{(k)}$, $(Z^{(l)})^{(k)}$ be the vector representation of $(v_i^{(l)})^{(k)}$ and $(z_i^{(l)})^{(k)}$ respectively, and let

$$\mathbf{V}^{(k)} = ((V^{(1)})^{(k)},\cdots,(V^{(N)})^{(k)}),\quad \mathbf{Z}^{(k)} = ((Z^{(1)})^{(k)},\cdots,(Z^{(N)})^{(k)}).$$

Then in vector form we may write (4.7) as

$$\begin{aligned}&\mathcal{A}^{(l)}(Z^{(l)})^{(k)} = \sum_{j=1}^{N} K_j^{(l)}(V^{(j)})^{(k-1)} + o^{(k)}(|h|^2),\quad l = 1,\ldots,n_0-1,\\&\overline{\mathcal{A}}^{(l)}(Z^{(l)})^{(k)} = \sum_{j=1}^{N} K_j^{(l)}(V^{(j)})^{(k-1)} + \sum_{j=1}^{N}\overline{K}_j^{(l)}(V^{(j)})^{(k-1)} + o(|h|),\\&\qquad l = n_0,\ldots,N,\\&(V^{(l)})^{(k)} = \hat{K}^{(l)}(Z^{(l)})^{(k)},\quad l = 1,\ldots,N,\end{aligned}\tag{4.8}$$

where $K_j^{(l)} = \mathrm{diag}(K_{1,j}^{(l)}, \cdots, K_{m^{(l)},j}^{(l)})$, $\overline{K}_j^{(l)} = \mathrm{diag}(\overline{K}_{1,j}^{(l)}, \cdots, \overline{K}_{m^{(l)},j}^{(l)})$ and $\hat{K}^{(l)} = \mathrm{diag}(\hat{K}_1^{(l)}, \cdots, \hat{K}_{m^{(l)}}^{(l)})$, are diagonal matrices. Let K, $\overline{K}$ and $\hat{K}$ be some upper bounds of $K_j^{(l)}$, $\overline{K}_j^{(l)}$ and $\hat{K}^{(l)}$, respectively. Then by the positive property of $(\mathcal{A}^{(l)})^{-1}$ and $(\overline{\mathcal{A}}^{(l)})^{-1}$ we have the estimates

$$\begin{aligned}
&|(Z^{(l)})^{(k)}| \leq (\mathcal{A}^{(l)})^{-1}[K \textstyle\sum_{j=1}^N |(V^{(j)})^{(k-1)}| + |o^{(k)}(|h|^2)|], \\
&|(Z^{(l)})^{(k)}| \leq (\overline{\mathcal{A}}^{(l)})^{-1}[(K + \overline{K}) \textstyle\sum_{j=1}^N |(V^{(j)})^{(k-1)}| + |o^{(k)}(|h|)|], \\
&|(V^{(l)})^{(k)}| \leq \hat{K}|(Z^{(l)})^{(k)}|.
\end{aligned}$$

Define

$$|\mathbf{Z}^{(k)}| = \sum_{l=1}^N |(Z^{(l)})^{(k)}|, \quad |\mathbf{V}^{(k)}| = \sum_{l=1}^N |(V^{(l)})^{(k)}|.$$

Then the above inequalities become

$$\begin{aligned}
&|(Z^{(l)})^{(k)}| \leq (\mathcal{A}^{(l)})^{-1}[K|\mathbf{V}^{(k-1)}| + |o^{(k)}(|h|^2)|], \\
&|(Z^{(l)})^{(k)}| \leq (\overline{\mathcal{A}})^{-1}[(K + \overline{K})|\mathbf{V}^{(k-1)}| + |o^{(k)}(|h|)|], \\
&|(V^{(l)})^{(k)}| \leq \hat{K}|(Z^{(l)})^{(k)}|.
\end{aligned} \tag{4.9}$$

It is well-known that given any $\epsilon_1 > 0$ there exists a matrix norm and a vector norm such that

$$\begin{aligned}
&||(\mathcal{A}^{(l)})^{-1}|| \leq (\gamma^{(l)} + \mu^{(l)} - \epsilon_1)^{-1} \equiv \rho^{(l)}, \ ||\mathcal{A}^{(l)}Y|| \leq \rho^{(l)}||Y||, \\
&||(\overline{\mathcal{A}}^{(l)})^{-1}|| \leq (\overline{\gamma}^{(l)} + \overline{\mu}^{(l)} - \epsilon_1)^{-1} \equiv \overline{\rho}^{(l)}, \ ||\overline{\mathcal{A}}^{(l)}\overline{Y}|| \leq \overline{\rho}^{(l)}||\overline{Y}||
\end{aligned} \tag{4.10}$$

for every $Y \in \mathbb{R}^{M^{(l)}}$ and $\overline{Y} \in \mathbb{R}^{\overline{M}^{(l)}}$, where $\mu^{(l)} \geq 0$ and $\overline{\mu}^{(l)} \geq 0$ are the respective principle eigenvalues of $\mathcal{A}^{(l)}$ and $\overline{\mathcal{A}}^{(l)}$ (cf. [25, 29]). We choose ϵ_1 sufficiently small (and choose $\gamma^{(l)} > \epsilon_1$ if $\mu^{(l)} = 0$) so that $\rho^{(l)} > 0$ for each $l = 1, \ldots, N$. A similar choice gives $\overline{\rho}^{(l)} > 0$. Let $\rho_0 \geq max\{\rho^{(l)}, \overline{\rho}^{(l)}\}$ for all l. Then by (4.9) and (4.10),

$$\begin{aligned}
&||(Z^{(l)})^{(k)}|| \leq \rho_0[K||\mathbf{V}^{(k-1)}|| + o^{(k)}(|h|^2)], \quad l = 1, \ldots, n_0 - 1, \\
&||(Z^{(l)})^{(k)}|| \leq \rho_0[(K + \overline{K})||\mathbf{V}^{(k-1)}|| + o^{(k)}(|h|)], \quad l = n_0, \ldots, N, \\
&||(V^{(l)})^{(k)}|| \leq \hat{K}||(Z^{(l)})^{(k)}||, \quad l = 1, \ldots, N,
\end{aligned} \tag{4.11}$$

where $o^{(k)}(|h|) \to 0$ as $|h| \to 0$ and

$$||\mathbf{V}^{(k)}|| = \sum_{l=1}^N ||(V^{(l)})^{(k)}||, \quad ||\mathbf{Z}^{(k)}|| = \sum_{l=1}^N ||(Z^{(l)})^{(k)}||.$$

Addition of the first two inequalities in (4.11) and also the last inequality in (4.11) over l yield

$$\begin{aligned}
&||\mathbf{Z}^{(k)}|| \leq \rho_0(2K + \overline{K})||\mathbf{V}^{(k-1)}|| + o(|h|), \\
&||\mathbf{V}^{(k)}|| \leq \hat{K}||\mathbf{Z}^{(k)}||,
\end{aligned} \tag{4.12}$$

where $o(|h|) =$ maximal $\{o^{(k)}(|h|), k \le k^*\}$. Let

$$r^{(k)} = ||\mathbf{V}^{(k)}||, \quad s^{(k)} = ||\mathbf{Z}^{(k)}||, \quad \overline{c} = \text{maximal } \{\rho_0(2K + \overline{K}), \hat{K}\}.$$

Then (4.12) is satisfied if

$$s^{(k)} \le \overline{c} r^{(k-1)} + o(|h|) \quad \text{and} \quad r^{(k)} \le \overline{c} s^{(k)}. \tag{4.13}$$

Consider $k = 1$. Since $r^{(0)} = ||\mathbf{V}^{(0)}|| = ||\underline{\mathbf{U}}^{(0)}(x) - \underline{\mathbf{U}}^{(0)}||$ and $\underline{\mathbf{U}}^{(0)}(x) = (\hat{u}(x_1), \cdots, \hat{u}(x_M))$, $\underline{\mathbf{U}}^{(0)} = (\hat{u}_1, \cdots, \hat{u}_M)$, where M is the total number of components of $\underline{\mathbf{U}}^{(0)}$, we see from (4.13) that for any $\epsilon_0^{'} > 0$ there exists δ_0' such that $r^{(0)} < \epsilon_0'$ and $o(|h|) < \epsilon'$ when $|h| < \delta_0'$. This implies that

$$s^{(1)} \le (\overline{c} + 1)\epsilon_0', \quad r^{(1)} \le \overline{c} s^{(1)} \le (\overline{c}^2 + \overline{c})\epsilon_0' \quad \text{when} \quad |h| < \delta_0'.$$

Using the relation (4.13), an induction argument gives

$$s^{(k)} \le (\overline{c}^{(k)} + \overline{c}^{(k-1)} + \cdots + 1)\epsilon_0', \quad r^{(k)} \le (\overline{c}^{(k+1)} + \overline{c}^{(k)} + \cdots + \overline{c})\epsilon_0',$$

when $|h| < \delta_0'$. Let $k \ge k^*$ be fixed. Then by choosing ϵ_0' sufficiently small there exists $\delta > 0$ such that $s^{(k)} + r^{(k)} < \epsilon/3$ when $|h| < \delta$. This is equivalent to

$$||\mathbf{V}^{(k)}|| + ||\mathbf{Z}^{(k)}|| < \epsilon/3 \quad \text{when} \quad |h| < \delta,$$

which ensures that the relation in (4.5) holds. This proves the convergence of the minimal solution. The proof for the maximal solution is similar and is omitted. □

If the minimal solution $(\underline{\mathbf{u}}(x), \underline{\mathbf{w}}(x))$ coincide with the maximal solution $(\overline{\mathbf{u}}(x), \overline{\mathbf{w}}(x))$, then Theorem 4.1 implies that both $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$ and $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$ converge to their common value $(\mathbf{u}^*(x_i), \mathbf{w}^*(x_i))$. This observation leads to the following.

Corollary 4.1. *Let the conditions in Theorem 4.1 hold. If, in addition, either* $\underline{\mathbf{u}}(x) = \overline{\mathbf{u}}(x)$ *or* $\underline{\mathbf{w}}(x) = \overline{\mathbf{w}}(x)$ *then* $(\underline{\mathbf{u}}(x), \underline{\mathbf{w}}(x)) = (\overline{\mathbf{u}}(x), \overline{\mathbf{w}}(x)) \equiv (\mathbf{u}^*(x), \mathbf{w}^*(x))$ *and* $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$*(or* $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$*) converges to* $(\mathbf{u}^*(x_i), \mathbf{w}^*(x_i))$ *in* $\overline{\Lambda}^*$ *as* $|h| \to 0$.

5. Applications

In this section, we given some applications of the existence theorem and the monotone iterative schemes to the model problems (1.1) and (1.2). These model problems have their own merits in the field of heat transfer and population growth. To avoid the evaluation of the integral term $I[(u_i^{(l)})^{(k-1)}]$ in the iteration process (2.12) we replace it by $(w_i^{(l)})^{(k-1)}$ to obtain an equivalent iteration process in the form

$$\begin{aligned}
&L[(w_i^{(l)})^{(k)}] = \gamma_i^{(l)}(w_i^{(l)})^{(k-1)} + f_i^{(l)}(\mathbf{u}^{(k-1)}), && l = 1, \ldots, N,\\
&(w_i^{(l)})^{(k)} = \eta_i^{(l)}, && l = 1, \ldots, n_0 - 1,\\
&B_i[(w_i^{(l)})^{(k)}] = \overline{\gamma}^{(l)}(w_i^{(l)})^{(k-1)} + g_i^{(l)}(\mathbf{u}^{(k-1)}), && l = n_0, \ldots, N,\\
&(u_i^{(l)})^{(k)} = q^{(l)}[(w_i^{(l)})^{(k)}], && l = 1, \ldots, N,
\end{aligned} \tag{5.1}$$

where $k = 1, 2, \ldots$, $(w_i^{(l)})^{(0)} = I[(u_i^{(l)})^{(0)}]$ and $(u_i^{(l)})^{(0)}$ is either $\hat{u}_i^{(l)}$ or $\tilde{u}_i^{(l)}$. The above iteration process simplifies the computation of the functions $F_i^{(l)}(\mathbf{u}^{(k-1)})$ and

$G_i^{(l)}(\mathbf{u}^{(k-1)})$ for each iteration k. The value of $(u_i^{(l)})^{(k)}$ in the last equation of (5.1) can be obtained by solving the equation $I[(u_i^{(l)})^{(k)}] = (w_i^{(l)})^{(k)}$ if the inverse function $q^{(l)}[w_i]$ cannot be explicitly given. In the application of the iteration process (5.1) for a particular problem where $D^{(l)}(u^{(l)})$, $f_i^{(l)}(\mathbf{u})$ and $g_i^{(l)}(\mathbf{u})$ are given we need to construct a pair of ordered lower and upper solutions and the determination of the functions $\gamma_i^{(l)}$ and $\overline{\gamma}_i^{(l)}$ from (2.8). We do this for both problem (1.1) and problem (1.2).

(A). The heat-transfer problem.

For the heat-transfer problem (1.1) we impose the following hypothesis:

(H_3) $D(u) = k_c + k_r u^3$, $c(x) > 0$, $p(x) > 0$ in $\overline{\Omega}$, $\sigma(x) \geq 0$ on $\partial\Omega$, and k_c, k_r, α and a_0 are positive constants with $\alpha \geq 1$

It is clear that problem (1.1) is a special case of (1.3) with $N = n_0 = 1$, $u^{(1)} = u$, $\mathbf{b}^{(1)} = 0$ and

$$\begin{aligned} D^{(1)}(u^{(1)}) \quad &= k_c + k_r u^3, \\ f^{(1)}(x, u^{(1)}) &= p(x) - c(x)u^\alpha, \\ g^{(1)}(x, u^{(1)}) &= \sigma(x)(a_0^4 - u^4). \end{aligned} \tag{5.2}$$

This implies that $w_i = I[u_i] = k_c u_i + (k_r/4)u_i^4$, and the finite difference system for problem (1.1) becomes

$$\begin{aligned} &-\Delta w_i = p_i - c_i u_i^\alpha, \quad (i \in \Lambda), \\ &\partial w_i/\partial\nu = \sigma_i(a_0^4 - u_i^4), \quad (i \in \partial\Lambda), \\ &u_i = q(w_i), \quad (i \in \overline{\Lambda}), \end{aligned} \tag{5.3}$$

where u_i is determined from the equation $k_c u_i + (k_r/4)u_i^4 = w_i$. By (5.1) the iteration process for problem (5.3) is given by

$$\begin{aligned} &-\Delta[w_i^{(k)}] + \gamma_i^{(1)} w_i^{(k)} = \gamma_i^{(1)} w_i^{(k-1)} + p_i - c_i(u_i^\alpha)^{(k-1)}, \quad (i \in \Lambda), \\ &B_i[w_i^{(k)}] + \overline{\gamma}_i^{(1)} w_i^{(k)} = \overline{\gamma}_i^{(1)} w_i^{(k-1)} + \sigma(a_0^4 - (u_i^4)^{(k-1)}), \quad (i \in \partial\Lambda), \\ &u_i^{(k)} = q(w_i^{(k)}), \quad (i \in \overline{\Lambda}), \end{aligned} \tag{5.4}$$

where $w_i^{(0)} = I[u_i^{(0)}]$ and $\gamma_i^{(1)}$, $\overline{\gamma}_i^{(1)}$ are any nonnegative functions (with $\overline{\gamma} \neq 0$) satisfying (2.8) with respect to the functions in (5.2). Since

$$\partial f^{(1)}/\partial u^{(1)} = -\alpha c(x)u^{\alpha-1}, \quad \partial g^{(1)}/\partial u^{(1)} = -4\sigma(x)u^3,$$

all the conditions in (H_1) are satisfied if $\gamma_i^{(1)}$ and $\overline{\gamma}_i^{(1)}$ are chosen to satisfy the condition

$$\gamma_i^{(1)}(k_c + k_r u_i^3) - \alpha c_i u_i^{\alpha-1} \geq 0, \overline{\gamma}_i^{(1)}(k_c + k_r u_i^3) - 4\sigma_i u_i^3 \geq 0 \text{ for } \hat{u}_i \leq u_i \leq \tilde{u}_i, (i \in \overline{\Lambda}), \tag{5.5}$$

where $\hat{u}_i$ and $\tilde{u}_i$ are the respective components of a pair of ordered lower and upper solutions $(\hat{u}_i, \hat{w}_i)$, $(\tilde{u}_i, \tilde{w}_i)$ of (5.3). It is easy to verify from Definition 2.1 (for

$N = n_0 = 1$) that $(\hat{u}_i, \hat{w}_i) = (0, 0)$ is a lower solution. To find an upper solution we let

$$\overline{P} = \text{maximal } \{(p(x)/c(x))^{1/\alpha}; x \in \overline{\Omega}\}, \quad I[\rho] = k_c\rho + (k_r/4)\rho^4 \tag{5.6}$$

and choose any constant $\rho \geq$ maximal $\{\overline{P}, a_0\}$. Then it is easy to verify that the constant $(\tilde{u}_i, \tilde{w}_i) = (\rho, I[\rho])$ is an upper solution. This shows that the pair

$$(\hat{u}_i, \hat{w}_i) = (0, 0), \quad (\tilde{u}_i, \tilde{w}_i) = (\rho, I[\rho]) \tag{5.7}$$

are ordered lower and upper solutions of (5.3). With this construction, condition (5.5) is satisfied if

$$\gamma_i^{(1)} \geq (\alpha c_i/k_c)\rho^{\alpha-1}, \quad \overline{\gamma}_i^{(1)} \geq (4\sigma_i\rho^3)/(k_c + k_r\rho^3). \tag{5.8}$$

By Theorem 2.1 (or Theorem 3.1) the finite difference problem (5.3) has a minimal solution $(\underline{u}_i, \underline{w}_i)$ and a maximal solution $(\overline{u}_i, \overline{w}_i)$ such that

$$(0, 0) \leq (\underline{u}_i, \underline{w}_i) \leq (\overline{u}_i, \overline{w}_i) \leq (\rho, I[\rho]).$$

The positive property of p_i ensures that $(\underline{u}_i, \underline{w}_i) > (0, 0)$ in $\overline{\Lambda}$ (cf. [21]). Moreover, by the relation $\partial f^{(1)}/\partial u = -\alpha c(x)u^{\alpha-1} \leq 0$ and $\partial g^{(1)}/\partial u = -4\sigma(x)u^3 \leq 0$ for $0 \leq u \leq \rho$ we conclude from Theorem 3.3 of [16] that $(\underline{u}_i, \underline{w}_i) = (\overline{u}_i, \overline{w}_i) \equiv (u_i^*, w_i^*)$ and (u_i^*, w_i^*) is the unique solution between $(0, 0)$ and $(\rho, I[\rho])$. Since ρ can be chosen arbitrarily large we conclude that (u_i^*, w_i^*) is the unique positive solution of (5.3). This implies that the minimal sequence $\{\underline{u}_i^{(k)}, \underline{w}_i^{(k)}\}$ and the maximal sequence $\{\overline{u}_i^{(k)}, \overline{w}_i^{(k)}\}$ converge monotonically to (u_i^*, w_i^*) and satisfy the relation

$$\begin{aligned}(0, 0) &\leq (\underline{u}_i^{(k)}, \underline{w}_i^{(k)}) \leq (\underline{u}_i^{(k+1)}, \underline{w}_i^{(k+1)}) \leq (u_i^*, w_i^*) \leq (\overline{u}_i^{(k+1)}, \overline{w}_i^{(k+1)}) \\ &\leq (\overline{u}_i^{(k)}, \overline{w}_i^{(k)}) \leq (\rho, I[\rho])\end{aligned} \tag{5.9}$$

for every $k = 1, 2, \ldots$. It is easy to see that the constant pair in (5.7) are also ordered lower and upper solutions of the corresponding continuous problem (5.3). This ensures that condition (4.3) is trivially satisfied. Hence by Theorem 4.1 and Theorem 4.2, the continuous problem has a unique positive solution $(u^*(x), w^*(x))$, and as $|h| \to 0$, the finite difference solution (u_i^*, w_i^*) converges to the continuous solution $(u^*(x_i), w^*(x_i))$ at every mesh point x_i in $\overline{\Lambda}^*$. To summarize the above conclusions, we have the following results for the heat-transfer problem (1.1).

Theorem 5.1. *Let Hypothesis (H_3) be satisfied, and let $\rho \geq max\{\overline{P}, a_0\}$ where $\overline{P}$ is given by (5.6). Then the following statements hold:*

(a) Problem (5.3) has a unique positive solution (u_i^*, w_i^*);

(b) The minimal sequence $\{\underline{u}_i^{(k)}, \underline{w}_i^{(k)}\}$ and the maximal sequence $\{\overline{u}_i^{(k)}, \overline{w}_i^{(k)}\}$ governed by (5.4) with $(\underline{u}^{(0)}, \underline{w}^{(0)}) = (0, 0)$ and $(\overline{u}_i^{(0)}, \overline{w}_i^{(0)}) = (\rho, I[\rho])$ converge to (u_i^*, w_i^*) and satisfy the relation (5.9), where $I[\rho]$ is given by (5.6);

(c) The continuous problem (1.1) has a unique positive solution $u^*(x)$, and as $|h| \to 0$, the finite difference solution (u_i^*, w_i^*) converges to $(u^*(x_i), w^*(x_i))$ at every mesh point $x_i \in \overline{\Lambda}^*$, where $w^*(x_i) = I[u^*(x_i)]$.

(B). The Lotka-Volterra cooperation system.

We next consider an extended finite difference system of the cooperation model (1.2) which is given by

$$\begin{aligned} -\Delta_p[u_i^m] &= u_i(a^{(1)} - b^{(1)}u_i + c^{(1)}v_i) + q_i^{(1)} \text{ in } \Lambda,\ u_i = 0 \text{ on } \partial\Lambda, \\ -\Delta_p[v_i^n] &= v_i(a^{(2)} - b^{(2)}v_i + c^{(2)}u_i) + q_i^{(2)} \text{ in } \Lambda,\ v_i = 0 \text{ on } \partial\Lambda, \end{aligned} \tag{5.10}$$

where $q_i^{(1)} \geq 0$, $q_i^{(2)} \geq 0$ are some nonnegative functions in Λ. The consideration of the above source functions is for the purpose of constructing a known continuous solution. It is clear that the above problem is a special case of (1.3) with $N = 2$, $n_0 = 3$, $\mathbf{b}^{(1)} = 0$, $(u^{(1)}, u^{(2)}) = (u, v)$, and

$$\begin{aligned} &D^{(1)}(u^{(1)}) = mu^{m-1}, \quad D^{(2)}(u^{(2)}) = nv^{n-1}, \quad \xi^{(1)} = \xi^{(2)} = 0, \\ &f^{(1)}(x, u^{(1)}, u^{(2)}) = u(a^{(1)} - b^{(1)}u + c^{(1)}v) + q^{(1)}(x), \\ &f^{(2)}(x, u^{(1)}, u^{(2)}) = v(a^{(2)} - b^{(2)}v + c^{(2)}u) + q^{(2)}(x). \end{aligned} \tag{5.11}$$

Since $w_i \equiv I^{(1)}[u_i^{(1)}] = u_i^m$, $z_i \equiv I^{(2)}[u_i^{(2)}] = v_i^n$, the transformed system of (5.10) becomes

$$\begin{aligned} &-\Delta_p[w_i] = u_i(a^{(1)} - b^{(1)}u_i + c^{(1)}v_i) + q_i^{(1)}, \quad (i \in \Lambda), \\ &-\Delta_p[z_i] = v_i(a^{(2)} - b^{(2)}v_i + c^{(2)}u_i) + q_i^{(2)}, \quad (i \in \Lambda), \\ &w_i = z_i = 0, \quad (i \in \partial\Lambda), \\ &u_i = w_i^{1/m}, \quad v_i = z_i^{1/n}, \quad (i \in \overline{\Lambda}). \end{aligned} \tag{5.12}$$

Recall from Definition 2.1 that lower and upper solutions of (5.12), denoted by $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i) \equiv ((\hat{u}_i, \hat{v}_i), (\hat{w}_i, \hat{z}_i))$ and $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{v}}_i) \equiv ((\tilde{u}_i, \tilde{v}_i), (\tilde{w}_i, \tilde{z}_i))$, are required to satisfy (5.12) with all the equality sign "=" replaced by the respective inequality sign "$\leq$" and "$\geq$". Let such a pair be given. Then from the relation

$$\begin{aligned} &\tfrac{\partial f^{(1)}}{\partial v}(u, v) = c_1 u \geq 0, \quad \tfrac{\partial f^{(2)}}{\partial u}(u, v) = c^{(2)}v \geq 0 \quad \text{for } (u, v) \geq (0, 0), \\ &\tfrac{\partial f^{(1)}}{\partial u}(u, v) = a^{(1)} - 2b^{(1)}u + c^{(1)}v, \quad \tfrac{\partial f^{(2)}}{\partial v}(u, v) = a^{(2)} - 2b^{(2)}v + c^{(2)}u, \end{aligned} \tag{5.13}$$

we see that all the conditions in Hypothesis (H_1) are satisfied by any $\gamma_i^{(1)}$, $\gamma_i^{(2)}$ that satisfy the relation

$$\begin{aligned} &\gamma_i^{(1)}(mu_i^{m-1}) \geq 2b^{(1)}u_i - a^{(1)} - c^{(1)}v_i, \\ &\gamma_i^{(2)}(nv_i^{n-1}) \ \ \geq 2b^{(2)}v_i - a^{(2)} - c^{(2)}u_i \text{ for } \ \hat{u}_i \leq u_i \leq \tilde{u}_i,\ \hat{v}_i \leq v_i \leq \tilde{v}_i. \end{aligned} \tag{5.14}$$

Using the functions $\gamma^{(1)}$, $\gamma^{(2)}$ determined from (5.14) we compute the minimal and maximal sequences $\{\hat{\mathbf{u}}^{(k)}, \hat{\mathbf{w}}^{(k)}\}$, $\{\tilde{\mathbf{u}}^{(k)}, \tilde{\mathbf{w}}^{(k)}\}$ from the iteration process

$$\begin{aligned} &-\Delta_p[w_i^{(k)}] + \gamma_i^{(1)} w_i^{(k)} = \gamma_i^{(1)} w_i^{(k-1)} + u_i^{(k-1)}(a^{(1)} - b^{(1)}u_i^{(k-1)} + c^{(1)}v_i^{(k-1)}) + q_i^{(1)}, \\ &-\Delta_p[z_i^{(k)}] + \gamma_i^{(2)} z_i^{(k)} = \gamma_i^{(2)} z_i^{(k-1)} + v_i^{(k-1)}(a^{(2)} - b^{(2)}v_i^{(k-1)} \\ &\qquad\qquad + c^{(2)}u_i^{(k-1)}) + q_i^{(2)}, \quad (i \in \Lambda), \\ &w_i^{(k)} = z_i^{(k)} = 0, \quad (i \in \partial\Lambda), \\ &u_i^{(k)} = (w_i^{(k)})^{1/m},\ v_i^{(k)} = (z_i^{(k)})^{1/n}, \quad (i \in \overline{\Lambda}). \end{aligned} \tag{5.15}$$

Hence to show the existence and the computation of a positive solution of (5.12) we need to find a pair of positive lower and upper solutions $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$. In fact, it suffices to find $\hat{\mathbf{u}}_i \equiv (\hat{u}_i, \hat{v}_i)$ and $\tilde{\mathbf{u}} \equiv (\tilde{u}_i, \tilde{v}_i)$ since $\hat{\mathbf{w}}_i = (I^{(1)}[\hat{u}_i], I^{(2)}[\hat{v}_i])$ and $\tilde{\mathbf{w}}_i = (I^{(1)}[\tilde{u}_i], I^{(2)}[\tilde{v}_i])$. Before doing this we show the existence of two semitrivial solutions of the system (5.10) for the case $q_i^{(1)} = q_i^{(2)} = 0$.

Lemma 5.1. *Let $m > 1$, $n > 1$ and $q_i^{(1)} = q_i^{(2)} = 0$. Then for any positive constants $a^{(l)}$, $b^{(l)}$ and $c^{(l)}$, $l = 1, 2$, problem (5.10) has two semitrivial solutions $(u_i^*, 0)$, $(0, v_i^*)$, where $u_i^* > 0$ and $v_i^* > 0$ in Λ.*

Proof. To show the existence of the semitrivial solution $(u_i^*, 0)$ we apply Theorem 2.1 to the scalar problem

$$\begin{aligned} &-\Delta_p[w_i] = u_i(a^{(1)} - b^{(1)}u_i) \quad \text{in} \quad \Lambda,\ w_i = 0 \text{ on } \ \partial\Lambda, \\ &u_i = w_i^{1/m}, \ (i \in \overline{\Lambda}). \end{aligned} \tag{5.16}$$

It is obvious that for any constant $\rho \geq a^{(1)}/b^{(1)}$, $(\tilde{u}_i, \tilde{w}_i) = (\rho, \rho^m)$ is an upper solution. To find a positive lower solution we let λ^* and ϕ_i be the smallest eigenvalue and its corresponding (normalized) positive eigenfunction of the eigenvalue problem

$$\Delta_p[\phi_i] + \lambda^*\phi_i = 0 \quad \text{in} \quad \Lambda, \quad \phi_i = 0 \quad \text{on} \quad \partial\Lambda, \tag{5.17}$$

where $\lambda^* > 0$. In terms of the matrix $A^{(1)}$, λ^* is the smallest eigenvalue of $A^{(1)}$ and $\Phi \equiv (\phi_1, \cdots, \phi_M)^T$ is the positive eigenvector. It is clear that for any small constant $\delta > 0$, $(\hat{u}_i, \hat{w}_i) = ((\delta\phi_i)^{1/m}, \delta\phi_i)$ is a lower solution of (5.16) if

$$-\Delta_p[\delta\phi_i] \leq (\delta\phi_i)^{1/m}(a^{(1)} - b^{(1)}(\delta\phi_i)^{1/m}).$$

In view of (5.17) the above inequality is equivalent to

$$\lambda^*(\delta\phi)^{1-1/m} \leq a^{(1)} - b^{(1)}(\delta\phi_i)^{1/m}.$$

Since $m > 1$ and $\phi_i > 0$ there exists a small constant $\delta^* > 0$ such that the above inequality holds for every $\delta \leq \delta^*$. This shows that the pair $(\hat{u}_i, \hat{w}_i) = ((\delta\phi_i)^{1/m}, \delta\phi_i)$ and (ρ, ρ^m) are ordered lower and upper solutions of (5.16). By Theorem 2.1(or Theorem 3.1), this problem has a positive solution (u_i^*, w_i^*) which ensures that $(u_i^*, 0)$ is a semitrivial solution of (5.10) (for $q_i^{(1)} = 0$). The proof for the semitrivial solution $(0, v_i^*)$ is similar. □

To show the existence of a positive solution to (5.12) we construct a pair of positive lower and upper solutions. The following lemma gives a positive lower solution.

Lemma 5.2. *Given any $m > 1$, $n > 1$, there exists $\delta^* > 0$ such that for every $\delta \leq \delta^*$ the function $(\hat{\mathbf{u}}_i, \hat{\mathbf{w}}_i)$, where*

$$\hat{\mathbf{u}}_i \equiv (\hat{u}_i, \hat{v}_i) = ((\delta\phi_i)^{1/m}, (\delta\phi_i)^{1/n}), \quad \hat{\mathbf{w}}_i \equiv (\hat{w}_i, \hat{z}_i) = (\delta\phi_i, \delta\phi_i), \tag{5.18}$$

is a positive lower solution of (5.12).

Proof. Since $\phi_i = 0$ on $\partial\Lambda$, the function $((\hat{u}_i, \hat{v}_i), (\hat{w}_i, \hat{z}_i))$ in (5.18) is a lower solution of (5.12) if

$$\begin{aligned} -\Delta_p[\delta\phi_i] &\leq (\delta\phi_i)^{1/m}[a^{(1)} - b^{(1)}(\delta\phi_i)^{1/m} + c^{(1)}(\delta\phi_i)^{1/n}] + q_i^{(1)}, \\ -\Delta_p[\delta\phi_i] &\leq (\delta\phi_i)^{1/n}[a^{(2)} - b^{(2)}(\delta\phi_i)^{1/n} + c^{(2)}(\delta\phi_i)^{1/m}] + q_i^{(2)}. \end{aligned}$$

By (5.17) and $q_i^{(l)} \geq 0$ for $l = 1, 2$, the above inequalities are satisfied if

$$\begin{aligned} \lambda^*(\delta\phi_i)^{1-1/m} &\leq a^{(1)} - b^{(1)}(\delta\phi_i)^{1/m} + c^{(1)}(\delta\phi_i)^{1/n}, \\ \lambda^*(\delta\phi_i)^{1-1/n} &\leq a^{(2)} - b^{(2)}(\delta\phi_i)^{1/n} + c^{(2)}(\delta\phi_i)^{1/m}. \end{aligned} \tag{5.19}$$

In view of $m > 1$, $n > 1$, and $0 \leq \phi \leq 1$ there exists a small constant $\delta^* > 0$ such that the above relation holds for every $\delta \leq \delta^*$. This proves the lemma. □

We next give some sufficient conditions for the construction of a positive upper solution $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i) = ((\tilde{u}_i, \tilde{v}_i), (\tilde{w}_i, \tilde{z}_i))$. This function is given in the form

$$(\tilde{u}_i, \tilde{v}_i) = ((\rho\psi_i)^{1/m}, (\rho\psi_i)^{1/n}), \quad (\tilde{w}_i, \tilde{z}_i) = (\rho\psi_i, \rho\psi_i) \tag{5.20}$$

for a sufficiently large constant ρ, where ψ_i is either the positive eigenfunction of (5.17) in a slightly large domain $\tilde{\Lambda}$ containing $\overline{\Lambda}$ or $\psi_i = 1$ in $\overline{\Lambda}$. In fact, we choose $\psi_i = 1$ and $\rho^{(l)} = \overline{\rho}^{(l)}$ for $l = 1, 2$, only if $b^{(1)}b^{(2)} > c^{(1)}c^{(2)}$, where $\overline{\rho}^{(1)}$ and $\overline{\rho}^{(2)}$ are given by

$$\begin{aligned} \overline{\rho}^{(1)} &= (\overline{a}^{(1)}b^{(2)} + \overline{a}^{(2)}b^{(1)})/(b^{(1)}b^{(2)} - c^{(1)}c^{(2)}), \\ \overline{\rho}^{(2)} &= (\overline{a}^{(1)}c^{(2)} + \overline{a}^{(2)}c^{(1)})/(b^{(1)}b^{(2)} - c^{(1)}c^{(2)}) \end{aligned} \tag{5.21}$$

and $\overline{a}^{(l)}$, $l = 1, 2$, are any constants satisfying $\overline{a}^{(l)} \geq a^{(l)} + q_i^{(l)}$. Specifically we have the following results.

Lemma 5.3. *Let $m > 1$, $n > 1$, and let one of the following conditions holds:*

$$\begin{aligned} &(a)\ 1/m + 1/n < 1; \qquad\qquad (b)\ m = n \text{ and } b^{(l)} \geq c^{(l)},\ l = 1, 2; \\ &(c)\ m = n = 2 \text{ and } c^{(l)} - b^{(l)} < \lambda^*;\ (d)\ b^{(1)}b^{(2)} > c^{(1)}c^{(2)}. \end{aligned} \tag{5.22}$$

Then there exists a constant ρ^* such that the function $((\tilde{u}_i, \tilde{v}_i), (\tilde{w}_i, \tilde{z}_i))$ in (5.20) is a positive upper solution of (5.12) for every $\rho \geq \rho^*$ if one of the conditions in (a), (b) and (c) holds. In case condition (d) is satisfied then the constant $((\rho^{(1)}, \rho^{(2)}), ((\rho^{(1)})^m, (\rho^{(2)})^n))$ is an upper solution, where $\rho^{(l)} = max\{1, \overline{\rho}^{(l)}\}$ and $\overline{\rho}^{(l)}$, $l = 1, 2$, are given by (5.21).

Proof. Since $\psi_i > 0$ in $\tilde{\Lambda}$ and $\tilde{\Lambda}$ contains $\overline{\Lambda}$ we see from (5.20) that $(\tilde{w}_i, \tilde{z}_i) > (0, 0)$ on $\partial\Lambda$. Hence the function in (5.20) is an upper solution of (5.12) if

$$\begin{aligned} -\Delta_p[\rho\psi_i] &\geq (\rho\psi_i)^{1/m}[a^{(1)} - b^{(1)}(\rho\psi_i)^{1/m} + c^{(1)}(\rho\psi_i)^{1/n}] + q_i^{(1)}, \\ -\Delta_p[\rho\psi_i] &\geq (\rho\psi_i)^{1/n}[a^{(2)} - b^{(2)}(\rho\psi_i)^{1/n} + c^{(2)}(\rho\psi_i)^{1/m}] + q_i^{(2)}. \end{aligned}$$

By (5.17) with ϕ_i replaced by ψ_i the above relation is equivalent to

$$\begin{aligned} \tilde{\lambda} &\geq (\rho\psi_i)^{1/m-1}[a^{(1)} - b^{(1)}(\rho\psi_i)^{1/m} + c^{(1)}(\rho\psi_i)^{1/n}] + q_i^{(1)}/(\rho\psi_i), \\ \tilde{\lambda} &\geq (\rho\psi_i)^{1/n-1}[a^{(2)} - b^{(2)}(\rho\psi_i)^{1/n} + c^{(2)}(\rho\psi_i)^{1/m}] + q_i^{(2)}/(\rho\psi_i), \end{aligned} \tag{5.23}$$

where $\tilde{\lambda} > 0$ is the smallest eigenvalue of (5.17) corresponding to ψ_i. Since $(\rho\psi_i)^{1/m-1} \to 0$ as $\rho \to \infty$ there exists a large constant ρ^* such that for every $\rho \geq \rho^*$ the inequalities in (5.23) hold if

$$\begin{aligned} &\lim_{\rho\to\infty}[c^{(1)}(\rho\psi_i)^{1/m+1/n-1} - b^{(1)}(\rho\psi_i)^{2/m-1}] < \tilde{\lambda}, \\ &\lim_{\rho\to\infty}[c^{(2)}(\rho\psi_i)^{1/m+1/n-1} - b^{(2)}(\rho\psi_i)^{2/n-1}] < \tilde{\lambda}. \end{aligned} \tag{5.24}$$

It is obvious that the above requirement is fulfilled by every $b^{(l)}$, $c^{(l)}$, $(l = 1, 2)$ if $1/m + 1/n < 1$. It is also fulfilled if $m = n$ and $c^{(l)} \le b^{(l)}$. This proves the lemma if condition (a) or condition (b) in (5.22) is satisfied. In the case $m = n = 2$ the requirement in (5.24) becomes $c^{(l)} - b^{(l)} < \tilde{\lambda}$ for $l = 1, 2$. Since $\tilde{\lambda}$ can be made arbitrarily close to λ^* by choosing $\tilde{\Lambda}$ sufficiently close to Λ, this requirement is fulfilled if condition (c) in (5.22) is satisfied. Finally if $b^{(1)}b^{(2)} > c^{(1)}c^{(2)}$ we seek a constant upper solution $(\tilde{\mathbf{u}}_i, \tilde{\mathbf{w}}_i)$ in the form

$$\tilde{\mathbf{u}}_i \equiv (\tilde{u}_i, \tilde{v}_i) = (\rho^{(1)}, \rho^{(2)}), \quad \tilde{\mathbf{w}}_i \equiv (\tilde{w}_i, \tilde{z}_i) = ((\rho^{(1)})^m, (\rho^{(2)})^n). \tag{5.25}$$

Clearly the above constant is an upper solution if

$$0 \ge \rho^{(1)}(a^{(1)} - b^{(1)}\rho^{(1)} + c^{(1)}\rho^{(2)}) + q_i^{(1)}, \quad 0 \ge \rho^{(2)}(a^{(2)} - b^{(2)}\rho^{(2)} + c^{(2)}\rho^{(1)}) + q_i^{(2)}$$

This relation is clearly satisfied if for each $l = 1, 2$, and any $\overline{a}^{(l)} \ge a^{(l)} + q_i^{(l)}$, $\rho^{(l)} \ge 1$ and satisfies the relation

$$b^{(1)}\rho^{(1)} - c^{(1)}\rho^{(2)} = \overline{a}^{(1)}, \quad b^{(2)}\rho^{(2)} - c^{(2)}\rho^{(1)} = \overline{a}^{(2)}.$$

Solving the above equations for $\rho^{(1)}$, $\rho^{(2)}$ gives $\rho^{(1)} = \overline{\rho}^{(1)}$, $\rho^{(2)} = \overline{\rho}^{(2)}$, where $\overline{\rho}^{(1)}$ and $\overline{\rho}^{(2)}$ are given by (5.21). This completes the proof of the lemma. □

Using the lower and upper solution in (5.18), (5.20) we choose the functions $\gamma^{(l)}$, $l = 1, 2$, that satisfy condition (5.14). Indeed since $a^{(1)} > 0$ and $a^{(2)} > 0$, this condition is trivially satisfied by any $\gamma_i^{(l)} > 0$ when $u_i = v_i = 0$. By continuity, we see that given any $\gamma_i^{(l)}$, say $\gamma_i^{(l)} \ge \beta_0$ for some $\beta_0 > 0$, there exists $\alpha_0 > 0$ such that (5.14) holds for $0 < u_i \le \alpha_0$ and $0 < v_i \le \alpha_0$. In the range $\alpha_0 \le u_i \le \rho^{(1)}$, $\alpha_0 \le v_i \le \rho^{(2)}$ for any $(\rho^{(1)}, \rho^{(2)}) \ge (\tilde{u}_i, \tilde{v}_i)$ we choose $\gamma^{(l)} \ge \beta^{(l)}$, $l = 1, 2$, where

$$\begin{aligned} \beta^{(1)} &= (2b^{(1)}\rho^{(1)} - a^{(1)} - c^{(1)}\alpha_0)/(m\alpha_0^{m-1}), \\ \beta^{(2)} &= (2b^{(2)}\rho^{(2)} - a^{(2)} - c^{(2)}\alpha_0)/(n\alpha_0^{n-1}). \end{aligned} \tag{5.26}$$

It follows by the choice of $\gamma_i^{(l)} \ge$ maximal $\{\beta_0, \beta^{(l)}\}$ that the condition (2.8) in $(H_2) - (ii)$ holds. Hence by Theorem 2.1 (or Theorem 3.1), problem (5.12) has a minimal solution $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i) \equiv ((\underline{u}_i, \underline{v}_i), (\underline{w}_i, \underline{z}_i))$ and a maximal solution $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i) \equiv ((\overline{u}_i, \overline{v}_i), (\overline{w}_i, \overline{z}_i))$. In particular, $(\underline{u}_i, \underline{v}_i)$ and $(\overline{u}_i, \overline{v}_i)$ are the respective minimal and maximal solutions of (5.10) and satisfy the relation

$$((\delta\phi_i)^{1/m}, (\delta\phi_i)^{1/n}) \le (\underline{u}_i, \underline{v}_i) \le (\overline{u}_i, \overline{v}_i) \le ((\rho\psi_i)^{1/m}, (\rho\psi_i)^{1/n}). \tag{5.27}$$

In the case $b^{(1)}b^{(2)} > c^{(1)}c^{(2)}$, the function $(\rho\psi_i, \rho\psi_i)$ in (5.25) should be replaced by $(\rho^{(1)}, \rho^{(2)})$ where $\rho^{(l)} =$ maximal $\{1, \overline{\rho}^{(l)}\}$, $l = 1, 2$. Moreover, Theorem 2.1 also implies that $(\underline{u}_i, \underline{v}_i)$ and $(\overline{u}_i, \overline{v}_i)$ can be computed from the iteration process (5.15) with the initial iteration given by (5.18) and (5.20), respectively.

In addition to the computation of minimal and maximal solutions of (5.12) we can also show the convergence of these solutions to the corresponding minimal and maximal solutions of the continuous problem (1.2). To see this we let $(\lambda_0, \phi_0(x))$ be the principal eigen-pair of (1.4). Upon replacing (λ^*, ϕ_i) by (λ_0, ϕ_0) (and $(\tilde{\lambda}, \psi_i)$ by $(\tilde{\lambda}_0, \tilde{\psi}_0)$) the pair in (5.18) and (5.20) are lower and upper solutions of problem

(1.2). This implies that problem (1.2) has a minimal solution $(\underline{u}(x), \underline{v}(x))$ and a maximal solution $(\overline{u}(x), \overline{v}(x))$ that satisfy the relation

$$((\delta\phi_0(x))^{1/m}, (\delta\phi_0(x))^{1/n}) \leq (\underline{u}(x), \underline{v}(x)) \leq (\overline{u}(x), \overline{v}(x)) \\ \leq ((\rho\tilde{\psi}_0(x))^{1/m}, (\rho\tilde{\psi}_0(x))^{1/n}).$$

(cf. [22]). Since the difference between ϕ_i and $\phi_0(x)$ (or ψ_i and $\tilde{\psi}_i(x)$) can be made arbitrarily small by taking $|h|$ small we see that condition (4.3) holds between these two pairs of lower and upper solutions. By Theorem 4.2 the minimal and maximal solutions $(\underline{\mathbf{u}}_i, \underline{\mathbf{w}}_i)$, $(\overline{\mathbf{u}}_i, \overline{\mathbf{w}}_i)$ of (5.12) converge to their respective minimal and maximal solutions $(\underline{\mathbf{u}}(x_i), \underline{\mathbf{w}}(x_i))$, $(\overline{\mathbf{u}}(x_i), \overline{\mathbf{w}}(x_i))$ of the transformed system of (5.10) at every mesh point x_i in $\overline{\Lambda}^*$, where $\underline{\mathbf{u}}(x_i) = (\underline{u}(x_i), \underline{v}(x_i))$ and $\overline{\mathbf{u}}(x_i) = (\overline{u}(x_i), \overline{v}(x_i))$. To summarize the above conclusions we have the following results for the cooperating model problem (5.10).

Theorem 5.2. *Let $m > 1$, $n > 1$, and let one of the conditions in (5.22) be satisfied. Then the following statements hold:*

(a) Problem (5.10) has the trivial solution $(0, 0)$ and two semitrivial solutions $(u_i, 0)$ and $(0, v_i)$ if $q_i^{(1)} = q_i^{(2)} = 0$;

(b) For any $q_i^{(1)} \geq 0$, $q_i^{(2)} \geq 0$, problem (5.10) has a positive minimal solution $(\underline{u}_i, \underline{v}_i)$ and a positive maximal solution $(\overline{u}_i, \overline{v}_i)$ that satisfy the relation (5.27). Moreover, if $(\underline{u}_i, \underline{v}_i) = (\overline{u}_i, \overline{v}_i)(\equiv (u_i^*, v_i^*))$ then (u_i^*, v_i^*) is the unique positive solution of (5.10);

(c) The minimal solution $(\underline{u}_i, \underline{v}_i)$ can be computed from (5.15) with $(u_i^{(0)}, v_i^{(0)}) = ((\delta\phi)^{1/m}, (\delta\phi_i)^{1/n})$ and $\gamma_i^{(l)} \geq max\{\beta_0, \beta^{(l)}\}$, $l = 1, 2$, where $\beta^{(l)}$ is given by (5.26);

(d) The maximal solution $(\overline{u}_i, \overline{v}_i)$ can be computed from (5.15) with $(u_i^{(0)}, v_i^{(0)}) = ((\rho\psi_i)^{1/m}, (\rho\psi_i)^{1/n})$ if one of the conditions (a), (b) and (c) in (5.22) holds, and with $(u_i^{(0)}, v_i^{(0)}) = (\rho^{(1)}, \rho^{(2)})$ if condition (d) holds, where $\rho^{(l)} = max\{1, \overline{\rho}^{(l)}\}$, $l = 1, 2$;

(e) As $|h| \to 0$, the minimal and maximal solutions $(\underline{u}_i, \underline{v}_i)$, $(\overline{u}_i, \overline{v}_i)$ converge to their respective minimal and maximal solutions $(\underline{u}(x_i), \underline{v}(x_i))$ and $(\overline{u}(x_i), \overline{v}(x_i))$ of the extended problem (1.2) at every mesh point $x_i \in \overline{\Lambda}^*$.

6. Numerical results

Based on the conclusions in Theorem 5.1 and Theorem 5.2 we compute numerical values of the positive solutions for both problems (1.1) and (1.2).

(A). The heat-transfer problem Consider problem (1.1) in a disk region of radius $R = 1$ for the case $\alpha = 1$ and $c_0(x) = c_0$, where the solution depends only on the radial direction. The physical problem may be considered as a fuel rod of a fuel assembly in a nuclear reactor where the temperature in the rod due to fission depends on the radial direction only (cf. [3, 17]). To demonstrate the accuracy and reliability of the monotone iterative schemes we construct a particular source function $p(r)$ so that the solution $u^*(r)$ of the continuous problem is explicitly known. The values of this continuous solution will be used to compare with the computed finite difference solution u_i^* at every mesh point r_i in a given partition $\overline{\Lambda}^*$. In the disk region the continuous problem (1.1) (for the case $\alpha = 1$) becomes

$$\begin{aligned} &-\tfrac{1}{r}\tfrac{d}{dr}(r(k_c + k_r u^3)\tfrac{du}{dr}) + c_0 u = p(r) \quad (0 < r < 1), \\ &\tfrac{du}{dr}(0) = 0, \quad (k_c + k_r u^3(1))\tfrac{du}{dr}(1) = \sigma_1(a_0^4 - u^4(1)), \end{aligned} \tag{6.1}$$

where $\sigma(0) = 0$, $\sigma_1 = \sigma(1)$ and $c(r) \equiv c_0 > 0$.

By letting $w = k_c u + (k_r/4)u^4$ the transformed system of (6.1) is given by

$$\begin{aligned} &\tfrac{1}{r} \cdot \tfrac{d}{dr}(r\tfrac{dw}{dr}) = f(r,u), \quad (0 < r < 1), \\ &\tfrac{dw}{dr}(0) = 0, \quad \tfrac{dw}{dr}(1) = g(u(1)), \end{aligned}$$

where $f(r,u) = p(r) - c_0 u$, $g(u) = \sigma_1(a_0^4 - u^4)$. It is easy to see by the central difference approximation (2.3) that the finite difference system of the above boundary-value problem is given by

$$\begin{aligned} &4w_0 - 4w_1 = h^2 f_0(u_0), \\ &-(1 - \tfrac{1}{2i})w_{i-1} + 2w_i + (1 + \tfrac{1}{2i})w_{i+1} = h^2 f_i(u_i), \;\; i = 1, \dots, N-1, \\ &-2w_{N-1} + 2w_N = h^2 f_N(u_N) + h(2 + \tfrac{1}{2N})g(u_N). \end{aligned}$$

In vector form, this system may be written as

$$AW = F(U) + G(U), \quad U = Q(W),$$

where

$$A = \begin{bmatrix} 4 & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & \cdot & -(1 - \frac{1}{2i}) & 2 & (1 + \frac{1}{2i}) & \cdot & 0 \\ 0 & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdots & 0 & -2 & 2 \end{bmatrix}$$

$$\begin{aligned} &F(U) = (f_0(u_0), \cdots, f_N(u_N))^T, \quad G(U) = (0, \cdots, 0, g(u_N))^T, \\ &Q(W) = (q(w_0), \cdots, q(w_N))^T \end{aligned}$$

and $q(w_i) = u_i$ which is determined from the equation $w_i = k_c u_i + (k_r/4)u^4$, for $i = 1, \dots, N$.

To construct a continuous solution of (6.1) for any physical parameters k_c, k_r, c_0, σ_1 and a_0 we choose

$$p(r) = 4[k_c + k_r(a - r^2)^2(a - 4r^2)] + c_0(a - r^2), \tag{6.2}$$

where $a > 1$ is a constant satisfying the relation

$$\sigma_1(a-1)^4 - 2k_r(a-1)^3 = \sigma_1 a_0^4 + 2k_c. \tag{6.3}$$

It is easy to verify that for any choice of $p(r)$ and $a > 1$ in (6.2) and (6.3), the solution of (6.1) is given by

$$u^*(r) = (a - r^2), \quad (0 \le r \le 1).$$

In particular, if we choose $a \ge 4$ then $p(r) \ge 0$ and is bounded by $4(k_c + k_r a^3) + c_0 a$ for $r \in [0, 1]$. This implies that the constant $\overline{P}$ in (5.6) becomes

$$\overline{P} = a + (4/c_0)(k_c + k_r a^3). \tag{6.4}$$

Using the above value of $\overline{P}$ and any $\rho \ge$ maximal $\{\overline{P}, a_0\}$ in (5.8) for $\gamma^{(1)}$ and $\gamma^{(2)}$ we compute the minimal and maximal sequences $\{\underline{u}_i^{(k)}, \underline{w}_i^{(k)}\}$, $\{\overline{u}_i^{(k)}, \overline{w}_i^{(k)}\}$ from the iteration process (5.1) (or any one of the iteration process in (3.6), (3.10) and (3.11)). The initial iterations for these sequences are $(\underline{u}_i^{(0)}, \underline{w}_i^{(0)}) = (0, 0)$ and $(\overline{u}_i^{(0)}, \overline{w}_i^{(0)}) = (\rho, I[\rho])$ where $I[\rho] = k_c\rho + (k_r/4)\rho^4$. The only requirement in the iteration process is that the constant a be chosen to satisfy $a \ge 4$ and the relation (6.3). In particular, if we choose $a = 4$ then $u^* = (4 - r^2)$ and the requirement (6.3) becomes

$$27(3\sigma_1 - 2k_r) = \sigma_1 a_0^4 + 2k_c. \tag{6.5}$$

The physical constants in the above relation can be arbitrarily chosen. For example, by choosing

$$k_c = 1,\ k_r = 1/64,\ c_0 = 8,\ \sigma_1 = 1,\ a_0^4 = 3^4 - (3^3/32 + 2) = 78.15625,$$

we see that condition (6.5) holds. Moreover, by (6.2) and (6.4) (with $\alpha = 1$)

$$p(r) = 4 + (1/4)(1 - r^2)(4 - r^2)^2 + 8(4 - r^2) \text{ and } \rho = \overline{P} = 5$$

and by (5.6), (5.8) we may choose

$$I[5] = 5 + (5/4)^4 = 7.4414,\ \gamma^{(1)} = 8,\ \overline{\gamma}^{(1)} = 250.$$

This leads to $(\tilde{u}_i, \tilde{w}_i) = (5, 7.4414)$. Using the above values of the parameters and the initial iterations $(\underline{u}_i^{(0)}, \underline{w}_i^{(0)}) = (0, 0)$, and $(\overline{u}_i^{(0)}, \overline{w}_i^{(0)}) = (5, 7.4414)$ we compute the sequences $\{\underline{u}_i^{(k)}, \underline{w}_i^{(k)}\}$, $\{\overline{u}_i^{(k)}, \overline{w}_i^{(k)}\}$ for various values of h from (5.1). Since for the heat-transfer problem the solution (u_i^*, w_i^*) is unique we use the stopping criterion

$$|\overline{u}_i^{(k)} - \underline{u}_i^{(k)}| + |\overline{w}_i^{(k)} - \underline{w}_i^{(k)}| < \epsilon$$

for various values of $\epsilon > 0$. Numerical values of $\underline{u}_i^{(k)}$ and $\overline{u}_i^{(k)}$ of the above sequences together with the continuous solution $u^*(r_i)$ for the case $h = 0.01$ and $\epsilon = 0.0001$ are given in Table 1. It is seen from this table that the monotone property of the minimal and maximal sequences are observed at every mesh point r_i, and after about 16 number of iterations, the values of $\underline{u}_i^{(k)}$ and $\overline{u}_i^{(k)}$ are very close and differ from the true solution $u^*(r_i)$ by less than 2 percent. Interest readers may choose different values of the physical parameters in (6.5) to compute the numerical solution u_i^* and compare it with the continuous solution $u^*(r_i) = a - r_i^2$. One may also choose a different source function $p(r)$ and a different continuous solution $u^*(r)$ of (6.1).

Remark 6.1. (a) When $\alpha = 4$ the function $f^{(1)}$ in (5.2) becomes $f^{(1)}(x, u^{(1)}) = c(x)(b_0^4(x) - u^4)$ with $b_0^4(x) = p(x)/c(x)$. This implies that the Newton's fourth-power radiation law applies also to the interior of the rod;

Table 1. Numerical Results of Example 1

Iteration N		x=0.1	x=0.2	x=0.4	x=0.6	x=0.8	x=1.0
	$\tilde{u}$	5.000000	5.000000	5.000000	5.000000	5.000000	5.000000
0	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\hat{u}$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
	$\overline{u}$	4.744023	4.712858	4.641680	4.523687	4.354267	4.124031
1	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	3.174488	3.160456	2.936469	2.488861	1.690137	0.347743
	$\overline{u}$	4.470324	4.428027	4.327544	4.165898	3.948667	3.690371
2	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	3.433045	3.362950	3.128945	2.688336	1.934653	0.692275
	$\overline{u}$	4.166897	4.124722	4.012457	3.831310	3.587396	3.292526
4	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	3.594452	3.527810	3.322344	2.949272	2.338642	1.356907
	$\overline{u}$	4.036968	3.995057	3.875503	3.679042	3.406969	3.061968
8	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	3.823022	3.771908	3.618385	3.354129	2.957748	2.385870
	$\overline{u}$	4.012747	3.970288	3.848051	3.646283	3.364777	3.003324
16	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	3.996364	3.953420	3.829117	3.623289	3.334566	2.960473
	$\overline{u}$	4.011516	3.969024	3.846637	3.644576	3.362549	3.000186
32	u^*	4.000000	3.960000	3.840000	3.640000	3.360000	3.000000
	$\underline{u}$	4.011471	3.968977	3.846585	3.644512	3.362466	3.000068

(b) If $k_r = 0$, (that is, if the heat transfer is due only to conduction) then $D(u) = k_c$ and problem (1.1) is semilinear. In this case the iteration process (5.4) is directly applicable with $w_i^{(k)} = k_c u_i^{(k)}$, and all the conclusions in Theorem 5.1 hold true for the semilinear problem (1.1).

(B). The cooperation system.

We next compute the positive minimal and maximal solutions of the cooperation system (1.2). Before doing this we consider the extended problem (5.12) in a one-dimensional domain $\Omega = (0, 1)$ by constructing a known continuous solution in the form

$$\begin{aligned} (u(x), v(x)) &= ((\sigma^{(1)}x(1-x))^{1/m}, (\sigma^{(2)}x(1-x))^{1/n}), \\ (w(x), z(x)) &= (\sigma^{(1)}x(1-x), \sigma^{(2)}x(1-x)), \end{aligned} \tag{6.6}$$

where $\sigma^{(l)}$, $l = 1, 2$, are some positive constants to be chosen. Since $w_{xx} = -2\sigma^{(1)}$, $z_{xx} = -2\sigma^{(2)}$ and $(w(0), z(0)) = (w(1), z(1)) = (0, 0)$ we see that the function in (6.6) is a solution of (5.12) if

$$\begin{aligned} q^{(1)}(x) &= 2\sigma^{(1)} - (\sigma^{(1)}x(1-x))^{1/m}[a^{(1)} - b^{(1)}(\sigma^{(1)}x(1-x))^{1/m} \\ &\quad + c^{(1)}(\sigma^{(2)}x(1-x))^{1/n}], \\ q^{(2)}(x) &= 2\sigma^{(2)} - (\sigma^{(2)}x(1-x))^{1/n}[a^{(2)} - b^{(2)}(\sigma^{(2)}x(1-x))^{1/n} \\ &\quad + c^{(2)}(\sigma^{(1)}x(1-x))^{1/m}]. \end{aligned} \tag{6.7}$$

To ensure the nonnegative property of $q^{(1)}(x)$ and $q^{(2)}(x)$ we choose $\sigma^{(l)} \geq 1$ such that

$$\frac{a^{(1)}}{(\sigma^{(1)})^{\alpha}} + \frac{c^{(1)}}{(\sigma^{(2)})^{\beta}} \leq 2 \quad \text{and} \quad \frac{a^{(2)}}{(\sigma^{(2)})^{\gamma}} + \frac{c^{(2)}}{(\sigma^{(1)})^{\beta}} \leq 2, \tag{6.8}$$

where $\alpha = 1-1/m$, $\beta = 1-(1/m+1/n)$ and $\gamma = 1-1/n$. In the case $1/m+1/n < 1$, we also choose $\underline{\sigma} \equiv$ minimal $(\sigma^{(1)}, \sigma^{(2)})$ to satisfy

$$\underline{\sigma} \geq [(a^{(l)} + c^{(l)})/2]^{1/\beta} \quad \text{for} \quad l = 1, 2. \tag{6.9}$$

It is easy to verify that under the conditions (6.8) and (6.9) the functions $q^{(1)}(x)$ and $q^{(2)}(x)$ in (6.7) are positive. In particular, by choosing

$$a^{(1)} = a^{(2)} = b^{(1)} = c^{(1)} = c^{(2)} = \sigma^{(1)} = 1, \quad b^{(2)} = \sigma^{(2)} = 2, \tag{6.10}$$

we see that (6.8) and (6.9) hold, and by (6.6) and (6.7),

$$\begin{aligned}
&(u(x), v(x)) = ((x(1-x))^{1/m}, (2x(1-x))^{1/n}),\\
&q^{(1)}(x) = 2 - (x(1-x))^{1/m}[1 - (x(1-x))^{1/m} + (2x(1-x))^{1/n}],\\
&q^{(2)}(x) = 4 - (2x(1-x))^{1/n}[1 - 2(2x(1-x))^{1/n} + (x(1-x))^{1/m}].
\end{aligned}$$

To find more explicit lower and upper solutions for numerical computations we observe that the principal eigen-pair of (5.17) are given by

$$\lambda^* = \frac{4}{h^2}\sin^2(\frac{\pi h}{2}), \quad \phi_i = \sin(i\pi h).$$

(cf. [24]). By lemma 5.2, a lower solution is given by

$$\begin{aligned}
&(\hat{u}, \hat{v}) = ((\delta \sin(i\pi h))^{1/m}, (\delta \sin(i\pi h))^{1/n}),\\
&(\hat{w}, \hat{z}) = (\delta \sin(i\pi h), \delta \sin(i\pi h)),
\end{aligned} \tag{6.11}$$

where $\delta > 0$ is required to satisfy (5.19). In view of $0 \leq \phi_i \leq 1$ it suffices to choose

$$\delta \leq [\frac{a^{(1)}}{\lambda^* + b^{(1)}}]^{\underline{m}} = [\frac{1}{\lambda^* + 1}]^{\underline{m}} \quad \text{and} \quad \delta \leq [\frac{a^{(2)}}{\lambda^* + b^{(2)}}]^{\underline{n}} = [\frac{1}{\lambda^* + 2}]^{\underline{n}}, \tag{6.12}$$

where $\underline{m}$ = minimal $\{1/m, 1-1/m\}$, $\underline{n}$ = minimal $\{1/n, 1-1/n\}$. For an upper solution we use the fact $b^{(1)}b^{(2)} > c^{(1)}c^{(2)}$ (from (6.10)) to find a constant upper solution from Lemma 5.3. It is easy to verify from (6.10) and (5.21) (with $\overline{a}^{(1)} = 3$, $\overline{a}^{(2)} = 5$) that $\overline{\rho}^{(1)} = 11$, $\overline{\rho}^{(2)} = 8$ and the constant upper solution is

$$(\tilde{u}_i, \tilde{v}_i) = (11, 8), \quad (\tilde{w}_i, \tilde{z}_i) = (11^m, 8^n). \tag{6.13}$$

With this pair of explicit lower and upper solutions we choose $\gamma^{(1)}$ and $\gamma^{(2)}$ from (5.14) to obtain the relation

$$\gamma_i^{(1)}(mu_i^{m-1}) \geq 2u_i - 1 - v_i, \quad \gamma^{(2)}(nv_i^{n-1}) \geq 4v_i - 1 - u_i$$

for $0 \leq u_i \leq 11$ and $0 \leq v_i \leq 8$. It is obvious that the above inequalities are trivially satisfied for $0 \leq u_i \leq 1/2$ and $0 \leq v_i \leq 1/4$. This leads to the choice of

$$\begin{aligned}
&\gamma_i^{(1)} \geq \frac{2}{m}\text{maximal } \{u_i^{2-m}; \frac{1}{2} \leq u_i \leq 11\},\\
&\gamma_i^{(2)} \geq \frac{4}{n}\text{maximal } \{v_i^{2-n}; \frac{1}{4} \leq v_i \leq 8\}.
\end{aligned} \tag{6.14}$$

Using a suitable choice of $\delta^{(l)}$ from (6.12) and $\gamma^{(l)}$ from (6.14), $(l = 1, 2)$, and the functions $(\hat{u}, \hat{v})$ and $(\tilde{u}, \tilde{v})$ in (6.11) and (6.13) as the initial iterations we compute the minimal and maximal sequences from (5.15) for various values of m and n. The stopping criterion for the monotone sequences is given by

$$|(u_i^{(k+1)} - u_i^{(k)})/u_i^{(k)}| + |(v_i^{(k+1)} - v_i^{(k)})/v_i^{(k)}| < \epsilon \tag{6.15}$$

for various $\epsilon > 0$, where $\{u_i^{(k)}, v_i^{(k)}\}$ stands for either the minimal sequence or the maximal sequence. Numerical values for the case $(m, n) = (1.5, 2.0)$ and $(h, \epsilon) = (10^{-2}, 10^{-6})$ are given in Table 2 and Table 3. It is seen from these tables that the monotone property of both minimal and maximal sequences are observed and the differences between the computed solutions and the true continuous solution are less than 1 percent.

Table 2. Numerical Results of Example 2, $\overline{u}$ and $\underline{u}$

Iteration N		x=0.1	x=0.2	x=0.4	x=0.6	x=0.8	x=1.0
	$\tilde{u}$	11.00000	11.00000	11.00000	11.00000	11.00000	11.00000
0	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\hat{u}$	0.000000	0.122075	0.168249	0.168249	0.122075	0.000000
	$\overline{u}$	0.000000	3.992098	5.127119	5.127119	3.992098	0.000000
1	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.231269	0.297754	0.297754	0.231269	0.000000
	$\overline{u}$	0.000000	1.717012	2.313905	2.313905	1.717012	0.000000
2	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.270939	0.352243	0.352243	0.270939	0.000000
	$\overline{u}$	0.000000	0.523087	0.701919	0.701919	0.523087	0.000000
4	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.292341	0.382194	0.382194	0.292341	0.000000
	$\overline{u}$	0.000000	0.301438	0.394927	0.394927	0.301438	0.000000
8	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.296046	0.387382	0.387382	0.296046	0.000000
	$\overline{u}$	0.000000	0.296252	0.387670	0.387670	0.296252	0.000000
12	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.296130	0.387499	0.387499	0.296130	0.000000
	$\overline{u}$	0.000000	0.296135	0.387506	0.387506	0.296135	0.000000
16	u^*	0.000000	0.294723	0.386196	0.386196	0.294723	0.000000
	$\underline{u}$	0.000000	0.296132	0.387502	0.387502	0.296132	0.000000

To compute the positive minimal and maximal solutions of (5.12) for the original problem where $q_i^{(1)} = q_i^{(2)} = 0$ we consider the system in a rectangular domain $\Omega_2 \equiv \{(x, y); 0 < x < L_1, 0 < y < L_2\}$. Let $\Omega_2' \equiv \{(x, y); -h_1 < x < L_1 + h_1, -h_2 < y < L_2 + h_2\}$ so that $\overline{\Omega}_2$ is contained in Ω_2'. Since the principal eigen-pair of (5.17) for $\Lambda = \Lambda_2$ and $\Lambda = \Lambda_2'$ are given, respectively, by

$$\begin{aligned} \lambda_2^* &= \tfrac{4}{h_1^2}\sin^2(\tfrac{\pi h_1}{2L_1}) + \tfrac{4}{h_2^2}\sin^2(\tfrac{\pi h_2}{2L_2}), & \phi_{jk} &= \sin(\tfrac{j\pi}{M_1})\sin(\tfrac{k\pi}{M_2}), \\ \tilde{\lambda}_2 &= \tfrac{4}{h_1^2}\sin^2(\tfrac{\pi h_1}{2(L_1+2h_1)}) + \tfrac{4}{h_2^2}\sin^2(\tfrac{\pi h_2}{2(L_2+2h_2)}), & \tilde{\phi}_{jk} &= \sin(\tfrac{j\pi}{\overline{M}_1})\sin(\tfrac{k\pi}{\overline{M}_2}), \end{aligned} \tag{6.16}$$

where $M_1 = L_1/h_1$, $M_2 = L_2/h_2$, $\overline{M}_1 = (L_1 + 2h_1)/h_1$ and $\overline{M}_2 = (L_2 + 2h_2)/h_2$, we see that the pair of lower and upper solutions in (5.18) and (5.20) become

$$\begin{aligned} (\hat{u}_i, \hat{v}_i) &= ((\delta \sin(\tfrac{j\pi}{M_1})\sin(\tfrac{k\pi}{M_2}))^{1/m}, (\delta \sin(\tfrac{j\pi}{M_1})\sin(\tfrac{k\pi}{M_2}))^{1/n}), \ ((j,k) \in \overline{\Lambda}_2), \\ (\tilde{u}_i, \tilde{v}_i) &= ((\rho \sin(\tfrac{j\pi}{\overline{M}_1})\sin(\tfrac{k\pi}{\overline{M}_2}))^{1/m}, (\rho \sin(\tfrac{j\pi}{\overline{M}_1})\sin(\tfrac{k\pi}{\overline{M}_2}))^{1/n}), \ ((j,k) \in \overline{\Lambda}_2') \end{aligned} \tag{6.17}$$

Table 3. Numerical Results of Example 2, $\overline{v}$ and $\underline{v}$

Iteration N		x=0.1	x=0.2	x=0.4	x=0.6	x=0.8	x=1.0
0	$\tilde{v}$	8.000000	8.000000	8.000000	8.000000	8.000000	8.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\hat{v}$	0.000000	0.206524	0.262702	0.262702	0.206524	0.000000
1	$\overline{v}$	0.000000	2.597357	3.157492	3.157492	2.597357	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.532995	0.650286	0.650286	0.532995	0.000000
2	$\overline{v}$	0.000000	1.145613	1.436870	1.436870	1.145613	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.561690	0.686744	0.686744	0.561690	0.000000
4	$\overline{v}$	0.000000	0.611150	0.751687	0.751687	0.611150	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.566821	0.693432	0.693432	0.566821	0.000000
8	$\overline{v}$	0.000000	0.568173	0.695218	0.695218	0.568173	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.567379	0.694170	0.694170	0.567379	0.000000
12	$\overline{v}$	0.000000	0.567410	0.694210	0.694210	0.567410	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.567392	0.694186	0.694186	0.567392	0.000000
16	$\overline{v}$	0.000000	0.567392	0.694187	0.694187	0.567392	0.000000
	v^*	0.000000	0.565685	0.692820	0.692820	0.565685	0.000000
	$\underline{v}$	0.000000	0.567392	0.694186	0.694186	0.567392	0.000000

In the case of $b_1 b_2 > c_1 c_2$, the upper solution in (6.17) should be replaced by $(\tilde{u}_i, \tilde{v}_i) = (\overline{\rho}^{(1)}, \overline{\rho}^{(2)})$ where $\overline{\rho}^{(1)}$ and $\overline{\rho}^{(2)}$ are the constants given by (5.21) with $\overline{a}^{(1)} = a^{(1)}$, $\overline{a}^{(2)} = a^{(2)}$. If we choose the same constants a_i, b_i and c_i as that in (6.10) then $(\tilde{u}_i, \tilde{v}_i) = (3, 2)$, and $(\hat{u}_i, \hat{v}_i)$ is given by (6.17). The constant δ in (6.17) is given by (6.12) with λ^* replaced by λ_2^* while $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ are given by (6.14) with the constant $(11, 8)$ replaced by $(3, 2)$.

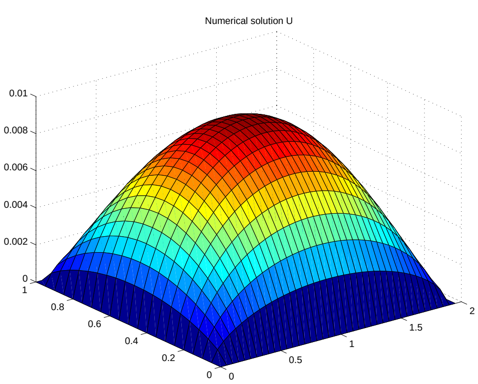

Figure 1. Numerical Solution u of Example 3

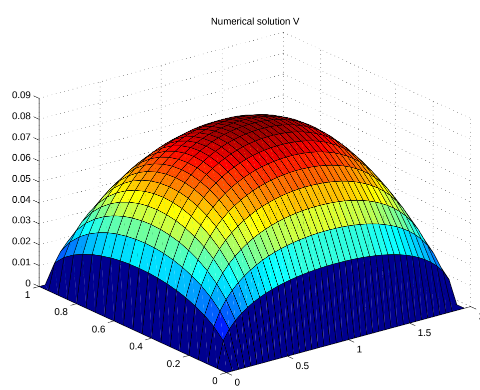

Figure 2. Numerical Solution v of Example 3

Using the lower solution $(\hat{u}_i, \hat{v}_i)$ in (6.17) and the upper solution $(\tilde{u}_i, \tilde{v}_i) = (3, 2)$ as the initial iterations in (5.15) we compute the minimal and maximal sequences $\{\underline{u}_i^{(k)}, \underline{v}_i^{(k)}\}$ and $\{\overline{u}_i^{(k)}, \overline{v}_i^{(k)}\}$. The stopping criterion is again given by (6.15). Numerical values of these sequences for the case $h_1 = h_2 = h$ and

$$(L_1, L_2) = (1, 2), \quad (m, n) = (1.5, 2.0), \quad (h, \epsilon) = (10^{-2}, 10^{-6})$$

Table 4. Numerical Results of Example 3, $\overline{u}$ and $\underline{u}$

Iteration N	(x, y)	(0.2, 0.5)	(0.2, 1.0)	(0.4, 0.5)	(0.4, 1.0)	(0.5, 0.5)	(0.5, 1.0)
0	$\tilde{u}$	3.000000	3.000000	3.000000	3.000000	3.000000	3.000000
	$\hat{u}$	0.002294	0.002890	0.003161	0.003983	0.003269	0.004118
1	$\overline{u}$	0.349739	0.402627	0.472531	0.545619	0.487250	0.562849
	$\underline{u}$	0.003441	0.004108	0.004646	0.005554	0.004791	0.005728
2	$\overline{u}$	0.154955	0.183628	0.211088	0.250432	0.217919	0.258576
	$\underline{u}$	0.003898	0.004626	0.005264	0.006257	0.005429	0.006453
4	$\overline{u}$	0.043264	0.051892	0.058871	0.070679	0.060771	0.072969
	$\underline{u}$	0.004601	0.005444	0.006217	0.007368	0.006412	0.007600
8	$\overline{u}$	0.011028	0.013123	0.014942	0.017803	0.015416	0.018370
	$\underline{u}$	0.005319	0.006295	0.007192	0.008523	0.007418	0.008793
16	$\overline{u}$	0.005998	0.007103	0.008112	0.009621	0.008367	0.009925
	$\underline{u}$	0.005626	0.006660	0.007608	0.009019	0.007847	0.009304
24	$\overline{u}$	0.005683	0.006728	0.007685	0.009112	0.007927	0.009400
	$\underline{u}$	0.005653	0.006692	0.007644	0.009063	0.007884	0.009349
32	$\overline{u}$	0.005657	0.006698	0.007651	0.009070	0.007891	0.009357
	$\underline{u}$	0.005655	0.006694	0.007647	0.009066	0.007887	0.009353

Table 5. Numerical Results of Example 3, $\overline{v}$ and $\underline{v}$

Iteration N	(x, y)	(0.2, 0.5)	(0.2, 1.0)	(0.4, 0.5)	(0.4, 1.0)	(0.5, 0.5)	(0.5, 1.0)
0	$\tilde{v}$	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
	$\hat{v}$	0.010481	0.012464	0.013332	0.015854	0.013671	0.016257
1	$\overline{v}$	0.354478	0.392931	0.443057	0.492244	0.453214	0.503687
	$\underline{v}$	0.037434	0.041656	0.046335	0.051674	0.047344	0.052814
2	$\overline{v}$	0.188231	0.211981	0.235952	0.266251	0.241486	0.272567
	$\underline{v}$	0.046501	0.051533	0.057560	0.063940	0.058813	0.065353
4	$\overline{v}$	0.090716	0.101494	0.112852	0.126540	0.115386	0.129418
	$\underline{v}$	0.055265	0.061176	0.068433	0.075938	0.069926	0.077619
8	$\overline{v}$	0.062257	0.069026	0.077149	0.085743	0.078840	0.087650
	$\underline{v}$	0.058831	0.065140	0.072866	0.080875	0.074458	0.082668
16	$\overline{v}$	0.059269	0.065633	0.073411	0.081492	0.075016	0.083299
	$\underline{v}$	0.059186	0.065538	0.073307	0.081372	0.074910	0.083176
24	$\overline{v}$	0.059198	0.065552	0.073322	0.081389	0.074925	0.083194
	$\underline{v}$	0.059193	0.065546	0.073316	0.081381	0.074918	0.083186
32	$\overline{v}$	0.059193	0.065547	0.073317	0.081382	0.074919	0.083187
	$\underline{v}$	0.059193	0.065546	0.073316	0.081382	0.074918	0.083187

are given in Table 4 and Table 5, and are sketched in Figure 1 and Figure 2. It is seen from these tables that the monotone property of the minimal and maximal sequences and the comparison relation in Theorem 2.1 (or in (3.7)) are observed at every point in the domain $\overline{\Lambda}_2$. It turns out that in this example the minimal solution $(\underline{u}_i, \underline{v}_i)$ coincide with the maximal solution $(\overline{u}_i, \overline{v}_i)$ and the problem has a unique solution between the pair of lower and upper solutions in (6.17). Interested readers may choose different values of (m, n) to compute the maximal and minimal solutions of the problem.

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