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MEAN-SQUARE EXPONENTIAL DICHOTOMY OF NUMERICAL SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract We present the ability of numerical simulations to reproduce the mean-square exponential dichotomy of stochastic differential equations. Under some conditions, we show that the mean-square exponential dichotomy of stochastic differential equations is equivalent to that of the numerical method for sufficient small step sizes.

Keywords Mean-square exponential dichotomy, Euler-Maruyama method, stochastic differential equation.

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1. Introduction

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a standard filtered probability space, i.e., $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration with $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets. For a matrix or a vector A, we use A^T to denote its transpose. Let $\omega(t) = (\omega_1(t), \dots \omega_m(t))^T$ be an m-dimensional Brownian motion defined on the space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$ and operator norm in $\mathbb{R}^{n\times m}$, i.e., $|A| = \sup\{|Ax| : |x| = 1\}$ if A is a matrix. In addition, let $L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$ denote the family of all $\mathscr{F}_s$-measurable $\mathbb{R}^n$-valued random variables, i.e., $\xi_s : \Omega \to \mathbb{R}^n$ such that

$$\mathbb{E}|\xi_s|^2 < \infty,$$

for all $s \geq 0$. Consider an n-dimensional Itô stochastic differential equation (SDE) (we refer the reader to [1, 5, 6, 15] for details about SDEs),

$$dy(t) = f(y(t))dt + g(y(t))d\omega(t), \quad t \geq 0, \tag{1.1}$$

with initial data $y(s) = \xi_s \in L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$. The following numerical method computes approximations $x_k \approx y(k\Delta t)$:

$$x_{k+1} = x_k + (1-\theta)f(x_k)\Delta t + \theta f(x_{k+1})\Delta t + \Delta t^{\frac{1}{2}} g(x_k)V_k, \tag{1.2}$$

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where $\Delta t > 0$ is the constant stepsize, $\theta \in [0, 1]$ is a fixed parameter and each V_k is an independent random variable of the form $N(0, 1)$. Such a method was called "stochastic theta method (STM)" in the literature [8, 9, 11]. In the deterministic case, $g \equiv 0$, (1.2) is called the theta method (TM) and the choice $\theta = 0$ gives the Euler-Maruyama method (EM)

$$x_{k+1} = x_k + f(x_k)\Delta t + \Delta t^{\frac{1}{2}} g(x_k) V_k, \tag{1.3}$$

which has been widely used [7]. The aim of this paper is to study the mean-square exponential dichotomy of numerical methods for stochastic differential equations. The classical notion of exponential dichotomy introduced by Perron in [17] plays an important role in the study of dynamical behaviors of differential equations in the deterministic case, particularly in what concerns the study of stable and unstable invariant manifolds, and therefore has attracted much attention. We refer to the books [3, 16] for details and further references related to exponential dichotomies. A more general exponential dichotomy, which is the so-called "nonuniform exponential dichotomy", has been introduced and developed by Barreira and Valls during the last few decades [2]. The concept of exponential dichotomy in mean-square for SDE has been introduced and studied in [13, 20, 21]. Mean-square and asymptotic stability of the stochastic theta method was studied by Higham in [8]. For more results about the numerical methods of SDE, we refer the reader to [4, 9, 10, 12, 14, 19]. We carry out numerical simulations using a numerical method with a small step size Δt, and we will try to answer two key questions: (Q1) If the SDE is exponential dichotomy in mean square, will the numerical method be exponential dichotomy in mean square for sufficiently small Δt? (Q2) If the numerical method is exponential dichotomy in mean square for small Δt, can we infer that the underlying SDE is exponential dichotomy in mean square? These two questions deal with an asymptotic ($t \to \infty$) property, and therefore they cannot be answered directly by applying traditional finite-time convergence results. Our approach was heavily motivated by the ideas contained in [8, 11, 18]. In [18], the concept of mean-square stability with respect to a linear test equation was studied, and a condition was derived to characterizes the mean-square stability. In [8], Higham studied a linear test equation with a multiplicative noise term and considered mean-square and asymptotic stability of a stochastic version of the theta method. In [11], it was proved that, under some suitable conditions, for sufficiently small sizes, the mean-square stability of the SDE and that of the numerical method is equivalent. The rest of this paper is organized as follows: The definitions of exponential dichotomy in mean square (EDMS) for the SDE and numerical method are given in Section 2. In Section 3, under some reasonable conditions, we show the equivalence, for sufficiently small step sizes, of the mean-square exponential dichotomy of the SDE and that of the method. We also present the result on exponential contraction as a special case of exponential dichotomy, and an example is presented to illustrate our result about exponential contraction.

2. Preliminaries

To ensure that (1.1) has a unique solution for any initial data $y(s) = \xi_s$, thoughout this paper we will always assume that the following two standard hypotheses hold. See [1, 5, 15] for details.

H1. (*Lipschitz condition*) for all $x, y \in \mathbb{R}^n$ and $t \in [0, T]$,

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq L|x - y|^2$$

for some constant $L > 0$, here $a \vee b$ means the maximum of a and b.

H2. $f(0) = g(0) = 0$.

Note that assumptions H1 and H2 implies the following linear growth condition

$$|f(x)|^2 \vee |g(x)|^2 \leq L|x|^2, \quad \text{for all } x \in \mathbb{R}^n.$$

We assume that the phase space $\mathbb{R}^n$ can be splitted into a direct sum, i.e.,

$$\mathbb{R}^n = X_1 \oplus X_2,$$

where X_1 is a linear subspace of $\mathbb{R}^n$, and X_2 is the complementary subspace of X_1.

Definition 2.1. The solution $y(t) \in L^2_{\mathscr{F}_t}(\Omega, \mathbb{R}^n)$ with initial data $\xi_s \in L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$ of SDE (1.1) is said to be EDMS if there exist positive constants M and a such that

$$\mathbb{E}|y(t)|^2 \leq Me^{-a(t-s)}\mathbb{E}|y(s)|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, \xi_s \in X_1; \tag{2.1a}$$

$$\mathbb{E}|y(s)|^2 \leq Me^{-a(t-s)}\mathbb{E}|y(t)|^2, \quad \forall\, t \geq s \geq 0,\ \forall\, \xi_s \in X_2. \tag{2.1b}$$

Stoica proved in [20] that there exists an exponential dichotomy in mean square of stochastic cocycles generated by stochastic differential equations. The following example proposed here is to illustrate the existence of EDMS for the solution of SDE.

Example 2.1. Consider the following equation

$$\begin{cases} dX = -bX dt + \sigma dW, \\ dY = bY dt + \sigma dW, \end{cases} \tag{2.2}$$

with initial data $(X(0), Y(0))$ independent of the Brownian motion, where $b > 0$ is a coefficient of friction, and σ is a diffusion coefficient. It is easy to verify that its solution is given as

$$\begin{cases} X(t) = e^{-bt}X(0) + \sigma \displaystyle\int_0^t e^{-b(t-t_1)} dW, \\ Y(t) = e^{bt}Y(0) + \sigma \displaystyle\int_0^t e^{b(t-t_1)} dW. \end{cases}$$

Note that

$$\begin{aligned} \mathbb{E}|X(t)|^2 &= \mathbb{E}\Big(e^{-2bt}X^2(0) + 2\sigma e^{-bt}X(0)\int_0^t e^{-b(t-t_1)}dW \\ &\quad + \sigma^2\big(\int_0^t e^{-b(t-t_1)}dW\big)^2\Big) \\ &= e^{-2bt}\mathbb{E}|X(0)|^2 + 2\sigma e^{-bt}\mathbb{E}|X(0)|\mathbb{E}\Big(\int_0^t e^{-b(t-t_1)}dW\Big) \\ &\quad + \sigma^2\int_0^t e^{-2b(t-t_1)}dt_1 \\ &= e^{-2bt}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2b}(1 - e^{-2bt}). \end{aligned}$$

From Condition 2.1 below, we know that $\mathbb{E}|X(s)|^2 > 0$ for all $s \geq 0$ and note that we focus on finite-time convergence, e.g., $t \in [0,T]$, thus we obtain that

$$\begin{aligned}\frac{\mathbb{E}|X(t)|^2}{\mathbb{E}|X(s)|^2} &= \frac{e^{-2bt}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2b}(1-e^{-2bt})}{e^{-2bs}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2b}(1-e^{-2bs})} \\ &\leq \frac{e^{-2bt}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2b}}{e^{-2bs}\mathbb{E}|X(0)|^2} \\ &\leq M_1 e^{-2b(t-s)}, \end{aligned} \tag{2.3}$$

with $t \geq s \geq 0$, where $M_1 = 1 + \frac{\sigma^2 e^{2bT}}{2b\mathbb{E}|X(0)|^2} > 0$. Similarly, we can get

$$E|Y(s)|^2 \leq M_2 e^{-2b(t-s)} E|Y(t)|^2 \quad \text{for } t \geq s \geq 0, \tag{2.4}$$

where $M_2 = 1 + \frac{\sigma^2 e^{2bT}}{2b\mathbb{E}|Y(0)|^2} > 0$. From (2.3) and (2.4), we know that the solution of SDE (2.2) is EDMS.

Now we define EDMS for a numerical method to a continuous-time (using interpolation) approximation $x(t)$ of SDE (1.1).

Definition 2.2. For a given step size Δt, a numerical method (1.2) is said to be EDMS on the SDE (1.1) if there exist positive constants N and b such that with initial data $\xi_s \in L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$,

$$\mathbb{E}|x(t)|^2 \leq N e^{-b(t-s)} \mathbb{E}|x(s)|^2, \quad \forall t \geq s \geq 0,\ \forall \xi_s \in X_1; \tag{2.5a}$$

$$\mathbb{E}|x(s)|^2 \leq N e^{-b(t-s)} \mathbb{E}|x(t)|^2, \quad \forall t \geq s \geq 0,\ \forall \xi_s \in X_2. \tag{2.5b}$$

In this paper, we wish to know whether the numerical method shares EDMS with the solution of SDE. Theorem 3.1 below resolves the issue positively for numerical methods that satisfy the following natural finite-time bound condition.

Condition 2.1. For sufficiently small Δt, the numerical method applied to (1.1) with initial condition $x(s) = y(s) = \xi_s$ satisfies, for any $T > 0$,

$$0 < \inf_{s \leq t \leq T} \mathbb{E}|x(t)|^2 \leq \sup_{s \leq t \leq T} \mathbb{E}|x(t)|^2 \leq B_{\xi_s, T}, \tag{2.6}$$

where $B_{\xi_s,T}$ depends on ξ_s and T but not on Δt, and

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t) - y(t)|^2 \leq \left(\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^2 \right) C_T \Delta t, \tag{2.7}$$

where C_T depends on T but not on ξ_s and Δt.

Under the hypotheses H1 and H2, Condition 2.1 is a natural condition, which has been explained in [11]. Moreover, C_T is an increasing function in T, see [11, Theorem A.4] for details. Such a fact will be used in this paper.

3. Main results

We will prove that EDMS of the numerical method is equivalent to that of SDE under Condition 2.1 and for sufficiently small step size, which means that we can investigate the EDMS of the SDE from careful numerical simulations. The main result reads as follows.

Theorem 3.1. *Suppose that a numerical method satisfies Condition* 2.1*. Then the solution of SDE* (1.1) *is EDMS if and only if there exists a* $\Delta t^* > 0$ *such that for any step size* $0 < \Delta t \leq \Delta t^*$*, the numerical method* (1.2) *is EDMS.*

Theorem 3.1 comes directly from the following two lemmas. The first lemma shows that if the SDE admits an exponential dichotomy in mean square, then the numerical method admits an exponential dichotomy in mean square for sufficiently small Δt.

Lemma 3.1. *Assume that the solution of SDE* (1.1) *is EDMS with* (2.1a) *and* (2.1b)*. Under the condition* 2.1*, there exists a* $\Delta t^* > 0$ *such that for any step size* $0 < \Delta t \leq \Delta t^*$*, the numerical method* (1.2) *is also EDMS with* (2.5a) *and* (2.5b)*, where* $b = \frac{1}{2}a$*,* $N = 2Me^{\frac{1}{2}aT}$ *and* $T = 1 + (4\log M)/a > 0$*.*

Proof. Firstly we consider the space X_1. Given $x \in X_1$, choose $T = 1 + (4\log M)/a > 0$ such that

$$Me^{-aT} \leq e^{-\frac{3}{4}aT}. \tag{3.1}$$

For any $\alpha_1 > 0$, it is easy to show that

$$\mathbb{E}|x(t)|^2 \leq (1+\alpha_1)\mathbb{E}|x(t) - y(t)|^2 + (1 + 1/\alpha_1)\mathbb{E}|y(t)|^2. \tag{3.2}$$

Using (2.7) and (2.1a) over $[0, 2T]$, we see that

$$\sup_{0\leq t\leq 2T} \mathbb{E}|x(t)|^2 \leq (1+\alpha_1) \sup_{0\leq t\leq 2T} \mathbb{E}|x(t)|^2 C_{2T}\Delta t + (1+1/\alpha_1)M\mathbb{E}|y(0)|^2. \tag{3.3}$$

If we take Δt sufficiently small, then we arrive at

$$\sup_{0\leq t\leq 2T} \mathbb{E}|x(t)|^2 \leq \frac{(1+1/\alpha_1)M\mathbb{E}|y(0)|^2}{1-(1+\alpha_1)C_{2T}\Delta t}. \tag{3.4}$$

Taking the supremum over $[T, 2T]$ in (3.2) and using condition 2.1, the bound (3.4) and condition (2.1a), we can obtain

$$\sup_{T\leq t\leq 2T} \mathbb{E}|x(t)|^2 \leq \frac{(1+\alpha_1)(1+1/\alpha_1)M\mathbb{E}|y(0)|^2}{1-(1+\alpha_1)C_{2T}\Delta t} C_{2T}\Delta t + (1+1/\alpha_1)M\mathbb{E}|y(0)|^2 e^{-aT}.$$

We rewrite the above inequality as

$$\sup_{T\leq t\leq 2T} \mathbb{E}|x(t)|^2 \leq R(\Delta t)\mathbb{E}|y(0)|^2, \tag{3.5}$$

where

$$R(\Delta t) = \frac{(1+\alpha_1)(1+1/\alpha_1)M}{1-(1+\alpha_1)C_{2T}\Delta t} C_{2T}\Delta t + (1+1/\alpha_1)Me^{-aT}. \tag{3.6}$$

Putting $\alpha_1 = 1/\sqrt{\Delta t}$ and using (3.1), we see that for sufficiently small Δt,

$$R(\Delta t) \leq 2\sqrt{\Delta t} C_{2T} M + (1+\sqrt{\Delta t})e^{-\frac{3}{4}aT}.$$

The right hand side (RHS) of the above inequality is equal to $e^{-(3/4)aT}$ when $\Delta t = 0$ and increases monotonically with Δt. Hence, by taking Δt sufficiently small, we have

$$R(\Delta t) \leq e^{-\frac{1}{2}aT},$$

together with (3.5), we obtain

$$\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}aT}\mathbb{E}|y(0)|^2 = e^{-\frac{1}{2}aT}\mathbb{E}|x(0)|^2,$$

which can be weakened to

$$\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}aT}\sup_{0\le t\le T}\mathbb{E}|x(t)|^2.$$

Now let $\hat{y}(t)$ be the solution to the SDE (1.1) for $t\in[T,\infty)$, with the initial condition that $\hat{y}(T)=x(T)$. Using the similar analysis above, we obtain

$$\mathbb{E}|x(t)|^2 \le (1+\alpha_1)\mathbb{E}|x(t)-\hat{y}(t)|^2 + (1+1/\alpha_1)\mathbb{E}|\hat{y}(t)|^2. \tag{3.7}$$

Taking the supremum over $[T,3T]$, and using the Markov property for the SDE, we may shift (2.1a) and condition 2.1 to $[T,3T]$, obtaining

$$\sup_{T\le t\le 3T}\mathbb{E}|x(t)|^2 \le (1+\alpha_1)\sup_{T\le t\le 3T}\mathbb{E}|x(t)|^2 C_{2T}\Delta t + (1+1/\alpha_1)M\mathbb{E}|x(T)|^2,$$

which gives

$$\sup_{T\le t\le 3T}\mathbb{E}|x(t)|^2 \le \frac{(1+1/\alpha_1)M\mathbb{E}|x(T)|^2}{1-(1+\alpha_1)C_{2T}\Delta t}.$$

Now we take the supremum over $[2T,3T]$ in (3.7) and obtain

$$\sup_{2T\le t\le 3T}\mathbb{E}|x(t)|^2 \le R(\Delta t)\mathbb{E}|x(T)|^2 \le e^{-\frac{1}{2}aT}\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2.$$

Continuing the above step, we can get

$$\begin{aligned}\sup_{iT\le t\le (i+1)T}\mathbb{E}|x(t)|^2 &\le e^{-\frac{1}{2}aT}\sup_{(i-1)T\le t\le iT}\mathbb{E}|x(t)|^2\\ &\ \vdots\\ &\le e^{-\frac{1}{2}(i-j)aT}\sup_{jT\le t\le (j+1)T}\mathbb{E}|x(t)|^2.\end{aligned} \tag{3.8}$$

Now let $\hat{y}(t)$ be the solution to the SDE (1.1) for $t\in[s,\infty)$, with the initial condition that $\hat{y}(s)=x(s)$, where $(j-1)T\le s\le jT\le t\le (j+1)T$. Using the same idea above over $[s,(j+1)T]$ in (3.3) for $\alpha_1 = 1/\sqrt{\Delta t}>0$. Note that C_T is an increasing function in T and using the Markov property for the SDE, we have

$$\begin{aligned}\sup_{jT\le t\le (j+1)T}\mathbb{E}|x(t)|^2 &\le \sup_{s\le t\le (j+1)T}\mathbb{E}|x(t)|^2\\ &\le \frac{(1+1/\alpha_1)M\mathbb{E}|\hat{y}(s)|^2}{1-(1+\alpha_1)C_{(j+1)T-s}\Delta t}\\ &\le \frac{(1+\sqrt{\Delta t})M\mathbb{E}|x(s)|^2}{1-(\Delta t+\sqrt{\Delta t})C_{2T}}.\end{aligned}$$

The RHS of the inequality is equal to $M\mathbb{E}|x(s)|^2$ when $\Delta t=0$ and increases monotonically with Δt. Thus for sufficiently small Δt, we have

$$\sup_{jT\le t\le (j+1)T}\mathbb{E}|x(t)|^2 \le 2M\mathbb{E}|x(s)|^2. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\le \sup_{iT\le t\le (i+1)T}\mathbb{E}|x(t)|^2\\ &\le 2Me^{-\frac{1}{2}(i-j)aT}\mathbb{E}|x(s)|^2\\ &\le 2Me^{-\frac{1}{2}a(t-s-T)}\mathbb{E}|x(s)|^2\end{aligned}$$

with $jT \le s \le (j+1)T \le iT \le t \le (i+1)T$. Therefore, (2.5a) holds with

$$b = \frac{1}{2}a, \quad \text{and} \quad N = 2Me^{\frac{1}{2}aT}.$$

Now we consider the space X_2. Given $x \in X_2$, using (2.7) and (2.1b) for any $\alpha_1 > 0$ over $[0, 2T]$ in (3.2), we see that

$$\sup_{0\le t\le 2T}\mathbb{E}|x(t)|^2 \le (1+\alpha_1)\sup_{0\le t\le 2T}\mathbb{E}|x(t)|^2 C_{2T}\Delta t + (1+1/\alpha_1)M\mathbb{E}|y(2T)|^2. \quad (3.10)$$

If we take Δt sufficiently small, then we arrive at

$$\sup_{0\le t\le 2T}\mathbb{E}|x(t)|^2 \le \frac{(1+1/\alpha_1)M\mathbb{E}|y(2T)|^2}{1-(1+\alpha_1)C_{2T}\Delta t}. \quad (3.11)$$

Now taking the supremum over $[0, T]$ in (3.2), using condition 2.1, the bound (3.11) and condition (2.1b), we obtain

$$\sup_{0\le t\le T}\mathbb{E}|x(t)|^2 \le R(\Delta t)\mathbb{E}|y(2T)|^2, \quad (3.12)$$

where $R(\Delta t)$ is represented by (3.6). Letting $\alpha_1 = 1/\sqrt{\Delta t}$ and using (3.1), we see that for sufficiently small Δt,

$$R(\Delta t) \le 2\sqrt{\Delta t}C_{2T}M + (1+\sqrt{\Delta t})e^{-\frac{3}{4}aT}.$$

The RHS of the above inequality is equal to $e^{-(3/4)aT}$ when $\Delta t = 0$ and increases monotonically with Δt. Hence, by taking Δt sufficiently small, we may ensure that

$$R(\Delta t) \le e^{-\frac{5}{8}aT}.$$

Combined (3.12) with the above inequality, we have

$$\sup_{0\le t\le T}\mathbb{E}|x(t)|^2 \le e^{-\frac{5}{8}aT}\mathbb{E}|y(2T)|^2 \le e^{-\frac{5}{8}aT}\sup_{T\le t\le 2T}\mathbb{E}|y(t)|^2.$$

In addition, for any $\alpha_2 > 0$, we have

$$\mathbb{E}|y(t)|^2 \le (1+\alpha_2)\mathbb{E}|x(t)-y(t)|^2 + (1+1/\alpha_2)\mathbb{E}|x(t)|^2. \quad (3.13)$$

Putting $\alpha_2 = 1/\sqrt{\Delta t}$, taking the supremum over $[T, 2T]$ in (3.13) and using (2.7), we will see

$$\begin{aligned}\sup_{T\le t\le 2T}\mathbb{E}|y(t)|^2 &\le ((1+\alpha_2)C_T\Delta t + (1+1/\alpha_2))\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\\ &\le ((\Delta t+\sqrt{\Delta t})C_T + (1+\sqrt{\Delta t}))\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\\ &\le e^{\frac{1}{8}aT}\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\end{aligned}$$

for sufficiently small Δt due to the obvious fact that $e^{\frac{1}{8}aT} > 1$.

Therefore,

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}aT} \sup_{T\le t\le 2T} \mathbb{E}|x(t)|^2.$$

Now let $\hat{y}(t)$ be the solution of the SDE (1.1) for $t \in [T,\infty)$, with the initial condition that $\hat{y}(T) = x(T)$. By the similar analysis above, we have

$$\mathbb{E}|x(t)|^2 \le (1+\alpha_1)\mathbb{E}|x(t)-\hat{y}(t)|^2 + (1+1/\alpha_1)\mathbb{E}|\hat{y}(t)|^2. \tag{3.14}$$

Taking the supremum over $[T, 3T]$, and using the Markov property for the SDE, we may shift (2.1b) and condition 2.1 to $[T, 3T]$, obtaining

$$\sup_{T\le t\le 3T} \mathbb{E}|x(t)|^2 \le (1+\alpha_1) \sup_{T\le t\le 3T} \mathbb{E}|x(t)|^2 C_{2T}\Delta t + (1+1/\alpha_1)M\mathbb{E}|y(3T)|^2,$$

which gives

$$\sup_{T\le t\le 3T} \mathbb{E}|x(t)|^2 \le \frac{(1+1/\alpha_1)M\mathbb{E}|y(3T)|^2}{1-(1+\alpha_1)C_{2T}\Delta t}.$$

Now, taking the supremum over $[T, 2T]$ in (3.14), we have

$$\sup_{T\le t\le 2T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}aT} \sup_{2T\le t\le 3T} \mathbb{E}|x(t)|^2.$$

Continuing this approach gives

$$\begin{aligned}\sup_{jT\le t\le (j+1)T} \mathbb{E}|x(t)|^2 &\le e^{-\frac{1}{2}aT} \sup_{(j+1)T\le t\le (j+2)T} \mathbb{E}|x(t)|^2 \\ &\vdots \\ &\le e^{-\frac{1}{2}(i-j)aT} \sup_{iT\le t\le (i+1)T} \mathbb{E}|x(t)|^2. \end{aligned} \tag{3.15}$$

Now let $\hat{y}(s)$ be the solution of the SDE (1.1) for $s \in [t,\infty)$, with the initial condition that $\hat{y}(t) = x(t) = \xi_t$, where $iT \le t \le (i+1)T \le s \le (i+2)T$. Using the same idea above over $[iT, t]$ in (3.10) for $\alpha_1 = 1/\sqrt{\Delta t} > 0$. Note that C_T is an increasing function and using the Markov property for the SDE, we can see

$$\begin{aligned}\sup_{iT\le t\le (i+1)T} \mathbb{E}|x(t)|^2 &\le \sup_{iT\le t\le s} \mathbb{E}|x(t)|^2 \\ &\le \frac{(1+1/\alpha_1)M\mathbb{E}|\hat{y}(s)|^2}{1-(1+\alpha_1)C_{s-iT_2}\Delta t} \\ &\le \frac{(1+\sqrt{\Delta t})M\mathbb{E}|\hat{y}(s)|^2}{1-(\Delta t+\sqrt{\Delta t})C_{2T}} \\ &\le \frac{4}{3}M\mathbb{E}|\hat{y}(s)|^2\end{aligned}$$

for sufficiently small Δt. In addition, From (2.6) and (2.7) of Condition 2.1, we have

$$\begin{aligned}\mathbb{E}|\hat{y}(s)|^2 &\le (1+1/\sqrt{\Delta t})\mathbb{E}|x(s)-\hat{y}(s)|^2 + (1+\sqrt{\Delta t})\mathbb{E}|x(s)|^2 \\ &\le (\Delta t+\sqrt{\Delta t})C_{2T}B_{\xi_s,2T} + (1+\sqrt{\Delta t})\mathbb{E}|x(s)|^2 \\ &\le \frac{3}{2}\mathbb{E}|x(s)|^2\end{aligned}$$

for sufficiently small Δt. Thus we have

$$\sup_{iT \le t \le (i+1)T} \mathbb{E}|x(t)|^2 \le 2M\mathbb{E}|x(s)|^2. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}\mathbb{E}|x(t)|^2 &\le \sup_{jT \le t \le (j+1)T} \mathbb{E}|x(t)|^2 \\ &\le 2Me^{-\frac{1}{2}(i-j)aT}\mathbb{E}|x(s)|^2 \\ &\le 2Me^{-\frac{1}{2}a(s-t-T)}\mathbb{E}|x(s)|^2\end{aligned}$$

with $jT \le t \le (j+1)T \le iT \le s \le (i+1)T$. Thus (2.5b) holds for the numerical method with

$$b = \frac{1}{2}a, \quad \text{and} \quad N = 2Me^{\frac{1}{2}aT}.$$

From the analysis above, we know that, if the solution y of SDE (1.1) is EDMS, then there is a $\Delta t^* > 0$ such that for any step size $0 < \Delta t \le \Delta t^*$, the numerical solution x with the same initial data is also EDMS. □

The following lemma shows that if the numerical method admits an exponential dichotomy in mean square for small Δt, then we can infer that the underlying SDE admits an exponential dichotomy in mean square.

Lemma 3.2. *Assume that Condition* 2.1 *holds and there exists a* $\Delta t^* > 0$ *such that for any step size* $0 < \Delta t \le \Delta t^*$, *the numerical method* (1.2) *for SDE* (1.1) *is EDMS, i.e., satisfies* (2.5a) *and* (2.5b), *where* Δt^* *satisfies*

$$C_{2\hat{T}}e^{b\hat{T}}(\Delta t^* + \sqrt{\Delta t^*}) + 1 + \sqrt{\Delta t^*} \le e^{\frac{1}{8}b\hat{T}} \tag{3.17}$$

and

$$C_{2\hat{T}}(\Delta t^* + \sqrt{\Delta t^*}) + \sqrt{\Delta t^*} \le 1. \tag{3.18}$$

Then the solution of SDE (1.1) *is also EDMS with* (2.5a) *and* (2.5b) *hold, where* $a = \frac{1}{2}b$, $M = 2Ne^{\frac{1}{2}b\hat{T}}$ *and* $\hat{T} = 1 + (4\log N)/b > 0$.

Proof. Firstly, we illustrate that assumptions are reasonable. it is easy to see that the left hand side (LHS) of (3.17) is equal to 1 when $\Delta t^* = 0$ and increases monotonically with Δt^*. Hence, by taking Δt^* sufficiently small, such that for any step size $0 < \Delta t \le \Delta t^*$, we have

$$C_{2\hat{T}}e^{b\hat{T}}(\Delta t + \sqrt{\Delta t}) + 1 + \sqrt{\Delta t} \le e^{\frac{1}{8}b\hat{T}}.$$

Similarly, the LHS of (3.18) is equal to 0 when $\Delta t^* = 0$ and increases monotonically with Δt^*. For any step size $0 < \Delta t \le \Delta t^*$ with Δt^* sufficiently small, we see that

$$C_{2\hat{T}}(\Delta t + \sqrt{\Delta t}) + \sqrt{\Delta t} \le 1.$$

Given $y \in X_1$, choose $\hat{T} = 1 + (4\log N)/b > 0$, so that

$$e^{-\frac{3}{4}b\hat{T}}N \le e^{-\frac{1}{2}b\hat{T}}, \tag{3.19}$$

and for any $\alpha_1 > 0$, we have

$$\mathbb{E}|y(t)|^2 \le (1+\alpha_1)\mathbb{E}|x(t) - y(t)|^2 + (1 + 1/\alpha_1)\mathbb{E}|x(t)|^2. \tag{3.20}$$

Using (2.7) and (2.5a) over $[\hat{T}, 2\hat{T}]$ in (3.20), we see that

$$\begin{aligned}\sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 &\le (1+\alpha_1) \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|x(t)-y(t)|^2 + (1+1/\alpha_1) \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2 \\ &\le (1+\alpha_1) \sup_{0\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2 C_{2\hat{T}}\Delta t + (1+1/\alpha_1) \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2 \\ &\le (1+\alpha_1) C_{2\hat{T}}\Delta t N\mathbb{E}|x(0)|^2 + (1+1/\alpha_1) N\mathbb{E}|x(0)|^2 e^{-b\hat{T}} \\ &= \left[(1+\alpha_1) C_{2\hat{T}}\Delta t e^{b\hat{T}} + (1+1/\alpha_1)\right] N\mathbb{E}|x(0)|^2 e^{-b\hat{T}}. \end{aligned} \tag{3.21}$$

Setting $\alpha_1 = 1/\sqrt{\Delta t}$ in (3.21), we obtain

$$\sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 \le \left[C_{2\hat{T}} e^{b_1\hat{T}} (\Delta t + \sqrt{\Delta t}) + 1 + \sqrt{\Delta t}\right] N\mathbb{E}|x(0)|^2 e^{-b\hat{T}}. \tag{3.22}$$

It follows from (3.17) and (3.19) that

$$\begin{aligned}\sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 &\le e^{-\frac{3}{4}b\hat{T}} N\mathbb{E}|x(0)|^2 \\ &= e^{-\frac{3}{4}b\hat{T}} N\mathbb{E}|y(0)|^2 \\ &\le e^{-\frac{1}{2}b\hat{T}} \mathbb{E}|y(0)|^2 \\ &\le e^{-\frac{1}{2}b\hat{T}} \sup_{0\le t\le \hat{T}} \mathbb{E}|y(t)|^2. \end{aligned} \tag{3.23}$$

Now let $\hat{x}(t)$ denote the approximation that arises from applying the numerical method (1.2) with $\hat{x}(\hat{T}) = y(\hat{T})$. Then using the same analysis in (3.21)-(3.23), we can prove

$$\sup_{2\hat{T}\le t\le 3\hat{T}} \mathbb{E}|y(t)|^2 \le e^{-\frac{1}{2}b\hat{T}} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2.$$

Continuing this approach gives

$$\begin{aligned}\sup_{i\hat{T}\le t\le (i+1)\hat{T}} \mathbb{E}|y(t)|^2 &\le e^{-\frac{1}{2}b\hat{T}} \sup_{(i-1)\hat{T}\le t\le i\hat{T}} \mathbb{E}|y(t)|^2 \\ &\vdots \\ &\le e^{-\frac{1}{2}(i-j)b\hat{T}} \sup_{j\hat{T}\le t\le (j+1)\hat{T}} \mathbb{E}|y(t)|^2. \end{aligned} \tag{3.24}$$

Now let $\hat{y}(t)$ be the solution to the SDE (1.1) for $t \in [s, \infty)$, with the initial condition that $\hat{y}(s) = x(s)$, where $(j-1)\hat{T} \le s \le j\hat{T} \le t \le (j+1)\hat{T}$. Under the condition (3.18) and C_T is an increasing function in T, using the same idea over $[s, (j+1)\hat{T}]$ in (3.21)-(3.23) for $\alpha_1 = 1/\sqrt{\Delta t} > 0$ and using the Markov property for the SDE, we can see

$$\begin{aligned}\sup_{j\hat{T}\le t\le (j+1)\hat{T}} \mathbb{E}|\hat{y}(t)|^2 &\le \sup_{s\le t\le (j+1)\hat{T}} \mathbb{E}|\hat{y}(t)|^2 \\ &\le \left[C_{(j+1)\hat{T}-s}(\Delta t + \sqrt{\Delta t}) + 1 + \sqrt{\Delta t}\right] N\mathbb{E}|x(s)|^2 \\ &\le \left[C_{2\hat{T}}(\Delta t + \sqrt{\Delta t}) + 1 + \sqrt{\Delta t}\right] N\mathbb{E}|\hat{y}(s)|^2. \end{aligned}$$

Thus we have

$$\sup_{j\hat{T}\le t\le (j+1)\hat{T}} \mathbb{E}|\hat{y}(t)|^2 \le 2N\mathbb{E}|\hat{y}(s)|^2. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\begin{aligned}\mathbb{E}|y(t)|^2 &\le \sup_{i\hat{T}\le t\le (i+1)\hat{T}} \mathbb{E}|y(t)|^2 \\ &\le 2Ne^{-\frac{1}{2}(i-j)b\hat{T}}\mathbb{E}|y(s)|^2 \\ &\le 2Ne^{-\frac{1}{2}b(t-s-\hat{T})}\mathbb{E}|y(s)|^2\end{aligned}$$

with $j\hat{T} \le s \le (j+1)\hat{T} \le i\hat{T} \le t \le (i+1)\hat{T}$. Then (2.1a) holds with

$$a = \frac{1}{2}b, \quad \text{and} \quad M = 2Ne^{\frac{1}{2}b\hat{T}}.$$

Now we consider $y \in X_2$, choose $\hat{T} = 1 + (4\log N)/b > 0$, so that

$$e^{-\frac{7}{8}b\hat{T}}N \le e^{-\frac{5}{8}b\hat{T}}. \tag{3.26}$$

Using (2.7) and (2.5b) over $[0, \hat{T}]$ in (3.20), we see that

$$\begin{aligned}\sup_{0\le t\le \hat{T}} \mathbb{E}|y(t)|^2 &\le (1+\alpha_1) \sup_{0\le t\le \hat{T}} \mathbb{E}|x(t)-y(t)|^2 + (1+1/\alpha_1) \sup_{0\le t\le \hat{T}} \mathbb{E}|x(t)|^2 \\ &\le (1+\alpha_1) \sup_{0\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2 C_{2\hat{T}}\Delta t + (1+1/\alpha_1) \sup_{0\le t\le \hat{T}} \mathbb{E}|x(t)|^2. \\ &\le (1+\alpha_1)C_{2\hat{T}}\Delta t N\mathbb{E}|x(2\hat{T})|^2 + (1+1/\alpha_1)N\mathbb{E}|x(2\hat{T})|^2 e^{-b\hat{T}} \\ &= \left[(1+\alpha_1)C_{2\hat{T}}\Delta t e^{b\hat{T}} + (1+1/\alpha_1)\right] N\mathbb{E}|x(2\hat{T})|^2 e^{-b\hat{T}}.\end{aligned} \tag{3.27}$$

Letting $\alpha_1 = 1/\sqrt{\Delta t}$ in (3.27), we obtain

$$\sup_{0\le t\le \hat{T}} \mathbb{E}|y(t)|^2 \le \left[C_{2\hat{T}}e^{b\hat{T}}(\Delta t + \sqrt{\Delta t}) + 1 + \sqrt{\Delta t}\right] N\mathbb{E}|x(2\hat{T})|^2 e^{-b\hat{T}}.$$

It follows from (3.17) and (3.26) that

$$\begin{aligned}\sup_{0\le t\le \hat{T}} \mathbb{E}|y(t)|^2 &\le e^{-\frac{7}{8}b\hat{T}}N\mathbb{E}|x(2\hat{T})|^2 \\ &\le e^{-\frac{5}{8}b\hat{T}} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2.\end{aligned} \tag{3.28}$$

In addition, for any $\alpha_2 > 0$, we have

$$\mathbb{E}|x(t)|^2 \le (1+\alpha_2)\mathbb{E}|x(t)-y(t)|^2 + (1+1/\alpha_2)\mathbb{E}|y(t)|^2. \tag{3.29}$$

Putting $\alpha_2 = 1/\sqrt{\Delta t}$ and taking the supremum over $[\hat{T}, 2\hat{T}]$ in (3.29) and using (2.7), we will see

$$\begin{aligned}\sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|x(t)|^2 &\le \frac{1+1/\alpha_2}{1-(1+\alpha_2)C_{\hat{T}}\Delta t} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 \\ &= \frac{1+\sqrt{\Delta t}}{1-(\Delta t+\sqrt{\Delta t})C_{\hat{T}}} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 \\ &\le e^{\frac{1}{8}b_2\hat{T}} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2\end{aligned}$$

for sufficiently small Δt. Combined the above inequality with (3.28), we obtain

$$\sup_{0\le t\le \hat{T}} \mathbb{E}|y(t)|^2 \le e^{-\frac{1}{2}b_2\hat{T}} \sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2. \tag{3.30}$$

Now let $\hat{x}(t)$ denote the approximation that arises from applying the numerical method (1.2) with $\hat{x}(\hat{T}) = y(\hat{T})$. Using the same analysis as in (3.28)-(3.30), we can prove

$$\sup_{\hat{T}\le t\le 2\hat{T}} \mathbb{E}|y(t)|^2 \le e^{-\frac{1}{2}b\hat{T}} \sup_{2\hat{T}\le t\le 3\hat{T}} \mathbb{E}|y(t)|^2.$$

Continuing this approach gives

$$\begin{aligned}\sup_{j\hat{T}\le t\le (j+1)\hat{T}} \mathbb{E}|y(t)|^2 &\le e^{-\frac{1}{2}b\hat{T}} \sup_{(j+1)\hat{T}\le t\le (j+2)\hat{T}} \mathbb{E}|y(t)|^2\\ &\vdots\\ &\le e^{-\frac{1}{2}(i-j)b\hat{T}} \sup_{i\hat{T}\le t\le (i+1)\hat{T}} \mathbb{E}|y(t)|^2. \end{aligned}\tag{3.31}$$

Now let $\hat{y}(s)$ be the solution to the SDE (1.1) for $s \in [t,\infty)$, with the initial condition that $\hat{y}(t) = x(t)$, where $i\hat{T} \le t \le (i+1)\hat{T} \le s \le (i+2)\hat{T}$. Under the condition (3.18) and C_T is an increasing function in T, using the same idea over $[i\hat{T}, s]$ in (3.27) for $\alpha_1 = 1/\sqrt{\Delta t} > 0$ and using the Markov property for the SDE, we can see

$$\begin{aligned}\sup_{i\hat{T}\le t\le (i+1)\hat{T}} \mathbb{E}|\hat{y}(t)|^2 &\le \sup_{i\hat{T}\le t\le s} \mathbb{E}|\hat{y}(t)|^2\\ &\le [(1+\alpha_1)C_{s-i\hat{T}}\Delta t + (1+1/\alpha_1)]N\mathbb{E}|x(s)|^2\\ &\le [(\Delta t+\sqrt{\Delta t})C_{2\hat{T}} + (1+\Delta t)]N\mathbb{E}|x(s)|^2\\ &\le \frac{3}{2}N\mathbb{E}|x(s)|^2\end{aligned}$$

for sufficiently small Δt. In addition, From (2.6) and (2.7) of Condition 2.1, we have

$$\begin{aligned}\mathbb{E}|x(s)|^2 &\le (1+1/\sqrt{\Delta t})\mathbb{E}|x(s)-\hat{y}(s)|^2 + (1+\sqrt{\Delta t})\mathbb{E}|\hat{y}(s)|^2\\ &\le (\Delta t+\sqrt{\Delta t})C_{2\hat{T}}B_{\xi_s,2\hat{T}} + (1+\sqrt{\Delta t})\mathbb{E}|\hat{y}(s)|^2\\ &\le \frac{4}{3}\mathbb{E}|\hat{y}(s)|^2,\end{aligned}$$

for sufficiently small Δt. Thus we have

$$\sup_{i\hat{T}\le t\le (i+1)\hat{T}} \mathbb{E}|\hat{y}(t)|^2 \le 2N\mathbb{E}|\hat{y}(s)|^2. \tag{3.32}$$

It follows from (3.31) and (3.32) that

$$\begin{aligned}\mathbb{E}|y(t)|^2 &\le \sup_{j\hat{T}\le t\le (j+1)\hat{T}} \mathbb{E}|y(t)|^2\\ &\le 2Ne^{-\frac{1}{2}(i-j)b\hat{T}}\mathbb{E}|y(s)|^2\\ &\le 2Ne^{-\frac{1}{2}b(s-t-\hat{T})}\mathbb{E}|y(s)|^2\end{aligned}$$

with $j\hat{T} \le t \le (j+1)\hat{T} \le i\hat{T} \le s \le (i+1)\hat{T}$. Then (2.1b) holds with

$$a = \frac{1}{2}b, \quad \text{and} \quad M = 2Ne^{\frac{1}{2}b\hat{T}}.$$

From the analysis above, we know that, if there exists a $\Delta t^* > 0$ such that for any step size $0 < \Delta t < \Delta t^*$, the numerical solution x for SDE (1.1) is EDMS, then the solution y of the SDE (1.1) with the same initial data is also EDMS. □

Proof of the Theorem 3.1. Let $\Delta t^* = \min\{T, \hat{T}\}$, it is easy to see that Theorem 3.1 hold, and the proof is complete. □

As a special case of exponential dichotomies in mean-square, next we consider the exponential contractions in mean-square.

Definition 3.1. The solution $y(t) \in L^2_{\mathscr{F}_t}(\Omega, \mathbb{R}^n)$ with initial data $\xi_s \in L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$ of SDE (1.1) is said to be exponential contraction in mean-square (ECMS) if there exist positive constants M and a such that

$$\mathbb{E}|y(t)|^2 \le Me^{-a(t-s)}\mathbb{E}|y(s)|^2, \quad \forall\, t \ge s \ge 0,\ \forall\, \xi_s \in \mathbb{R}^n.$$

Definition 3.2. For a given step size Δt, a numerical method is said to be ECMS on the SDE (1.1) if there exist positive constants N and b such that with initial data $\xi_s \in L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$,

$$\mathbb{E}|x(t)|^2 \le Ne^{-b(t-s)}\mathbb{E}|x(s)|^2, \quad \forall t \ge s \ge 0,\ \forall \xi_s \in \mathbb{R}^n.$$

The following result is a direct consequence of Theorem 3.1.

Theorem 3.2. *Suppose that a numerical method satisfies Condition* 2.1*. Then the solution of SDE* (1.1) *is ECMS if and only if there exists a* $\Delta t^* > 0$ *such that for any step size* $0 < \Delta t \le \Delta t^*$*, the numerical method is ECMS.*

Finally in this section, we present one example to illustrate the equivalence of ECMS between the numerical simulation and the solution of SDE.

Example 3.1. (Langevin's equation(see e.g., [4,5] for details))Consider the following equation

$$dX = -bXdt + \sigma dW, \tag{3.33}$$

with initial data $X(0)$, independent of the Brownian motion, where $b > 0$ is a coefficient of friction, and σ is a diffusion coefficient. From Example 2.1, it is easy to verify that the solution of Langevin's equation satisfies

$$\mathbb{E}|X(t)|^2 \le M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2, \quad \forall t \ge s \ge 0,$$

where $M_1 = 1 + \frac{\sigma^2 e^{2bT}}{2b\mathbb{E}|X(0)|^2} > 0$, which means that the solution of SDE (3.33) is ECMS.

Now we apply the (EM) method (1.3) to the linear SDE (3.33). In Figure 1, we consider (3.33) with $b = 2$, $\sigma = 0.2$, $X(0) = 1$, and let a solid blue line denote the mean of 10000 paths in mean square. We can see in Figure 1 that although $X(t)$ is nonsmooth along individual paths, its sample average appears to be smooth. In addition, we can compute the maximum *discrepancy* between the sample average and the exact expected value over all points t_i, i.e.,

$$\text{discr=norm(mean}(X^2) - (e^{-2bt}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2b}(1 - e^{-2bt})), \quad 'inf').$$

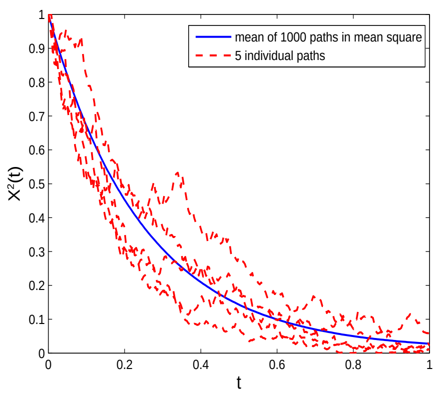

Figure 1. The mean function averaged over 10000 discretized paths in mean square and along 5 individual paths.

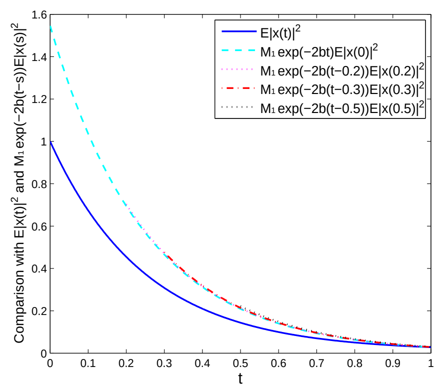

Figure 2. Comparison with $\mathbb{E}|X(t)|^2$ and $M_1 e^{-2b(t-s)}\mathbb{E}|y(s)|^2$ with s being chosen from 0, 0.2, 0.3 and 0.5 respectively.

Table 1. Calculation of $min(s)$

s	0	0.2	0.3	0.5
min(s)	1.8316e-004	5.3018e-004	8.4011e-004	0.0020

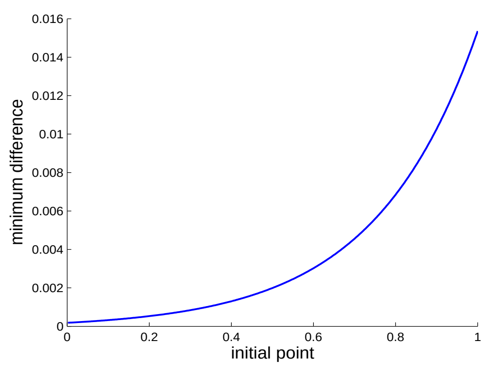

Figure 3. Minimum difference between $M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2$ starting from s and $\mathbb{E}|X(t)|^2$.

We can compute that $discr = 0.0027$. Increasing the number of samples to 50000 reduces $discr$ to 0.0011. In Figure 2, we compare $\mathbb{E}|X(t)|^2$ with $M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2$, where $M_1 = 1 + \frac{\sigma^2 e^{2bT}}{2b\mathbb{E}|X(0)|^2}$ and s is chosen as 0, 0.2, 0.3 and 0.5. To illustrate the numerical solution of (3.33) is ECMS, let

$$\min(s) = \min(M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2 - \mathbb{E}|X(t)|^2)$$

denote the minimum difference between $M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2$ starting from s and $\mathbb{E}|X(t)|^2$. If we can compute that the numerical simulation $\mathbb{E}|X(t)|^2$ is located below $M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2$, that is, $min(s) > 0$, then we can obtain the numerical solution of (3.33) is ECMS. Table 1 is based on the calculation of $min(s)$. Further, we compare the minimum difference between $M_1 e^{-2b(t-s)}\mathbb{E}|X(s)|^2$ starting from s and $\mathbb{E}|X(t)|^2$ in Figure 3, where s is chosen over all discrete points t_i in the interval $[0, 1]$.

From the analysis and computation above, we know that the ECMS of numerical method (1.2) is equivalent to the ECMS of SDE (3.33).

References

[1] L. Arnold, *Stochastic Differential Equations: theory and applications*, New York, 1974.

[2] L. Barreira and C. Valls, *Stability of Nonautonomous Differential Equations*, Lecture Notes in Mathematics, vol. 1926, Springer, 2008.

[3] W. A. Coppel, *Dichotomy in Stability Theory,* Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.

[4] J. M. DeLaurentis and B. A. Boughton, *An asymptotic analysis of a generalized Langevin equation*, Stochastic Process. Appl., 33(1989), 275–284.

[5] L. C. Evans, *An Introduction to Stochastic Differential Equations*, Amer. Math. Soc., 2012.

[6] A. Friedman, *Stochastic differential equations and applications*, Stochastic d-ifferential equations, 75–148, C.I.M.E. Summer Sch. 77, Springer, Heidelberg, 2010.

[7] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems*, 2nd ed., Springer-Verlag, Berlin, 1996.

[8] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numerical Anal., 38(2000), 753–769.

[9] D. J. Higham,*An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Rev., 43(2001), 525–546.

[10] D. J. Higham, X. Mao and A. M. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations,* SIAM J. Numer. Anal., 40(2002), 1041–1063

[11] D. J. Higham, X. Mao and A. M. Stuart, *Exponential mean-square stability of numerical solutions to stochastic differential equations*, LMS J. Comput. Math., 6(2003), 297–313.

[12] D. J. Higham, X. Mao and C. G. Yuan, *Preserving exponential mean-square stability in the simulation of hybrid stochastic differential equations*, Numer. Math., 108(2007), 295–325.

[13] P. E. Kloeden and T. Lorenz, *Mean-square random dynamical systems*, J. Differential Equations, 253(2012), 1422–1438.

[14] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.

[15] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, 1997.

[16] J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces,* in: Pure and Applied Mathematics, vol. 21, Academic Press, 1966.

[17] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z., 32(1930), 703–728.

[18] Y. Saito and T. Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33(1996), 2254–2267.

[19] H. Schurz, *Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise,* Stochastic Anal. Appl., 14(1996), 313–354.

[20] D. Stoica, *Uniform exponential dichotomy of stochastic cocycles*, Stochastic Process. Appl., 120(2010), 1920–1928.

[21] D. Stoica and M. Megan, *On nonuniform dichotomy for stochastic skew-evolution semiflows in Hilbert spaces*, Czechoslovak Math. J., 62(2012)(137), 879–887.