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MORE RESULTS ON HERMITE-HADAMARD TYPE INEQUALITY THROUGH (α, m)-PREINVEXITY

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Abstract We establish various inequalities for n-times differentiable mappings that are connected with illustrious Hermite-Hadamard integral inequality for mapping whose absolute values of derivatives are (α, m)-preinvex function. The new integral inequalities are then applied to some special means.

Keywords Hermite-Hadamard type inequality, preinvex function, invex set, power-mean inequality, Holder's integral inequality, (α, m)-preinvex function.

MSC(2010) 26D07, 26D10, 26D99.

1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamarad inequality, due to its rich geometrical significance and applications, which is stated in [17] as:
Let $f : I \subset R \to R$ be a convex function defined on the interval $I$of real numbers and $a, b \in I$, with $a < b$. Then f satisfies the following well-known Hermite Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In many areas of analysis, applications of Hermite-Hadamard inequality appeared for different classes of functions with and without weights; see for convex functions [3,5,6,15,16,18–20,27]. In recent years, the classical convexity has been generalized and extended in a diverse manner. One of them is the preinvexity, introduced by Weir et al. [27]as a significant generalization of convex function. Many researchers have studied the basic properties of the preinvex function and their role in optimization theory, variational inequalities and equilibrium problems. Let us recall some definitions and known results concerning invexity and preinvexity.

Definition 1.1 ([29]). A set $K \subseteq R^n$ is said to be invex with respect to $\eta : K \times K \to R^n$, if

$$x + t\eta(y, x) \in K, \quad \forall x, y \in K, \quad t \in [0, 1]. \tag{1.2}$$

The invex set K is also called a η-connected set.

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Definition 1.2 ([27]). Let $K \subseteq R$ be an invex set with respect to $\eta : K \times K \to R^n$. A function $f : K \to R$ is said to be preinvex with respect to η, if

$$f\left(x + t\eta\left(y, x\right)\right) \leq \left(1 - t\right) f\left(x\right) + tf\left(y\right), \forall x, y \in K, t \in \left[0, 1\right]. \tag{1.3}$$

The function f is said to be preconcave if and only if $-f$ is preinvex. It is to be noted that every convex function is preinvex with respect to the map $\eta\left(x, y\right) = x - y$, but the converse is not true.

Noor [19], established the following Hermite-Hadamard's inequality utilizing preinvex function which follows as:

Theorem 1.1 ([19]). *Let $f : [a, a + \eta\left(b, a\right)] \to (0, \infty)$ be an open preinvex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + \eta\left(b, a\right)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta\left(b, a\right)}{2}\right) \leq \frac{1}{\eta\left(b, a\right)} \int_{a}^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{1.4}$$

In similar manner to Noor methodology, inequalities for differentiable convex mappings associated with the right-hand side of Hermite-Hadamard's inequality was verified by Barani, Ghazanfari and Dragomir, by means of the following illustration:

Theorem 1.2 ([2]). *Let $f : [a, a + \eta\left(b, a\right)] \to (0, \infty)$ be an open preinvex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K^0$ with $a < a + \eta\left(b, a\right)$. Then the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta\left(b, a\right))}{2} - \frac{1}{\eta\left(b, a\right)} \int_{a}^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta\left(b, a\right)}{8} \left\{\left|f'\left(a\right)\right| + \left|f'\left(b\right)\right|\right\}. \tag{1.5}$$

Theorem 1.3 ([2]). *Let $K \subseteq R$ be an open invex subset with respect to $\eta : K \times K \to R$. Suppose that $f : K \to R$ is a differentiable function. Assume $p \in R$ with $p > 1$. If $|f'|^{\frac{p}{(p-1)}}$ is preinvex on K, then for every $a, b \in K$ with $\eta\left(b, a\right) \neq 0$, the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta\left(b, a\right))}{2} - \frac{1}{\eta\left(b, a\right)} \int_{a}^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta\left(b, a\right)}{2\left(1 + p\right)^{1/p}} \left\{\frac{\left|f'\left(a\right)\right|^{\frac{p}{(p-1)}} + \left|f'\left(b\right)\right|^{\frac{p}{(p-1)}}}{2}\right\}^{\frac{p-1}{p}}. \tag{1.6}$$

Recently, much attention has been given to theory of convex functions by many researchers. Consequently the classical concept of convex functions has been extended and generalized in different directions using various novel ideas, readers are directed to [8–14, 21–24]. In this paper we establish various inequalities for n-times differentiable mappings that are connected with illustrious Hermite-Hadamard integral inequality for mapping whose absolute values of derivatives are (α, m)-preinvex function. The new integral inequalities are then applied to some special means.

2. Main results

The following essential definitions and lemmas play a key role to establish our main results:

Definition 2.1 ([3]). Let $K \subseteq R$ be an invex set with respect to $\eta : K \times K \to R^n$. Suppose that $f : K \to R$ is said to be (α, m)-preinvex with respect to η, if for all $x, y \in K, t \in [0,1]$ and $(\alpha, m) \in (0,1] \times (0,1]$,

$$f\left(x + t\eta\left(y, x\right)\right) \leq +\left(1 - t^{\alpha}\right) f\left(x\right) + mt^{\alpha} f\left(\frac{y}{m}\right).$$

The function f is said to be (α, m)-preconcave if and only if $-f$ is (α, m)-preinvex.

Lemma 2.1. *Let $I \subseteq R$ be an open invex subset with respect to $\eta : I \times I \to R_+$. Suppose $f : I \to R$ is a function such that $f^{(n)}$ exist on I, for $n \in N$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, then for every $a, b \in I$ with $\eta(b,a) > 0$, the following inequality holds:*

$$\begin{aligned}
&-\frac{f(a) + f(a + \eta(b,a))}{2} \\
&-\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a)) \\
=&\frac{(-1)^n (\eta(b,a))^n}{2n!} \int_0^1 \lambda^{n-1}(n - 2\lambda)\left(f^{(n)}(a + \lambda\eta(b,a))\right) d\lambda. \qquad (2.1)
\end{aligned}$$

Lemma 2.2. *Let $I \subseteq R$ be an open invex subset with respect to $\eta : I \times I \to R_+$. Suppose $f : I \to R$ is a function such thatf$^{(n)}$ exist on I, for $n \in N$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, then for every $a, b \in I$ with $\eta(b,a) > 0$, the following inequality holds:*

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\
=&\frac{(-1)^{n+1}(\eta(b,a))^n}{n!} \int_0^1 P_n(\lambda)\left(f^{(n)}(a + \lambda\eta(b,a))\right) d\lambda, \qquad (2.2)
\end{aligned}$$

where $P_n(\lambda) = \begin{cases} \lambda^n, & \lambda \in [0, \frac{1}{2}], \\ (\lambda - 1)^n, & \lambda \in [\frac{1}{2}, 1]. \end{cases}$

Now we are in position to establish our first result for functions whose nth derivatives in absolute values are (α, m)-preinvex.

Theorem 2.1. *Let f be defined as in Lemma 2.1. If $\left|f^{(n)}\right|^q$ for $q \geq 1$, is (α, m)-preinvex on I, for $n \in N$ with $n \geq 2$, then for every $a, b \in I$ with $\eta(b,a) > 0$ and* *for some* $(\alpha, m) \in (0,1]^2$, *we have the following inequality*

$$\begin{aligned} &\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx - \sum_{k=2}^{n-1} \tfrac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}\left(a+\eta\left(b,a\right)\right)\right| \\ &\le \tfrac{(\eta(b,a))^n}{2n!} \left(\tfrac{n-1}{n+1}\right)^{1-1/q} \left[U_2 \left|f^{(n)}\left(a\right)\right|^q + mU_1 \left|f^{(n)}\left(\tfrac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{2.3}$$

where $U_1 = \frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}$ *and* $U_2 = \frac{n\alpha(n+\alpha)-\alpha(\alpha+1)}{(n+1)(n+\alpha)(n+\alpha+1)}$.

Proof. By using Lemma 2.1 and (α, m)-preinvexity of $\left|f^{(n)}\right|$, we have

$$\begin{aligned} &\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx - \sum_{k=2}^{n-1} \tfrac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right| \\ &\le \frac{(\eta\left(b,a\right))^n}{2n!} \int\limits_0^1 P_n(\lambda) \left|f^{(n)}\left(a+\lambda\eta\left(b,a\right)\right)\right| d\lambda \\ &= \frac{(\eta\left(b,a\right))^n}{2n!} \int\limits_0^1 \lambda^{n-1}\left(n-2\lambda\right) \left\{\left(1-\lambda^\alpha\right)\left|f^{(n)}\left(a\right)\right| + m\lambda^\alpha \left|f^{(n)}\left(\frac{b}{m}\right)\right|\right\} d\lambda. \end{aligned}$$

By simple calculations, we have

$$\int\limits_0^1 \lambda^{n+\alpha+1}\left(n-2\lambda\right) d\lambda = \frac{n\left(n-1\right)+\alpha\left(n-2\right)}{\left(n+\alpha\right)\left(n+\alpha+1\right)}, \tag{2.4}$$

$$\int\limits_0^1 \lambda^{n-1}\left(n-2\lambda\right)\left(1-\lambda^\alpha\right) d\lambda = \frac{\left(n+\alpha+1\right)\left(n\alpha-\alpha\right)+2\alpha}{\left(n+1\right)\left(n+\alpha\right)\left(n+\alpha+1\right)}. \tag{2.5}$$

Combining the above inequalities (2.4), and (2.5), we obtain (2.3). This completes the proof. □

Corollary 2.1. *If* $n = 2$, *in Theorem 2.1, then we have the following inequality:*

$$\begin{aligned} &\left|\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2} - \frac{1}{\eta\left(b,a\right)} \int\limits_a^{a+\eta(b,a)} f(x)dx\right| \\ &\le \frac{(\eta\left(b,a\right))^n}{2} \left(\frac{1}{3}\right)^{1-1/q} \left[\frac{2}{(\alpha+2)(\alpha+3} \left|f^{('')}\left(a\right)\right|^q + \frac{\alpha}{3(\alpha+2)} m \left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

Theorem 2.2. *Let* f *be defined as in Lemma 2.1. If* $\left|f^{(n)}\right|^q$, *for* $q \ge 1$ *is* $(\alpha, m)-$ *preinvex on* I, *for* $n \in N$ *with* $n \ge 2$, *then for every* $a, b \in I$ *with* $\eta\left(b,a\right) > 0$ *and for some* $(\alpha, m) \in (0,1]^2$, *we have the following inequality:*

$$\begin{aligned} &\left|\tfrac{f(a)+f(a+\eta(b,a))}{2} - \tfrac{1}{\eta(b,a)} \int\limits_a^{a+\eta(b,a)} f(x)dx - \sum_{k=2}^{n-1} \tfrac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}\left(a+\eta\left(b,a\right)\right)\right| \\ &\le \tfrac{(\eta(b,a))^n}{2n!} \left(n-1\right)^{1-1/q} \left[U_4 \left|f^{(n)}\left(a\right)\right|^q + mU_3 \left|f^{(n)}\left(\tfrac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}, \end{aligned} \tag{2.7}$$

where $U_3 = \left(\frac{2}{q+1} - \frac{2}{q+\alpha+1} - U_4\right)$ *and* $U_4 = \left(\frac{2}{q+\alpha+1} - \frac{2}{q+\alpha+2}\right)$.

Proof. By using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right. \\
&\left.-\sum_{k=2}^{n-1}\frac{(-1)^{k}(k-1)(\eta(b,a))^{k}}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right| \\
\leq&\frac{(\eta(b,a))^{n}}{2n!}\left(\int_{0}^{1}(n-2\lambda)\right)^{1-1/q} \\
&\times\int_{0}^{1}\lambda^{q(n-1)}(n-2\lambda)\left((1-\lambda^{\alpha})\left|f^{(n)}(a)\right|^{q}+m\lambda^{\alpha}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}\right)d\lambda \\
=&\frac{(\eta(b,a))^{n}}{2n!}(n-1)^{1-1/q}\left[U_3\left|f^{(n)}(a)\right|^{q}+mU_4\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned}$$

Using the convexity of $|f'|$, we have

$$\int_{0}^{1}\lambda^{n+\alpha+1}(n-2\lambda)(1-\lambda^{\alpha})d\lambda=\left(\frac{n}{nq-q+1}-\frac{2}{nq-q+\alpha+1}-U_4\right), \quad (2.8)$$

$$\int_{0}^{1}\lambda^{q(n-1)}(n-2\lambda)\lambda^{\alpha}d\lambda=\left(\frac{n}{nq-q+\alpha+1}-\frac{2}{nq-q+\alpha+2}\right). \quad (2.9)$$

Combing the above inequalities (2.8), and (2.9), we obtain (2.7). This completes the proof. □

Corollary 2.2. *If $n = 2$ in Theorem 2.2, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right| \\
\leq&\frac{(\eta(b,a))^{2}}{2^{2-1/q}}\left[U_3\left|f''(a)\right|^{q}+mU_4\left|f''\left(\frac{b}{m}\right)\right|^{q}\right]^{\frac{1}{q}},
\end{aligned}$$

where $U_3=\left(\frac{2}{q+1}-\frac{2}{q+\alpha+1}-U_4\right)$ *and* $U_4=\left(\frac{2}{q+\alpha+1}-\frac{2}{q+\alpha+2}\right)$.

Corollary 2.3. *If we take $q = 1$, $\alpha = 1$ and $m = 1$ in Corollary 2.2 we get,*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right|\leq\frac{(\eta(b,a))^{2}}{12}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Theorem 2.3. *Let $I \subseteq R$ be an open invex subset with respect to $\eta : I \times I \to R_+$ and suppose $f : I \to R$ is a function on I,with $\eta(b, a) > 0$ and $f^{(n)}$ is integrable on*

$[a, a+\eta(b,a)]$, for $n \in N$ with $n \geq 2$. If $\left|f^{(n)}\right|^q$, for $q \geq 1$, is (α, m) −preinvex on I, for $n \in N$ with ≥ 2, the following inequality holds:

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$

$$\leq \frac{(\eta(b,a))^n}{2^n n!\,(np+1)} \left[\frac{\alpha\left|f^{(n)}(a)\right|^q + m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}. \tag{2.10}$$

Proof. Using Lemma 2.1 and (α, m)-preinvexity of $\left|f^{(n)}\right|^q$,we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$

$$\leq \frac{(\eta(b,a))^n}{n!} \int_0^1 |P_n(\lambda)| \left|f^{(n)}(a+\lambda\eta(b,a))\right| d\lambda$$

$$\leq \frac{(\eta(b,a))^n}{n!} \left(\int_0^1 |P_n(\lambda)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(n)}(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{\frac{1}{q}}$$

$$\leq \frac{(\eta(b,a))^n}{n!} \left(\int_0^1 |P_n(\lambda)|^p\right)^{\frac{1}{p}} \left[\int_0^1 (1-\lambda^\alpha)\left|f^{(n)}(a)\right| + m\int_0^1 \lambda^\alpha \left|f^{(n)}\left(\frac{b}{m}\right)\right|\right] d\lambda.$$

Using the (α, m)-preinvexity of $|f^n|$, we have

$$\int_0^1 |P_n(\lambda)|^p = \int_0^{1/2} \lambda^{np} d\lambda + \int_{1/2}^1 (1-\lambda)^{np}\,d\lambda = \frac{1}{2^{np}(np+1)}, \tag{2.11}$$

$$\int_0^1 \left|f^{(n)}(a+\lambda\eta(b,a))\right|^q d\lambda \leq \int_0^1 (1-\lambda^\alpha)\left|f^{(n)}(a)\right| + m\int_0^1 \lambda^\alpha \left|f^{(n)}\left(\frac{b}{m}\right)\right|$$

$$= \frac{\alpha\left|f^{(n)}(a)\right|^q + m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q}{\alpha+1}. \tag{2.12}$$

Combing the above inequalities (2.11), and (2.12), we obtain (2.10). This completes the proof. □

Corollary 2.4. *If $n = 2$, $\alpha = 1$ and $m = 1$, in Theorem 2.3, then we have the following inequality:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$

$$\leq \frac{(\eta(b,a))^2}{8(2p+1)} \left[\frac{|f''(a)|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{3}\right]^{\frac{1}{q}}. \tag{2.13}$$

Theorem 2.4. *Let $I \subseteq R$ be an open invex subset with respect to $\eta : I \times I \to R_+$. Suppose $f : I \to R$ is a function on I, with $\eta(b,a) > 0$ and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$, for $n \in N$ with $n \geq 1$. If $\left|f^{(n)}\right|^q$, for $q > 1$ is (α, m)-preinvex on I, for $n \in N$ with $n \geq 1$, then following inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{2^n n!}\left(\frac{1}{2^{np+1}(np+1)}\right)^{1/p}\left[\begin{array}{c}\left(V_1\left|f^{(n)}(a)\right|^q + mV_2\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{1/q} \\ +\left(V_3\left|f^{(n)}(a)\right|^q + mV_4\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\end{array}\right], \tag{2.14}$$

where

$$V_1 = \frac{2^\alpha(\alpha+1) - 1}{(\alpha+1)2^{\alpha+1}}, \qquad V_2 = \frac{1}{(\alpha+1)2^{\alpha+1}},$$
$$V_3 = \frac{\alpha.2^{\alpha+1} - 2^\alpha(\alpha+1) + 1}{(\alpha+1)2^{\alpha+1}}, \qquad V_4 = \frac{\left(2^{\alpha+1} - 1\right)}{(\alpha+1)2^{\alpha+1}}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Power Mean inequality and by Lemma 2.2, we get

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{n!}\left(\int_0^{1/2} \lambda^{np} d\lambda\right)^{1/p}\left(\int_0^{1/2} f^{(n)}\left|(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}$$
$$+\frac{(\eta(b,a))^n}{n!}\left(\int_{1/2}^{1} (1-\lambda)^{np} d\lambda\right)^{1/p}\left(\int_{1/2}^{1} f^{(n)}\left|(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}.$$

Also the (α, m)-preinvexity of $\left|f^{(n)}\right|^q$ implies that

$$\int_0^{1/2} \lambda^{np} d\lambda = \int_{1/2}^{1} (1-\lambda)^{np} d\lambda = \frac{1}{2^{np}(np+1)}, \tag{2.15}$$

$$\int_0^{1/2} f^{(n)}\left|(a+\lambda\eta(b,a))\right|^q d\lambda \leq \int_0^{1/2}\left[(1-\lambda^\alpha)\left|f^{(n)}(a)\right|^q + m\lambda^\alpha\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right] d\lambda$$
$$= V_1\left|f^{(n)}(a)\right|^q + mV_2\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q, \tag{2.16}$$

$$\int_{1/2}^{1} f^{(n)} \left|(a+\lambda\eta(b,a))\right|^q d\lambda \leq \int_{1/2}^{1} \left[(1-\lambda^\alpha)\left|f^{(n)}(a)\right|^q + m\lambda^\alpha \left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right] d\lambda$$
$$= V_3 \left|f^{(n)}(a)\right|^q + mV_4 \left|f^{(n)}\left(\frac{b}{m}\right)\right|^q. \tag{2.17}$$

Combing the above inequalities (2.15), (2.16), and (2.17), we obtain (2.14). This completes the proof. □

Corollary 2.5. *If $\alpha = 1$, $m = 1$ and $n = 2$ in Theorem 2.4, then we have the following inequality:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{8}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{1/p}\left[\begin{array}{l}\left(\frac{3}{8}\left|f''(a)\right|^q + \frac{1}{8}\left|f''(b)\right|^q\right)^{1/q}\\ +\left(\frac{1}{8}\left|f''(a)\right|^q + \frac{3}{8}\left|f''(b)\right|^q\right)^{1/q}\end{array}\right].$$

Theorem 2.5. *Let $I \subseteq R$ be an open invex subset with respect to $\eta : I \times I \to R_+$. Suppose $f : I \to R$ is a function on I, with $\eta(b,a) > 0$, and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$, for $n \in N$ with $n \geq 1$. If $\left|f^{(n)}\right|^q$ for $q \geq 1$, is (α, m) −preinvex on I, for $n \in N$ with $n \geq 1$, then the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{2^n n!}\left(\frac{1}{2^{np+1}(np+1)}\right)^{1/p}\left[\begin{array}{l}\left(V_1\left|f^{(n)}(a)\right|^q + mV_2\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\\ +\left(V_3\left|f^{(n)}(a)\right|^q + mV_4\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right)^{1/q}\end{array}\right], \tag{2.18}$$

where

$$D = \left[\frac{1}{(\alpha+1)2^{\alpha+1}} - E\right], \qquad E = \frac{1}{(\alpha+n+1)2^{\alpha+n+1}},$$
$$F = \left[\frac{1}{(\alpha+1)2^{\alpha+1}} - G\right], \qquad G = \left[B(\alpha+1, n+1) - B\left(\frac{1}{2}; \alpha+1, n+1\right)\right],$$
$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1},$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Holder's inequality and by Lemma 2.2, we get

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n}{n!}\left(\int_0^{1/2}\lambda^n d\lambda\right)^{1-1/q}\left(\int_0^{1/2}f^{(n)}\left|(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}$$
$$+\frac{(\eta(b,a))^n}{n!}\left(\int_{1/2}^{1}(1-\lambda)^n d\lambda\right)^{1-1/q}\left(\int_{1/2}^{1}f^{(n)}\left|(a+\lambda\eta(b,a))\right|^q d\lambda\right)^{1/q}.$$

Also the (α, m)-preinvexity of $\left|f^{(n)}\right|^q$ implies that

$$\int_0^{1/2}\lambda^{np}d\lambda=\int_{1/2}^{1}(1-\lambda)^{np}\,d\lambda=\frac{1}{2^{np}(np+1)},\tag{2.19}$$

$$\int_0^{1/2}\lambda^n\left|f^{(n)}(a+\lambda\eta(b,a))\right|^q d\lambda$$
$$\leq\int_0^{1/2}\left[\lambda^n(1-\lambda^\alpha)\left|f^{(n)}(a)\right|^q+m\lambda^{\alpha+n}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]d\lambda$$
$$=D\left|f^{(n)}(a)\right|^q+E\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q,\tag{2.20}$$

$$\int_{1/2}^{1}(1-\lambda)^n\left|f^{(n)}(a+\lambda\eta(b,a))\right|^q d\lambda$$
$$\leq\int_{1/2}^{1}\left[(1-\lambda)^n(1-\lambda^\alpha)\left|f^{(n)}(a)\right|^q+m\lambda^\alpha(1-\lambda)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]d\lambda$$
$$=F\left|f^{(n)}(a)\right|^q+Gm\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q.\tag{2.21}$$

Combing the above inequalities (2.19), (2.20), and (2.21), we obtain (2.18). This completes the proof. □

Corollary 2.6. *If $\alpha = 1$, $m = 1$ and $n = 2$ in Theorem 2.5, then we have the following inequality:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{8}\left(\frac{1}{2^{2p+1}(2p+1)}\right)^{1/p}\left[\begin{array}{l}\left(\frac{5}{192}\left|f''(a)\right|^q+\frac{3}{192}\left|f''(b)\right|^q\right)^{1/q}\\+\left(\frac{3}{192}\left|f''(a)\right|^q+\frac{5}{192}\left|f''(b)\right|^q\right)^{1/q}\end{array}\right].$$

3. Application to some special means

Definition 3.1 ([4]). A function $M : R_+^2 \to R_+$, is called a mean function, if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) = M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

Let us recall the following means for arbitrary real numbers a and b.

1. The Arithmetic mean

$$A = A(a, b) = \frac{a+b}{2}, a, b \geq 0.$$

2. The Geometric mean

$$G = G(a, b) = \sqrt{ab}, a, b \geq 0.$$

3. The Power mean

$$P_r = P_r(a, b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}, a, b \geq 0, \; r \geq 1.$$

4. The Harmonic mean

$$H = H(a, b) = \frac{2ab}{a+b}, \; a, \; b \geq 0.$$

5. Generalized-logarithmic mean

$$L_n(a, b) = \begin{cases} a, & \text{if } a = b, \\ \left[\dfrac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right]^{\frac{1}{n}}, & \text{if } a \neq b. \end{cases}$$

6. Identric mean

$$I(a, b) = \begin{cases} a, & \text{if } a = b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right), & \text{if } a \neq b. \end{cases}$$

$$L = L(a, b) = \begin{cases} a, & \text{if } a = b, \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b. \end{cases}$$

Now utilizing outcomes of Section 2, some new inequalities are derived for the above means.

It is well known that L_P is monotonic nondecreasing over $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

Now let a and b be positive real numbers such that $a < b$. Consider the function $a < b$. $M : M(b,a) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to R$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

If $\eta(b,a) = M(b,a)$ in (2.6), and also with $n = 2$ in (2.10), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f(x)dx\right|$$
$$\leq \frac{(M(b,a))^n}{2}\left(\frac{1}{3}\right)^{1-1/q}\left[\frac{2}{(\alpha+2)(\alpha+3}\left|f^{('')}(a)\right|^q + \frac{\alpha}{3(\alpha+2)}m\left|f^{(n)}\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}. \tag{3.1}$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right) - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(M(b,a))^2}{8(2p+1)}\left[\frac{|f''(a)|^q + m\left|f''\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right]^{\frac{1}{q}}. \tag{3.2}$$

For $q \geq 1$. Letting $M = A,\ G,\ P_r,\ H,\ L_n,\ I,\ L$ in (3.1), and in (3.2), we can get the required inequalities and the details are left to the interested reader.

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