

# PULLBACK EXPONENTIAL ATTRACTORS FOR NONAUTONOMOUS DYNAMICAL SYSTEM IN SPACE OF HIGHER REGULARITY\*

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**Abstract** Under what condition, a process which exists a  $(E, E)$ -pullback exponential attractor implies the existence of  $(E, V)$ -pullback exponential attractor when  $V$  embedded in  $E$ ? We answer this question in this paper. As an application of this result, we prove the existence of pullback exponential attractor for a nonlinear reaction-diffusion equation with a polynomial growth nonlinearity in  $L^q(\Omega)$  ( $\forall q \geq 2$ ) and  $H_0^1(\Omega)$ .

**Keywords** Nonautonomous dynamical system, fractal dimension, pullback exponential attractors, reaction diffusion equation.

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## 1. Introduction

Pullback attractor (see [1–3, 7, 10, 12, 14]) is a suitable concept to describe the long time behavior of infinite dimensional nonautonomous dynamical systems or process generated by nonautonomous partial differential equations, which is a family of compact invariant sets attracting all bounded subsets of the phase space (see Definition 2.3). However, a pullback attractor attracts any bounded set of phase space, but the attraction to it may be arbitrarily slow, in order to describe the speed of attraction, the concept of pullback exponential attractors (see [4, 6, 8]) is put forward (see Definition 2.5) and some methods were given to prove the existence of pullback exponential attractors.

As far as we know, only a few articles [4, 6, 8, 11] study the existence of pullback exponential attractors; for too many equations it is difficulty to establish the pullback exponential attractors by these methods. For example, we can not obtain the existence of pullback exponential attractors in  $H_0^1$  for nonautonomous reaction diffusion equations with a polynomial growth nonlinearity (see 4.1) by these methods when  $2 \leq p < \infty$  ( $n \leq 2$ ),  $2 \leq p \leq \frac{n}{n-2} + 1$  ( $n \geq 3$ ). Motivated by these problem and some ideas in [3, 6], under what condition, a process which exists a  $(E, E)$ -pullback exponential attractor implies the existence of  $(E, V)$ -pullback exponential attractor when  $V$  embedded in  $E$ ? We answer this question in this paper. As application of our new method, we discuss the nonautonomous reaction diffusion equation and

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get the existence of pullback exponential attractors in  $L^q(\Omega)(\forall q \geq 2)$  and  $H_0^1(\Omega)$  and improve the results of [6, 8].

The paper is organized as follows: In section 2, we recall some basic concepts about pullback attractors and pullback exponential attractors for a process. In section 3, under the condition that  $V$  is embedded into  $E$ , we provided a new method to verify the existence of pullback exponential in  $V$  when the process has a pullback exponential attractor in  $E$ . In section 4, we apply our result to prove the existence of pullback exponential attractors for nonautonomous reaction diffusion system.

## 2. Preliminaries

Let  $X$  be a complete metric space,  $B(X)$  be the set of all bounded subsets of  $X$ , and a two-parameter family of mappings  $\{U(t, \tau) | t \geq \tau\} = \{U(t, \tau) : t \geq \tau, t, \tau \in \mathbb{R}\}$  act on  $X : U(t, \tau) : X \rightarrow X, t \geq \tau, \tau \in \mathbb{R}$ .

**Definition 2.1.** A two-parameter family of mappings  $\{U(t, \tau)\}$  is said to be a process in  $X$ , if

- (1)  $U(t, s)U(s, \tau) = U(t, \tau), \forall t \geq s \geq \tau,$
- (2)  $U(\tau, \tau) = Id,$  is the identity operator,  $\tau \in \mathbb{R}.$

The pair  $(U(t, \tau), X)$  is generally referred to as a nonautonomous dynamical system. If  $x \rightarrow U(t, \tau)x$  is a continuous in  $X$ , we say that the process is continuous process; if  $U(t, \tau)x_n \rightarrow U(t, \tau)x$  as  $x_n \rightarrow x$ , we say that the process is a norm-to-weak continuous process. Obviously, continuous process is also a norm-to-weak continuous process.

**Definition 2.2.** A family of bounded sets  $\{B(t) | t \in \mathbb{R}\} \subset B(X)$  is called pullback absorbing sets for the process  $(U(t, \tau), X)$  if for any  $t \in \mathbb{R}$ , and any bounded set  $B \subset X$ , there exists a  $\tau_0(t, B) \leq t$  such that  $U(t, \tau)B \subset B(t)$  for all  $\tau \leq \tau_0$ .

**Definition 2.3.** ([1–3, 14]) The family  $\mathcal{A} = \{\mathcal{A}(t) | t \in \mathbb{R}\} \subset B(X)$  is said to be a pullback attractor for  $U(t, \tau)$  if

- (1)  $\mathcal{A}(t)$  is compact for all  $t \in \mathbb{R}$ ;
- (2)  $\mathcal{A}$  is invariant, i.e.,  $U(t, \tau)\mathcal{A}(\tau) = \mathcal{A}(t)$  for all  $t \geq \tau$ ;
- (3)  $\mathcal{A}$  is pullback attracting, i.e.,  $\lim_{\tau \rightarrow -\infty} dist((U(t, \tau)B, \mathcal{A}(t)) = 0,$  for all  $B \in B(X),$  and  $t \in \mathbb{R}$ ;
- (4) if  $\{C(t)\}_{t \in \mathbb{R}}$  is another family of closed attracting sets, then  $\mathcal{A}(t) \subset C(t)$  for all  $t \in \mathbb{R}$

Here  $dist(\cdot, \cdot)$  denotes the non-symmetric Hausdorff distance between sets in  $X$ ; that is  $dist(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$

**Definition 2.4.** ([2, 13]) For any  $\varepsilon > 0$ , let  $n(\mathcal{M}, \varepsilon, X)$  denote the minimum number of ball of  $X$  of radius  $\varepsilon$  which is necessary to cover  $\mathcal{M}$ . The fractal dimension of  $\mathcal{M}$ , which is also called the capacity of  $\mathcal{M}$ , is the number

$$\dim_f(\mathcal{M}, X) = \lim_{\varepsilon \rightarrow 0^+} \frac{\ln n(\mathcal{M}, \varepsilon, X)}{\ln \frac{1}{\varepsilon}}.$$

**Definition 2.5.** ([4, 6, 8]) Let  $\{U(t, \tau) | t \geq \tau\}$  be a process in a metric space  $X$ . We call the family  $\mathcal{M} = \{\mathcal{M}(t) | t \in \mathbb{R}\}$  a pullback exponential attractor for  $U(t, \tau)$  if

- (1) The sets  $\mathcal{M}(t) \in B(X)$  are compact in  $X, \forall t \in \mathbb{R}$ ;
- (2) It is positively semi-invariant, that is

$$U(t, \tau)\mathcal{M}(\tau) \subset \mathcal{M}(t), \forall t \geq \tau ;$$

- (3) The fractal dimension of  $\mathcal{M}(t)$  are uniformly bounded in  $X$ , that is, there exists  $F > 0$  such that

$$\dim_f \mathcal{M}(t) \leq F, \forall t \in \mathbb{R};$$

- (4) The sets  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  pullback exponential attracts bounded subsets of  $X$ ; that is, there exist constants  $k, l > 0$ , for every bounded subset  $B \in B(X)$  and  $t \in \mathbb{R}$  such that

$$\text{dist}(U(t, \tau)B, \mathcal{M}(t)) \leq ke^{-l(t-\tau)}.$$

### 3. Pullback exponential attractors in space of higher regularity

Let  $(E, \|\cdot\|_E)$  and  $(V, \|\cdot\|_V)$  be two Banach spaces such that  $V$  is embedded into  $E$ ,  $\{U(t, \tau) | t \geq \tau\}$  be a process in  $(E, E)$  and  $(E, V)$ ,  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  be a  $(E, E)$ -pullback exponential attractor.

Next we will show that the process exists a  $(E, V)$ -pullback exponential attractor. For the purpose, we will need the following lemmas.

**Lemma 3.1.** *Let  $B$  be a bounded set in  $E$ ,  $S$  is a map from  $E$  to  $V$ . Suppose that there exist  $k, \alpha > 0$  such that  $\|S(u) - S(v)\|_V \leq k\|u - v\|_E^\alpha$  for any  $u, v \in B$ , and that the fractal dimension of  $B$  is bounded in  $E$ , i.e., there exists a positive constant  $d > 0$  such that  $\dim_f(B, E) \leq d$ . Then the fractal dimension of  $S(B)$  is bounded in  $V$  and  $\dim_f(S(B), V) \leq \frac{d}{\alpha}$ .*

**Proof.** By the definition of fractal dimension, we have

$$\dim_f(B, E) = \lim_{\varepsilon \rightarrow 0^+} \frac{\ln n(B, \varepsilon, E)}{\ln \frac{1}{\varepsilon}} \leq d,$$

we get by for any  $\varepsilon > 0$ , there exist  $u_1, u_2, \dots, u_N \in E$ , such that  $B \subset \cup_{i=1}^N B(u_i, \varepsilon)$  in  $E$  and  $N \leq n(B, \varepsilon, E)$ . Thus, for any  $u \in B$ , there exists  $u_i \in E$ , such that

$$\|u - u_i\|_E < \varepsilon.$$

By the assumption, we have

$$\|S(u) - S(u_i)\|_V \leq k\|u - u_i\|_E^\alpha \leq k\varepsilon^\alpha.$$

This shows that  $S(B)$  be covered by a  $k\varepsilon^\alpha$ -net in  $V$ , i.e.,  $S(B) \subset \cup_{i=1}^N B(S(u_i), k\varepsilon^\alpha)$  in  $V$ , and  $N \leq n(B, \varepsilon, E)$ , by the definition of fractal dimension, we get

$$\dim_f(S(B), V) \leq \lim_{\varepsilon \rightarrow 0^+} \frac{\ln n(S(B), k\varepsilon^\alpha, V)}{\ln \frac{1}{k\varepsilon^\alpha}} \leq \lim_{\varepsilon \rightarrow 0^+} \frac{\ln n(B, \varepsilon, E)}{\ln \frac{1}{k\varepsilon^\alpha}} \leq \frac{d}{\alpha}.$$

□

**Lemma 3.2.** *Let the assumptions of Lemma 3.1 hold and let  $B$  be a compact set in  $E$ . Then  $S(B)$  is a compact set in  $V$ .*

The result is obvious, so we omit the proof.

**Theorem 3.1.** *Let  $(E, \|\cdot\|_E)$  and  $(V, \|\cdot\|_V)$  be two Banach spaces,  $\{U(t, \tau) | t \geq \tau\}$  be a process in  $(E, E)$  and  $(E, V)$ ,  $B$  is a positively semi-invariant bounded absorbing set in  $E \cap V$ , i.e.,  $U(t, \tau)B \subset B$ , for any  $t \geq \tau$ ,  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  be a  $(E, E)$ -pullback exponential attractor, and suppose that the process  $\{U(t, \tau) | t \geq \tau\}$  satisfies the condition with constants  $k, \alpha > 0$ ,*

$$\|u - v\|_V \leq k\|u - v\|_E^\alpha, \forall u, v \in B \text{ and } t \geq \tau. \tag{3.1}$$

Then the process  $\{U(t, \tau) | t \geq \tau\}$  exists a  $(E, V)$ -pullback exponential attractor.

**Proof.**  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  is a  $(E, E)$ -pullback exponential attractor, we have to so  $\mathcal{M}(t) \subset B$ . Next we will prove that  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  is also a  $(E, V)$ -pullback exponential attractor.

(Positively semi-invariant) Since  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  is a  $(E, E)$ -pullback exponential attractor, we get  $U(t, \tau)\mathcal{M}(\tau) \subset \mathcal{M}(t)$ . Obviously, the result is hold true in  $V$ .

(Compactness)  $\mathcal{M}(t)$  is a compact set in  $E$ ,  $\mathcal{M}(t) = U(t, t)\mathcal{M}(t)$ , by the condition of (3.1), we get

$$\|u - v\|_V = \|U(t, t)u - U(t, t)v\|_V \leq k\|u - v\|_E^\alpha, \forall u, v \in \mathcal{M}(t) \text{ and } t \geq \tau. \tag{3.2}$$

By Lemma 3.2, we get  $\mathcal{M}(t)$  is a compact set in  $V$ .

(Uniformly bounded of fractal dimension) By the definition of pullback exponential attractors, for any  $t \in \mathbb{R}$ , the fractal dimension of  $\mathcal{M}(t)$  is uniformly bounded in  $E$ .  $U(t, t)\mathcal{M}(t) = \mathcal{M}(t)$ , By (3.2) and Lemma 3.1, the fractal dimension of  $\mathcal{M}(t)$  is uniformly bounded in  $V$ .

(Pullback exponential attraction)  $\{\mathcal{M}(t) | t \in \mathbb{R}\}$  is a  $(E, E)$ -pullback exponential attractor, by the Definition 2.5, we conclude that there exist  $\lambda, l > 0$  such that

$$\text{dist}_E(U(t, \tau)B, \mathcal{M}(t)) \leq \lambda e^{-l(t-\tau)},$$

so by (3.1) and for any  $u \in B, v \in \mathcal{M}(t)$ ,  $\|U(t, \tau)u - v\|_V = \|U(t, t)U(t, \tau)u - U(t, t)v\|_V \leq k\|U(t, \tau)u - v\|_E^\alpha$ , we get

$$\text{dist}_V(U(t, \tau)B, \mathcal{M}(t)) \leq k(\text{dist}_E(U(t, \tau)B, \mathcal{M}(t)))^\alpha \leq k\lambda^\alpha e^{-l\alpha(t-\tau)}.$$

□

## 4. The existence of pullback exponential attractors for nonautonomous reaction diffusion equation

As an application of the Theorem 3.1, we prove the existence of the pullback exponential attractors in  $L^q(\Omega)$  and  $H_0^1(\Omega)$  for the process generated by the solution of the following non-autonomous reaction diffusion equation:

$$\begin{cases} u_t - \Delta u + f(u) = g(t), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u(\tau) = u_\tau. \end{cases} \tag{4.1}$$

Where  $f \in C^1(\mathbb{R}, \mathbb{R})$ ,  $g(\cdot) \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ ,  $\Omega$  is a bounded open subset of  $\mathbb{R}^n$  and there exist  $p \geq 2$ ,  $c_i > 0$ ,  $i = 1, \dots, 5, l > 0$  such that

$$c_1|u|^p - c_2 \leq f(u)u \leq c_3|u|^p + c_4, \quad (4.2)$$

$$f'(u) \geq -l, \quad |f'(u)| \leq c_5(1 + |u|^{p-2}) \quad (4.3)$$

for all  $u \in \mathbb{R}$ .

Denote  $H = L^2(\Omega)$  with norm  $|\cdot|$  and scalar product  $(\cdot, \cdot)$ ,  $H_0^1(\Omega)$  with norm  $\|\cdot\|$ ,  $|\cdot|_k$  denote the norm of  $L^k(\Omega)$ ,  $c_i$  denote constants which may change from line to line and even in the same line.

Suppose that the function  $g(t)$  is translation bounded in  $L^3_{loc}(\mathbb{R}; L^3(\Omega))$ , that is

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |g(s)|_3^3 ds < \infty. \quad (4.4)$$

By (4.4), there exists a constant  $c_6 > 0$  such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |g(s)|^2 ds < c_6, \quad (4.5)$$

and for some  $\lambda > 0$ ,

$$e^{-\lambda t} \int_{\tau}^t e^{\lambda s} |g(s)|^2 ds \leq \sum_{n=0}^{\infty} \int_{t-(n+1)}^{t-n} e^{-\lambda(t-s)} |g(s)|^2 ds \leq \frac{c_6 e^{\lambda}}{e^{\lambda} - 1} = c_7. \quad (4.6)$$

**Lemma 4.1.** (*[2, 5, 13]*) *Let the assumption (4.2) and (4.3) hold and  $g(t) \in L^2_{loc}(\mathbb{R}, H)$ . Then for any initial data  $u_{\tau} \in L^2(\Omega)$  and any  $T \geq \tau$ , there exists a unique solution  $u$  for (4.1) which satisfies*

$$u \in L^2(\tau, T; H_0^1) \cap L^p(\tau, T; L^p(\Omega)).$$

*If furthermore,  $u_{\tau} \in H_0^1$ , then*

$$u \in \mathcal{C}([\tau, T]; H_0^1) \cap L^2(\tau, T; H^2(\Omega)).$$

By Lemma 4.1, we can define the process  $\{U(t, \tau) | t \geq \tau\}$  as follows:

$$U(t, \tau)u_{\tau} : [\tau, +\infty) \times L^2(\Omega) \rightarrow L^p; \quad U(t, \tau)u_{\tau} : [\tau, +\infty) \times L^2(\Omega) \rightarrow H_0^1.$$

**Lemma 4.2.** *Assume that  $f$  and  $g$  satisfies (4.2), (4.3) and (4.5),  $u(t)$  be a weak solution of (4.1). Then for all  $t \geq \tau$  we have the following inequalities:*

$$|u(t)|^2 \leq e^{-\lambda(t-\tau)} |u_{\tau}|^2 + c_8 \quad (4.7)$$

and

$$\int_{\tau}^t e^{\lambda s} (\|u(s)\|^2 + 2c_1 |u|_p^p) ds \leq (1 + \lambda(t - \tau)) e^{\lambda \tau} |u_{\tau}|^2 + c_9 e^{\lambda t} + \lambda^{-1} \int_{\tau}^t e^{\lambda s} |g(s)|^2 ds. \quad (4.8)$$

**Proof.** Taking inner product of (4.1) with  $u$  in  $H$ , we have

$$\frac{1}{2} \frac{d}{dt} |u|^2 + \|u\|^2 + (f(u), u) = (g(t), u),$$

by (4.2), we obtain

$$\frac{d}{dt} |u|^2 + 2\|u\|^2 + 2c_1 |u|_p^p \leq 2c_2 |\Omega| + 2|g(t)| |u|,$$

thanks to Poincaré inequality  $\|u\| \geq \lambda|u|$  and Cauchy's inequality, we have

$$\frac{d}{dt} |u|^2 + \lambda|u|^2 \leq 2c_2 |\Omega| + \lambda^{-1} |g(t)|^2 \quad (4.9)$$

and

$$\frac{d}{dt} |u|^2 + \|u\|^2 + 2c_1 |u|_p^p \leq 2c_2 |\Omega| + \lambda^{-1} |g(t)|^2. \quad (4.10)$$

Applying the Gronwall's Lemma and (4.6) in (4.9), we get

$$|u(t)|^2 \leq e^{-\lambda(t-\tau)} |u_\tau|^2 + c_8. \quad (4.11)$$

Inequality (4.7) is proved.  $\square$

Now let us verify inequality (4.8). Multiplying (4.11) by  $e^{\lambda t}$  and integrating from  $\tau$  to  $t$ , we get

$$\int_\tau^t e^{\lambda s} |u(s)|^2 ds \leq (t-\tau) e^{\lambda \tau} |u_\tau|^2 + c_{10} e^{\lambda t}. \quad (4.12)$$

By (4.10), we find

$$\frac{d}{dt} (e^{\lambda t} |u(t)|^2) + e^{\lambda t} (\|u(t)\|^2 + 2c_1 |u|_p^p) \leq \lambda e^{\lambda t} |u(t)|^2 + 2c_2 |\Omega| e^{\lambda t} + \lambda^{-1} e^{\lambda t} |g(t)|^2,$$

integrating and using (4.12), we get (4.8) holds.

**Lemma 4.3.** *Assume that  $f$  and  $g$  satisfy (4.2), (4.3), (4.5), and let  $u(t)$  be a weak solution associated with Eq.(4.1). Then the following inequality holds for  $t > \tau$ :*

$$|u|^2 + \|u(t)\|^2 + |u|_p^p \leq c_{11} \left(1 + \frac{1}{t-\tau}\right) (1+t-\tau) e^{-\lambda(t-\tau)} |u_\tau|^2 + c_{12} \left(\frac{1}{t-\tau} + 1\right). \quad (4.13)$$

Moreover, for any bounded  $D \subset H$ , there exist  $T, r > 0$ , such that

$$|U(t, \tau) u_\tau|^2 + \|U(t, \tau) u_\tau\|^2 + |U(t, \tau) u_\tau|_p^p \leq r$$

for any  $u_\tau \in D$  and  $t - \tau \geq T$ .

**Proof.** Let  $F(s) = \int_0^s f(\tau) d\tau$ , then by (4.2), we deduce that

$$\tilde{c}_1 |s|^p - \tilde{c}_2 \leq F(s) \leq \tilde{c}_3 |s|^p + \tilde{c}_4. \quad (4.14)$$

Combining (4.10), (4.14), we get

$$\frac{d}{dt} |u|^2 + \|u\|^2 + c_{13} \int_\Omega F(u) dx \leq \lambda^{-1} |g(t)|^2 + c_{14}. \quad (4.15)$$

Multiply (4.1) by  $u_t$ , we have

$$|u_t|^2 + \frac{1}{2} \frac{d}{dt} (\|u\|^2 + 2 \int_{\Omega} F(u) dx) = (g(t), u_t),$$

since  $|(g(t), u_t)| \leq |g(t)| |u_t| \leq \frac{|g(t)|^2}{2} + \frac{|u_t|^2}{2}$ , we obtain

$$\frac{d}{dt} (\|u\|^2 + 2 \int_{\Omega} F(u) dx) \leq |g(t)|^2. \quad (4.16)$$

Combining (4.15), (4.16), we have

$$\frac{d}{dt} (|u|^2 + \|u\|^2 + 2 \int_{\Omega} F(u) dx) + \|u\|^2 + c_{13} \int_{\Omega} F(u) dx \leq (1 + \lambda^{-1}) |g(t)|^2 + c_{14}. \quad (4.17)$$

By Poincaré inequality, we get

$$\|u\|^2 + c_{13} \int_{\Omega} F(u) dx \geq \frac{\|u\|^2}{2} + \frac{\lambda}{2} |u|^2 + c_{13} \int_{\Omega} F(u) dx \geq c_{15} (|u|^2 + \|u\|^2 + 2 \int_{\Omega} F(u) dx). \quad (4.18)$$

Let  $G(u) = |u|^2 + \|u\|^2 + 2 \int_{\Omega} F(u) dx$ , by (4.17), (4.18), we obtain

$$\frac{d}{dt} G(u) + c_{15} G(u) \leq c_{16} |g(t)|^2 + c_{14}, \quad (4.19)$$

inequality (4.19) implies that

$$\frac{d}{dt} ((t - \tau) e^{\lambda t} G(u)) \leq (1 + (\lambda - c_{15})(t - \tau)) G(u) e^{\lambda t} + (c_{14} + c_{16} |g(t)|^2) (t - \tau) e^{\lambda t},$$

integrating, we get

$$(t - \tau) e^{\lambda t} G(u) \leq (1 + c_{17}(t - \tau)) \int_{\tau}^t G(u) e^{\lambda s} ds + c_{18}(t - \tau) e^{\lambda t} + c_{16}(t - \tau) \int_{\tau}^t e^{\lambda s} |g(s)|^2 ds,$$

using (4.8), (4.12), we obtain the inequality (4.13).  $\square$

**Lemma 4.4.** *Assume that (4.2), (4.3) and (4.4) hold. Then the following inequality holds for  $t > \tau_0$*

$$|u(t)|_{2p-2}^{2p-2} \leq c_{19} \left(1 + \frac{1}{t - \tau_0}\right) ((1 + t - \tau_0) e^{-\lambda(t - \tau_0)} |u_{\tau_0}|^2 + e^{-\lambda(t - \tau_0)} |u_{\tau_0}|_p^p), \quad (4.20)$$

here  $u(t) = U(t, \tau) u_{\tau}, u_{\tau_0} = U(\tau_0, \tau) u_{\tau}$  for any  $t \geq \tau_0 > \tau$ .

By the assumption and for some  $\lambda > 0$ , we obtain

$$\sup_{t > \tau} e^{-\lambda t} \int_{\tau}^t e^{\lambda s} |g(s)|^{\frac{3p-4}{p-1}} ds < \infty. \quad (4.21)$$

**Proof.** Multiplying (4.1) with  $|u|^{p-2} u$ , we obtain

$$\frac{1}{p} \frac{d}{dt} |u|_p^p + (p-1) \int_{\Omega} |u|^{p-2} |\nabla u|^2 dx + \int_{\Omega} f(u) |u|^{p-2} u dx = \int_{\Omega} g(t) |u|^{p-2} u dx \quad (4.22)$$

We deduce from (4.2) that

$$f(u)|u|^{p-2}u \geq c'_1|u|^{2p-2} - c'_2. \quad (4.23)$$

By Young's inequality, we have

$$\left| \int_{\Omega} g(t)|u|^{p-2}u dx \right| \leq \frac{c'_1}{2}|u|_{2p-2}^{2p-2} + \frac{1}{2c'_1}|g(t)|^2, \quad (4.24)$$

and using (4.22) and (4.23), we get

$$\frac{d}{dt}|u|_p^p + c_{20}|u|_{2p-2}^{2p-2} \leq c_{21}(1 + |g(t)|^2)$$

and

$$\frac{d}{dt}(e^{\lambda t}|u|_p^p) + c_{20}e^{\lambda t}|u|_{2p-2}^{2p-2} \leq \lambda e^{\lambda t}|u|_p^p + c_{21}e^{\lambda t}(1 + |g(t)|^2),$$

integrating and using (4.8), we deduce that

$$\int_{\tau_0}^t e^{\lambda s}|u|_{2p-2}^{2p-2} ds \leq c_{22}((1 + t - \tau_0)e^{\lambda \tau_0}|u_{\tau_0}|^2 + e^{\lambda \tau_0}|u_{\tau_0}|_p^p + e^{\lambda t} + \int_{\tau_0}^t e^{\lambda s}|g(s)|^2 ds). \quad (4.25)$$

Multiply (4.1) with  $|u|^{2p-4}u$ , we obtain

$$\frac{1}{2p-2} \frac{d}{dt}|u|_{2p-2}^{2p-2} + (2p-3) \int_{\Omega} |u|^{2p-4}|\nabla u|^2 dx + \int_{\Omega} f(u)|u|^{2p-4}u dx = \int_{\Omega} g(t)|u|^{2p-4}u dx. \quad (4.26)$$

We deduce from (4.2) that

$$f(u)|u|^{2p-4}u \geq c''_1|u|^{3p-4} - c''_2. \quad (4.27)$$

Using Young's inequality

$$\left| \int_{\Omega} g(t)|u|^{2p-4}u dx \right| \leq \frac{c''_1}{2}|u|_{3p-4}^{3p-4} + c_{23}|g(t)|^{\frac{3p-4}{\frac{3p-4}{p-1}}}. \quad (4.28)$$

Using (4.27) and (4.28), we have

$$\frac{d}{dt}|u|_{2p-2}^{2p-2} \leq c_{24}(1 + |g(t)|^{\frac{3p-4}{\frac{3p-4}{p-1}}})$$

and

$$\frac{d}{dt}((t - \tau_0)e^{\lambda t}|u|_{2p-2}^{2p-2}) \leq (1 + \lambda(t - \tau_0))e^{\lambda t}|u|_{2p-2}^{2p-2} + c_{24}(t - \tau_0)e^{\lambda t}(1 + |g(t)|^{\frac{3p-4}{\frac{3p-4}{p-1}}}),$$

integrating and using (4.21) and (4.25), we obtain estimate (4.20).  $\square$

**Theorem 4.1.** *Assume that (4.2), (4.3) and (4.4) hold, then the process generated by the solution of Eq.(4.1) have a bounded absorbing set  $B \subset H \cap H_0^1 \cap L^p \cap L^{2p-2}$ , and  $U(t, \tau)B \subset B$ .*

**Proof.** By Lemma 4.3 and Lemma 4.4, the process  $U(t, \tau)$  generated by Eq. (4.1) has a bounded absorbing set  $B_0$  in  $H \cap H_0^1 \cap L^p \cap L^{2p-2}$ , that is, for any bounded  $D \subset H$ , there exists  $T > 0$ , such that  $U(t, t - \tau)D \subset B_0$  for any  $\tau \geq T$ . For  $B_0$ , there exists  $T_0 > 0$ , such that  $U(t, t - \tau)B_0 \subset B_0$  for any  $\tau \geq T_0$ . Let  $B = \cup_{t \in \mathbb{R}} \cup_{\tau \geq T_0} U(t, t - \tau)B \subset B_0$ . Obviously,  $B$  is a positively semi-invariant bounded absorbing set for the process  $U(t, \tau)$  and there exists  $r > 0$  such that  $|u(x)| \leq r$ ,  $\|u(x)\| \leq r$ ,  $|u(x)|_p^p \leq r$  and  $|u(x)|_{2p-2}^{2p-2} \leq r$  for any  $u(x) \in B$ .  $\square$



**Theorem 4.2.** *Assume that (4.2),(4.3) hold, and let for any positive integer  $m$ ,  $g(t)$  is translation bounded in  $L^m_{loc}(\mathbb{R}; L^m(\Omega))$ , i.e.,  $\sup_{t \in \mathbb{R}} \int_t^{t+1} |g(s)|^m ds < \infty$ , then the process generated by the solution of Eq.(4.1) have a bounded absorbing set  $B \subset L^{(m-1)p-2(m-2)}(\Omega)$  and  $U(t, \tau)B \subset B$ .*

Through induction argument and using the same proof as in Lemma 4.4, we can prove this result.

Next we will assume that the function  $g(t)$  is normal ([9]) in  $L^2_{loc}(\mathbb{R}; H)$ , that is, for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} |g(s)|^2 ds < \varepsilon.$$

This condition guarantees that the process  $U(t, \tau)$  generated by the solution of Eq.(4.1) exists a  $(H, H)$ -pullback exponential attractors when  $2 \leq p < \infty (n \leq 2)$ ,  $2 \leq p \leq \frac{n}{n-2} + 1 (n \geq 3)$ , the proof is the same as in Theorem 4.2 of [8], so we omit.

**Theorem 4.3.** *Assume that (4.2),(4.3) hold,  $g(t)$  is normal in  $L^2_{loc}(\mathbb{R}; H)$  and translation bounded in  $L^m_{loc}(\mathbb{R}; L^m(\Omega))$ , and let  $2 \leq p < \infty (n \leq 2)$ ,  $2 \leq p \leq \frac{n}{n-2} + 1 (n \geq 3)$ ,  $q = (m-1)p - 2(m-2)$ , then the process  $U(t, \tau)$  generated by the solution of Eq.(4.1) have a pullback exponential attractor in  $L^q(\Omega)$  for any positive integer  $m \geq 2$ .*

**Proof.** By hölder inequality, we obtain

$$|u(t) - v(t)|_q = \left( \int_{\Omega} |u - v|^{q-1} |u - v| dx \right)^{\frac{1}{q}} \leq |u - v|_{2q-2}^{\frac{q-1}{q}} |u - v|^{\frac{1}{q}}.$$

By Theorem 4.2, there exists  $k > 0$  such that

$$|u(t) - v(t)|_q \leq k |u - v|^{\frac{1}{q}}.$$

By Theorem 3.1, Theorem 4.3 holds. □

**Theorem 4.4.** *Assume that (4.2),(4.3) and (4.4) hold,  $g'(t) \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$  and*

$$\sup_{t \in \mathbb{R}} \int_t^{t+1} |g'(s)|^2 ds < \infty. \tag{4.29}$$

*Then for the bounded absorbing set  $B$  in Theorem 4.1, there exists a positive constants  $T_B$  such that*

$$|u_t(s)|^2 = \left| \frac{d}{dt} U(t, \tau) u_{\tau} \right|^2 \leq M \text{ for any } u_{\tau} \in B \text{ and } t - \tau \geq T_B.$$

**Proof.** By differentiating (4.1) in time and denoting  $v = u_t$ , we have

$$v_t - \Delta v + f'(u)v = g'(t). \tag{4.30}$$

Multiplying the above equality by  $v$  and using (4.3), we obtain

$$\frac{d}{dt} |v|^2 \leq 2l|v|^2 + \lambda^{-1} |g'(t)|^2 \tag{4.31}$$

and

$$\frac{d}{dt}((t-\tau)e^{\lambda t}|v|^2) \leq (1+(\lambda+2l)(t-\tau))e^{\lambda t}|v|^2 + \lambda^{-1}(t-\tau)e^{\lambda t}|g'(t)|^2, \quad (4.32)$$

integrating (4.32) from  $\tau$  to  $t$ , we get

$$|v(t)|^2 \leq ((\lambda+2l) + \frac{1}{t-\tau})e^{-\lambda t} \int_{\tau}^t e^{\lambda s}|v(s)|^2 ds + c_{25}. \quad (4.33)$$

Multiplying equation (4.1) by  $u_t$ , we obtain

$$|u_t|^2 + \frac{1}{2} \frac{d}{dt}(\|u\|^2 + 2 \int_{\Omega} F(u) dx) = (g(t), u_t) \leq \frac{1}{2}(|u_t|^2 + |g(t)|^2),$$

that is

$$|u_t|^2 + \frac{d}{dt}(\|u\|^2 + 2 \int_{\Omega} F(u) dx) \leq |g(t)|^2$$

and

$$e^{\lambda t}|u_t|^2 + \frac{d}{dt}e^{\lambda t}(\|u\|^2 + 2 \int_{\Omega} F(u) dx) \leq \lambda e^{\lambda t}(\|u\|^2 + 2 \int_{\Omega} F(u) dx) + e^{\lambda t}|g(t)|^2. \quad (4.34)$$

Integrating (4.34) from  $\tau$  to  $t$ , we have

$$\begin{aligned} & \int_{\tau}^t e^{\lambda s}|v(s)|^2 ds \\ & \leq e^{\lambda \tau}(\|u_{\tau}\|^2 + 2 \int_{\Omega} F(u_{\tau}) dx) + \lambda \int_{\tau}^t e^{\lambda s}(\|u\|^2 + 2 \int_{\Omega} F(u) dx) ds + \int_{\tau}^t e^{\lambda s}|g(s)|^2 ds. \end{aligned} \quad (4.35)$$

Using (4.8), (4.14) and (4.35) in (4.33), we obtain

$$|v(t)|^2 \leq c_{26}(1 + \frac{1}{t-\tau})[e^{-\lambda(t-\tau)}((1+(t-\tau))|u_{\tau}|^2 + |u_{\tau}|_p^p) + 1]. \quad (4.36)$$

□

**Theorem 4.5.** *Assume that (4.2), (4.3), (4.4) and (4.29) hold,  $g(t)$  is normal in  $L_{loc}^2(\mathbb{R}; H)$ , and let  $2 \leq p < \infty (n \leq 2), 2 \leq p \leq \frac{n}{n-2} + 1 (n \geq 3)$ , then the process  $U(t, \tau)$  generated by the solution of Eq.(4.1) has a pullback attractor in  $H_0^1(\Omega)$ .*

**Proof.** Let  $B$  be the positively semi-invariant bounded absorbing set in Theorem 4.1,  $u(t) = U(t, \tau)u_{\tau}$  and  $v(t) = U(t, \tau)v_{\tau}$  to be solutions associated with Eq.(4.1) with initial data  $u_{\tau}, v_{\tau} \in B$ . Let  $w(t) = u(t) - v(t)$ , we have

$$w_t - \Delta w + f(u) - f(v) = 0. \quad (4.37)$$

Multiplying the above equality by  $w$ , we obtain

$$(w_t, w) + \|w\|^2 + (f(u) - f(v), w) = 0$$

and

$$\|w\|^2 \leq |w_t| |w| + (\int_{\Omega} |f(u) - f(v)|^2 dx)^{\frac{1}{2}} |w|. \quad (4.38)$$

From (4.2), we deduce that

$$\left(\int_{\Omega} |f(u) - f(v)|^2 dx\right)^{\frac{1}{2}} \leq c_{25}(1 + |u|_{2p-2}^{p-1} + |v|_{2p-2}^{p-1}). \quad (4.39)$$

We infer from Theorem 4.1 and Theorem 4.4 that

$$\|u(t) - v(t)\|^2 \leq c_{26}|u(t) - v(t)|,$$

that is

$$\|u(t) - v(t)\| \leq c_{27}|u(t) - v(t)|^{\frac{1}{2}},$$

therefore, by Theorem 3.1 that the process  $U(t, \tau)$  generated by the solution of Eq.(4.1) have a pullback exponential attractor in  $H_0^1(\Omega)$ .  $\square$

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