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APPLICATIONS OF FRACTIONAL COMPLEX TRANSFORM AND $\left(\frac{G'}{G}\right)$-EXPANSION METHOD FOR TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract In this paper, the fractional complex transform and the $\left(\frac{G'}{G}\right)$-expansion method are employed to solve the time-fractional modfied Korteweg–de Vries equation (fmKdV), Sharma-Tasso-Olver, Fitzhugh-Nagumo equations, where G satisfies a second order linear ordinary differential equation. Exact solutions are expressed in terms of hyperbolic, trigonometric and rational functions. These solutions may be useful and desirable to explain some nonlinear physical phenomena in genuinely nonlinear fractional calculus.

Keywords The $\left(\frac{G'}{G}\right)$-expansion method, exact solutions, fractional differential equation, modified Riemann–Liouville derivative.

MSC(2010) 34A08, 35R11, 37H10, 26A33, 35Q68.

1. Introduction

The fractional calculus (fractional derivatives and fractional integrals) has been recognized as an effective modeling methodology for researchers. Fractional differential equations are generalization of ordinary differential equations to arbitrary (noninteger) order. In recent decades, fractional differential equations capture nonlocal relations in space and time with power law memory kernels. Due to extensive applications in engineering and science, research in fractional differential equations has become intense around the world. Some aspects of the fractional differential equations have been investigated by many authors [24,32,35]. Among the investigations for fractional differential equations, research for seeking exact solutions solutions of time-fractional differential equations is an important topic, which can also provide valuable reference for other related research. In recently, some effective methods for fractional calculus were appeared in open literature, such as the exp-function method [5, 45], the fractional sub-equation method [14, 16, 38, 44], the $(G\prime/G)$-expansion method [4, 9, 46], the fractional homotopy analysis method [10–12], the Jacobi elliptic function method [13] and the first integral method [26]. Based on these methods, a variety of fractional differential equations have been investigated

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and solved. However, there are quite a few direct approaches to exact solutions of nonlinear equations. For example, the transformed rational function method and the multiple exp-function approach provide the most powerful approaches to traveling wave and multiple wave solutions.

The fractional complex transform [17,18] is the simplest approach, it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Recently, the fractional complex transform has been suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. The $(G\prime/G)$-expansion method [20-23] can be used to construct the exact solutions for fractional differential equations. The present paper investigates for the applicability and efficiency of the $(G\prime/G)$-expansion method on time-fractional differential equations. The aim of this paper is to extend the application of the $(G\prime/G)$-expansion method to obtain exact solutions to some fractional differential equations in broad science and technology area.

In this paper, we will apply the $(G\prime/G)$-expansion method for solving fractional partial differential equations in the sense of modified Riemann–Liouville derivative by Jumarie [20, 21]. The Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n\le\alpha<n+1, \quad n\ge 1. \end{cases} \tag{1.1}$$

We list some important properties for the modified Riemann–Liouville derivative as follows:

$$D_t^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}, \gamma>0, \tag{1.2}$$

$$D_t^\alpha(cf(t)) = cD_t^\alpha f(t), c = \text{constant}, \tag{1.3}$$

$$D_t^\alpha\{af(t)+bg(t)\} = aD_t^\alpha f(t) + bD_t^\alpha g(t), \tag{1.4}$$

where a and b constant.

$$D_t^\alpha c = 0, \quad c = \text{constant}. \tag{1.5}$$

The rest of this paper is organized as follows. In Section 2, we give the defination of the $(G\prime/G)$-expansion method for solving time-fractional partial differential equations. Then in Section 3-5, we use this method to construct exact solutions for the time-fractional fmKdV, Sharma-Tasso-Olver, Fitzhugh-Nagumo equations. Some conclusions are presented in Section 6.

2. The $\left(\frac{G'}{G}\right)$-expansion method for Fractional Partial Differential Equations

In the following, we give the main steps of the $\left(\frac{G'}{G}\right)$-expansion method.

1. Suppose that a fractional partial differential equation, say in the independent

variables $t, x_1, x_2, ..., x_n$ are given by

$$P(u_1, ..., u_k, \frac{\partial u_1}{\partial t}, ..., \frac{\partial u_k}{\partial t}, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_1}, ..., \frac{\partial u_1}{\partial x_n}, ..., \frac{\partial u_k}{\partial x_n},$$
$$D_t^{\alpha} u_1, ..., D_t^{\alpha} u_k, D_{x_1}^{\beta} u_1, ..., D_{x_1}^{\beta} u_k, ..., D_{x_n}^{\beta} u_1, ..., D_{x_n}^{\beta} u_k, ...) = 0, \tag{2.1}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, 2, 3, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

2. Li and He [17] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$U_i(\xi) = u_i(t, x_1, x_2, ..., x_n), \qquad \xi = \xi(t, x_1, x_2, ..., x_n),$$
$$\xi = \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \frac{\tau x_1^{\beta}}{\Gamma(1+\beta)} + \frac{\delta x_2^{\gamma}}{\Gamma(1+\gamma)} + ... + \frac{\psi x_n^{\varphi}}{\Gamma(1+\varphi)}, \tag{2.2}$$

where $c, \tau, \delta, ..., \psi$ are non zero arbitrary constants.

By using the chain rule

$$D_t^{\alpha} u = \sigma_t' \frac{dU}{d\xi} D_t^{\alpha} \xi,$$
$$D_x^{\alpha} u = \sigma_x' \frac{dU}{d\xi} D_x^{\alpha} \xi, \tag{2.3}$$

where σ_t' and σ_x' are called the sigma indexes see [7], without loss of generality we can take $\sigma_t' = \sigma_x' = l$, where l is a constant.

Substituting (2.2) with (2.3) and (1.2) into (2.2), equation (2.2) can be reduced into an ODE;

$$Q(U_1, ..., U_k, U_1', ..., U_k', U_{11}'', ..., U_{kk}'', ...) = 0, \tag{2.4}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate Eq. (2.4) term by term one or more times.

3. Assume that the solution of equation (2.4) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \tag{2.5}$$

where a_i $(i = 0, \pm 1, \pm 2, ..., \pm m)$ are constants, while $G(\xi)$ satisfies the following second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.6}$$

with λ and μ are being constants.

4. The positive integer m can be determined by considering the homogeneous balance between the highest order derivaives and the nonlinar terms appearing in equation (2.4).

5. Substituting equation (2.5) into equation (2.4) and using equation (2.6) collecting all terms with the same order of $(G\prime/G)$ together. Then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i $(i = 0, \pm 1, \pm 2, ..., \pm m)$, $c, \tau, \delta, ..., \psi, \mu$ and λ.

6. Solving the equations system in step 5, and using equation then substituting a_i $(i = 0, \pm 1, \pm 2, ..., \pm m)$, $c, \tau, \delta, ..., \psi, \mu, \lambda$ and the general solutions of equation (2.6) into equation (2.5), we can get a variety of exact solutions of equation (2.1).

The adopted $(G\prime/G)$-expansion method is also actually the expansion method using a special Riccati equation, since $G\prime/G$ satisfies a Riccati equation if G solves (2.6) in the manuscript. All explicit and exact solutions to general Riccati equations are presented in [29]. More generally, Frobenius integrable decompositions [30] and the invariant subspace method [28] will help in solving nonlinear equations.

3. Time-fractional fmKdV equation

Firstly, we consider the following time fractional fmKdV equation [25]

$$D_t^{\alpha} u + u^2 u_x + u_{xxx} = 0, \quad t > 0, \ 0 < \alpha \leq 1, \tag{3.1}$$

with the initial conditions as

$$u(x, 0) = \frac{4\sqrt{2} k \sin^2(kx)}{3 - \sin^2(kx)}, \tag{3.2}$$

where k is arbitrary constant, α is a parameter describing the order of the fractional time-derivative. By using homotopy perturbation method (HPM), Hashim et al. [1] have found approximate analytical solutions for fmKdV. By using differential transform method, Kurulay and Bayram [25] have found new approximate analytical solutions for fmKdV. In recent years, Guner and Cevikel applied the exp-function method to this equation and obtained new exact solutions [15]. When $\alpha = 1$, the fractional fmKdV equation reduces to the mKdV equation. The mKdV equation appears in many fields such as acoustic waves in certain anharmonic lattices, Alfvén waves in a collisionless plasma, transmission lines in Schottky barrier, models of traffic congestion, ion acoustic solitons, elastic media, shallow water model, plasma science, biophysics etc [40]. By using the variational iteration method, Inc [19] has found exact and numerical solutions and compared with those obtained by Adomian decomposition method. Lastly, the extended tanh method was successfully used to establish solitary wave solutions to this equation [2].

For our purpose, we introduce the following transformations;

$$u(x, t) = U(\xi), \qquad \xi = \nu x - \frac{c t^{\alpha}}{\Gamma(1 + \alpha)}, \tag{3.3}$$

where c and ν are a non-zero constants.

Substituting (3.3) with (2.3) and (1.2) into (3.1), we can know that (3.1) reduced into an ODE

$$-cU' + \nu U^2 U' + \nu^3 U''' = 0, \tag{3.4}$$

where $U' = \dfrac{dU}{d\xi}$. By once time integrating we find

$$\xi_0 - cU + \nu \frac{U^3}{3} + \nu^3 U'' = 0, \tag{3.5}$$

where ξ_0 is a integration constant.

Using the ansatz (3.5), for the linear term of highest order U'' with the highest order nonlinear term U^3. By simple calculation, we have balancing U'' with U^3 in (3.5) gives

$$m+2=3m, \tag{3.6}$$

so that

$$m=1. \tag{3.7}$$

Suppose that the solutions of (3.5) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi)=a_0+a_1\left(\tfrac{G'}{G}\right), \quad a_1\neq 0. \tag{3.8}$$

By using Eq. (2.6), from Eq. (3.8) we have

$$U''(\xi)=2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2+(2a_1\mu+a_1\lambda^2)\left(\frac{G'}{G}\right)+a_1\lambda\mu, \tag{3.9}$$

and

$$U^3(\xi)=a_1^3\left(\frac{G'}{G}\right)^3+3a_0a_1^2\left(\frac{G'}{G}\right)^2+3a_0^2a_1\left(\frac{G'}{G}\right)+a_0^3. \tag{3.10}$$

Substituting Eq. (3.8)-(3.10) into Eq. (3.5), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i=0,...,3)$ and set it to zero we obtain the system

$$\begin{aligned}
&\tfrac{1}{3}\nu a_1^3+2\nu^3a_1=0,\\
&\nu a_0a_1^2+3\nu^3a_1\lambda=0,\\
&\nu a_0^2a_1-ca_1+\nu^3a_1\lambda^2+2\nu^3a_1\mu=0,\\
&-ca_0+\tfrac{1}{3}\nu a_0^3+\nu^3a_1\lambda\mu+\xi_0=0.
\end{aligned} \tag{3.11}$$

Solving this system by simple calculation gives

$$a_0=\pm\tfrac{\lambda}{2}i\nu\sqrt{6}, \quad a_1=\pm i\nu\sqrt{6}, \quad c=\tfrac{\nu^3}{2}\left(-\lambda^2+4\mu\right), \quad \xi_0=0, \tag{3.12}$$

where λ and μ are arbitrary constants.

By using Eq. (3.12), expression (3.8) can be written as

$$U(\xi)=\pm\tfrac{\lambda}{2}i\nu\sqrt{6}\pm i\nu\sqrt{6}\left(\tfrac{G'}{G}\right). \tag{3.13}$$

Substituting general solutions of Eq. (2.6) into Eq. (3.13) we have two types of exact solutions of the time fractional fmKdV equation as follows:

When $\lambda^2-4\mu>0$,

$$U_{1,2}(\xi)=\pm i\nu\frac{\sqrt{6\lambda^2-24\mu}}{2}\left(\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}\right), \tag{3.14}$$

where $\xi=x-\frac{\nu^3\left(-\lambda^2+4\mu\right)}{2\Gamma(1+\alpha)}t^\alpha$ and C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$U_{3,4}(\xi) = \pm i\nu \frac{\sqrt{-6\lambda^2 + 24\mu}}{2} \left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} \right), \tag{3.15}$$

where $\xi = x - \frac{\nu^3\left(-\lambda^2+4\mu\right)}{2\Gamma(1+\alpha)} t^\alpha$ and C_1, C_2 are arbitrary constants.

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}$ becomes

$$u_{1,2}(x,t) = \pm i\nu \frac{\lambda\sqrt{6}}{2} \coth \frac{\lambda}{2} \left(x - \frac{\nu^3\left(-\lambda^2 + 4\mu\right)}{2\Gamma(1+\alpha)} t^\alpha \right), \tag{3.16}$$

and $U_{3,4}$ becomes

$$u_{3,4}(x,t) = \pm \nu \frac{\lambda\sqrt{6}}{2} \cot \frac{\lambda}{2} \left(x - \frac{\nu^3\left(-\lambda^2 + 4\mu\right)}{2\Gamma(1+\alpha)} t^\alpha \right). \tag{3.17}$$

which are the solitary wave solutions of the time fractional fmKdV equation.

Remark 3.1. Comparing our results to the Guner's results [15] it can be seen that these results are new.

4. Time-fractional Sharma-Tasso-Olver equation

Secondly, we consider the nonlinear fractional Sharma-Tasso-Olver equation [36]

$$D_t^\alpha u + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0, \quad t > 0,\ 0 < \alpha \leq 1, \tag{4.1}$$

where a is a arbitrary constant and subject to the initial condition

$$u(x,0) = -\sqrt{2B_0} \tan \left(\frac{\sqrt{2B_0}}{2} x \right), \tag{4.2}$$

where a and B_0 are arbitrary constants, α is a parameter describing the order of the fractional time-derivative. The function $u(x,t)$ is assumed to be a causal function of time. Song et al. have obtained the approximate analytical solutions of eq.(4.1) with the variational iteration method, the adomian decomposition method and the homotopy perturbation method. Esen et al. [6] have obtained the approximate analytical solutions of this equation with the homotopy analysis method (HAM). Guner and Cevikel [15] have used the Exp-function method to find the traveling wave solutions of (4.1). Equation (4.1) has been investigated in [47] using the fractional sub-equation method. By the improved (G'/G)-expansion method, Zayed et al. [41] obtained abundant new exact solutions for the fractional STO equations. In the case of $\alpha = 1$, Eq. (4.1) reduces to the classical nonlinear STO equation.

We use the following transformations:

$$u(x,t) = U(\xi), \qquad \xi = x - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{4.3}$$

where $c \neq 0$ is a constant.

Substituting (4.3) with (2.3) and (1.2) into (4.1), equation (4.1) can be reduced into an ODE,

$$-cU' + 3a(U')^2 + 3aU^2U' + 3aUU'' + aU''' = 0, \tag{4.4}$$

where $U' = \dfrac{dU}{d\xi}$.

Integrating equation (4.4) with respect to ξ yields

$$\xi_0 - cU + 3aUU' + aU^3 + aU'' = 0 \tag{4.5}$$

where ξ_0 is a constant of integration.

By the same procedure as illustrated in the section 3, we can determine value of m by balancing U^3 and U'' in Eq.(4.3). We find $m = 1$. We can suppose that the solutions of Eq. (4.1) is of the form

$$U(\xi) = a_0 + a_1\left(\tfrac{G'}{G}\right), \quad a_1 \neq 0. \tag{4.5}$$

By using Eq. (4.5) and (2.6) it is derived that

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu, \tag{4.6}$$

and

$$U^3(\xi) = a_1^3\left(\frac{G'}{G}\right)^3 + 3a_0a_1^2\left(\frac{G'}{G}\right)^2 + 3a_0^2a_1\left(\frac{G'}{G}\right) + a_0^3. \tag{4.7}$$

Substituting Eq. (4.5)-(4.7) into Eq. (4.1), collecting the coefficients of $\left(\tfrac{G'}{G}\right)^i$ $(i = 0, ..., 3)$ and set it to zero we obtain the system

$$\begin{aligned}
&2aa_1 + aa_1^3 - 3aa_1 = 0,\\
&-3aa_0a_1 + 3aa_1\lambda + 3aa_0a_1^2 - 3aa_1^2\lambda = 0,\\
&-3aa_0a_1\lambda - 3aa_1^2\mu - ca_1 + 3aa_0^2a_1 + 2aa_1\mu + aa_1\lambda^2 = 0,\\
&-3aa_0a_1\mu + \xi_0 - ca_0 + aa_0^3 + aa_1\lambda\mu = 0.
\end{aligned} \tag{4.8}$$

We can solve this system by symbolic computation get sets of solutions.

Case 1:

$$a_0 = \lambda, \quad a_1 = 2, \quad a = a, \quad c = a\lambda^2 - 4a\mu, \quad \xi_0 = 0, \tag{4.9}$$

where λ and μ are arbitrary constants. By using Eq. (4.9), expression (4.5) can be written as

$$U(\xi) = \lambda + 2\left(\frac{G'}{G}\right). \tag{4.10}$$

Substituting general solutions of Eq. (2.6) into Eq. (4.10) we have three types of exact solutions of the time-fractional Sharma-Tasso-Olver equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$U_1(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right), \tag{4.11}$$

where$\xi = x - \frac{a\lambda^2-4a\mu}{\Gamma(1+\alpha)}t^{\alpha}$ and C_1, C_2 are arbitrary constants. When $\lambda^2 - 4\mu < 0$,

$$U_2(\xi) = \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right), \tag{4.12}$$

where $\xi = x - \dfrac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)}t^{\alpha}$ and C_1, C_2 are arbitrary constants

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then U_1 becomes

$$u_1(x,t) = \lambda \tanh \frac{\lambda}{2}\left(x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)}t^{\alpha}\right), \tag{4.13}$$

and U_2 becomes

$$u_2(x,t) = -i\lambda \tan \frac{\lambda}{2}\left(x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)}t^{\alpha}\right). \tag{4.14}$$

which are the solitary wave solutions of the time-fractional Sharma-Tasso-Olver equation.

Case 2:

$$\begin{aligned} &a_0 = a_0, a_1 = 1, && c = -3aa_0\lambda - a\mu + 3aa_0^2 + a\lambda^2, \\ &a = a, && \xi_0 = 2aa_0\mu - 3aa_0^2\lambda + 2aa_0^3 + a_0 a\lambda^2 - a\lambda\mu, \end{aligned} \tag{4.15}$$

where λ and μ are arbitrary constants.

Substituting Eq. (4.9) into Eq. (4.5) yields

$$U(\xi) = a_0 + \left(\frac{G'}{G}\right). \tag{4.16}$$

Substituting general solutions of Eq. (2.6) into Eq. (4.10) we have three types of exact solutions of the time-fractional Sharma-Tasso-Olver equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$U_1(\xi) = a_0 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right), \tag{4.17}$$

where $\xi = x + \dfrac{3aa_0\lambda + a\mu - 3aa_0^2 - a\lambda^2}{\Gamma(1+\alpha)}t^{\alpha}$.

When $\lambda^2 - 4\mu < 0$,

$$U_2(\xi) = a_0 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right), \tag{4.18}$$

where $\xi = x + \frac{3aa_0\lambda + a\mu - 3aa_0^2 - a\lambda^2}{\Gamma(1+\alpha)}t^{\alpha}$.

When $\lambda^2 - 4\mu = 0$,

$$U_3(\xi) = a_0 - \frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi}. \tag{4.19}$$

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then U_1 becomes

$$u_1(x,t) = a_0 - \frac{\lambda}{2} + \frac{\lambda}{2}\tanh\frac{\lambda}{2}\left(x + \frac{3aa_0\lambda - 3aa_0^2 - a\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right), \tag{4.20}$$

and U_2 becomes

$$u_2(x,t) = a_0 - \frac{\lambda}{2} - i\frac{\lambda}{2}\tan\frac{\lambda}{2}\left(x + \frac{3aa_0\lambda - 3aa_0^2 - a\lambda^2}{\Gamma(1+\alpha)}t^\alpha\right). \tag{4.21}$$

which are the solitary wave solutions of the time-fractional Sharma-Tasso-Olver equation.

Remark 4.1. We note that the exact solutions established in (4.13), (4.14), (4.19), (4.20) and (4.21) are new exact solutions to the time-fractional Sharma-Tasso-Olver equation.

5. Time fractional Fitzhugh-Nagumo equation

Thirdly, we take into account the fractional Fitzhugh-Nagumo equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\psi), \quad t > 0,\ 0 < \alpha \leq 1, x \in R, \tag{5.1}$$

which is an important nonlinear reaction-diffusion equation, applied to model the transmission of nerve impulses [8,33], and also used in biology and the area of population genetics in circuit theory [37]. Pandir and Tandoğan, applied the modified trial equation method to obtain analytical solutions of the time-fractional Fitzhugh-Nagumo equation [34]. Khan et al. employed the homotopy perturbation method (HPM) to obtain approximate analytical solutions of the time-fractional reaction-diffusion equation of the Fisher type [22]. Merdan is using He's variational iteration method, obtain the analytical solutions of nonlinear time-fractional reaction-diffusion equations of the Fisher type [31]. For $\alpha = 1$, Eq. (5.1) reduces to the classical nonlinear Fitzhugh-Nagumo equation. By using some methods, many researchers have tried to obtain the exact solutions of this equation. For example; by using Hirota method, Kawahara and Tanaka [23] have found new exact solutions of Eq. (5.1); by using the first integral method, Li and Guo [27] have obtained a series of new exact solutions of the Fitzhugh–Nagumo equation. When $\psi = -1$, the Fitzhugh-Nagumo equation reduces to the real Newell-Whitehead equation.

For our goal, we present the following transformation

$$u(x,t) = U(\xi), \qquad \xi = cx - \frac{mt^\alpha}{\Gamma(1+\alpha)}, \tag{5.2}$$

where c and m are non zero constants.

Then by use of Eq. (5.2) with (2.3) and (1.2), Eq. (5.1) can be turned into an ODE

$$mU' + c^2U'' + U(1-U)(U-\psi) = 0, \tag{5.3}$$

where $U' = \dfrac{dU}{d\xi}$.

Using the ansatz (5.3), for the linear term of highest order U'' with the highest order nonlinear term U^3. By simple calculation, we have balancing U'' with U^3 in (5.3) gives

$$n+2=3n, \tag{5.4}$$

so that

$$n=1. \tag{5.5}$$

Suppose that the solutions of (5.3) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi)=a_0+a_1\left(\tfrac{G'}{G}\right), \quad a_1\neq 0. \tag{5.6}$$

By using Eq. (2.6), from Eq. (5.6) we have

$$U'(\xi)=-a_1\left(\frac{G'}{G}\right)^2-a_1\lambda\left(\frac{G'}{G}\right)-a_1\mu, \tag{5.7}$$

$$U''(\xi)=2a_1\left(\frac{G'}{G}\right)^3+3a_1\lambda\left(\frac{G'}{G}\right)^2+(2a_1\mu+a_1\lambda^2)\left(\frac{G'}{G}\right)+a_1\lambda\mu, \tag{5.8}$$

and

$$U^3(\xi)=a_1^3\left(\frac{G'}{G}\right)^3+3a_0a_1^2\left(\frac{G'}{G}\right)^2+3a_0^2a_1\left(\frac{G'}{G}\right)+a_0^3. \tag{5.9}$$

Substituting Eq. (5.6)-(5.9) into Eq. (5.3), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ $(i=0,...,3)$ and set it to zero we obtain the system

$$\begin{aligned}
&-ma_1\mu+c^2a_1\lambda\mu+a_0^2-a_0\psi-a_0^3+a_0^2\psi=0,\\
&-ma_1\lambda+c^2a_1\lambda^2+2c^2a_1\mu+2a_0a_1-3a_0^2a_1+2a_0a_1\psi-a_1\psi=0,\\
&-ma_1+3c^2a_1\lambda-3a_0a_1^2+a_1^2+a_1^2\psi=0,\\
&2c^2a_1-a_1^3=0.
\end{aligned} \tag{5.10}$$

Solving this system by sybolic computation gives

$$\begin{aligned}
&a_0=\tfrac{4\mu-\lambda^2\pm\sqrt{\lambda^4-4\lambda^2\mu}}{2(4\mu-\lambda^2)}, && a_1=\tfrac{\mp3\lambda^2+(1-2\psi)\sqrt{\lambda^4-4\lambda^2\mu}}{\lambda\left(\mp4\mu\mp\lambda^2\pm2\lambda^2\psi\mp8\mu\psi+3\sqrt{\lambda^4-4\lambda^2\mu}\right)},\\
&c=\pm\sqrt{\tfrac{1}{2\lambda^2-8\mu}}, && m=\pm\tfrac{\sqrt{\lambda^4-4\lambda^2\mu}(2\psi-1)}{2\lambda(4\mu-\lambda^2)},
\end{aligned} \tag{5.11}$$

where λ and μ are arbitrary constants.

By using Eq. (5.11), expression (5.7) can be written as

$$U(\xi)=\tfrac{4\mu-\lambda^2\pm\sqrt{\lambda^4-4\lambda^2\mu}}{2(4\mu-\lambda^2)}+\tfrac{\mp3\lambda^2+(1-2\psi)\sqrt{\lambda^4-4\lambda^2\mu}}{\lambda\left(\mp4\mu\mp\lambda^2\pm2\lambda^2\psi\mp8\mu\psi+3\sqrt{\lambda^4-4\lambda^2\mu}\right)}\left(\tfrac{G'}{G}\right). \tag{5.12}$$

Substituting general solutions of Eq. (2.6) into Eqs. (3.13) we have two types of exact solutions of the time fractional Fitzhugh-Nagumo equation as follows:

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} U_{1,2}(\xi) = & \tfrac{4\mu-\lambda^2\pm\sqrt{\lambda^4-4\lambda^2\mu}}{2(4\mu-\lambda^2)} + \tfrac{\mp 3\lambda^2+(1-2\psi)\sqrt{\lambda^4-4\lambda^2\mu}}{\lambda\left(\mp 4\mu\mp\lambda^2\pm 2\lambda^2\psi\mp 8\mu\psi+3\sqrt{\lambda^4-4\lambda^2\mu}\right)} \times \\ & \left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\frac{C_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi} - \frac{\lambda}{2} \right), \end{aligned} \tag{5.13}$$

where $\xi = \pm\sqrt{\frac{1}{2\lambda^2-8\mu}}x - \frac{\pm\sqrt{\lambda^4-4\lambda^2\mu}(2\psi-1)t^\alpha}{2\lambda(4\mu-\lambda^2)\Gamma(1+\alpha)}$ and C_1, C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} U_{3,4}(\xi) = & \tfrac{4\mu-\lambda^2\pm\sqrt{\lambda^4-4\lambda^2\mu}}{2(4\mu-\lambda^2)} + \tfrac{\mp 3\lambda^2+(1-2\psi)\sqrt{\lambda^4-4\lambda^2\mu}}{\lambda\left(\mp 4\mu\mp\lambda^2\pm 2\lambda^2\psi\mp 8\mu\psi+3\sqrt{\lambda^4-4\lambda^2\mu}\right)} \times \\ & \left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\frac{-C_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi} - \frac{\lambda}{2} \right), \end{aligned} \tag{5.14}$$

where $\xi = \pm\sqrt{\frac{1}{2\lambda^2-8\mu}}x - \frac{\pm\sqrt{\lambda^4-4\lambda^2\mu}(2\psi-1)t^\alpha}{2\lambda(4\mu-\lambda^2)\Gamma(1+\alpha)}$ and C_1, C_2 are arbitrary constants.

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}$ becomes

$$u_{1,2}(x,t) = \tfrac{-\lambda^2\pm\lambda^2}{2(-\lambda^2)} + \tfrac{\mp 3\lambda^2+(1-2\psi)\lambda^2}{2(\mp\lambda^2\pm 2\lambda^2\psi+3\lambda^2)}\left(\coth\frac{\lambda}{2}\left(\pm\sqrt{\tfrac{1}{2\lambda^2}}x - \tfrac{\pm\lambda^2(2\psi-1)t^\alpha}{2\lambda(-\lambda^2)\Gamma(1+\alpha)}\right) - 1\right), \tag{5.15}$$

and $U_{3,4}$ becomes

$$u_{3,4}(x,t) = \tfrac{-\lambda^2\pm\lambda^2}{2(-\lambda^2)} + i\tfrac{\mp 3\lambda^2+(1-2\psi)\lambda^2}{2(\mp\lambda^2\pm 2\lambda^2\psi+3\lambda^2)}\left(\tan\frac{\lambda}{2}\left(\pm\sqrt{\tfrac{1}{2\lambda^2}}x - \tfrac{\pm\lambda^2(2\psi-1)t^\alpha}{2\lambda(-\lambda^2)\Gamma(1+\alpha)}\right) + i\right). \tag{5.16}$$

which are the solitary wave solutions of the time fractional Fitzhugh-Nagumo equation.

Remark 5.1. Comparing our results to the Pandir's results [34] it can be seen that these results are new.

6. Conclusion

In this work, $(G\prime/G)$-expansion method is extended to solve the time-fractional fmKdV, Sharma-Tasso-Olver, Fitzhugh-Nagumo equations. As a result, some exact solutions are obtained including the hyperbolic function solutions, trigonometric function solutions and rational solutions. The work emphasized our belief that the method is a reliable technique to handle nonlinear fractional differential equations and fractional complex transform with help of $(G\prime/G)$-expansion method offer significant advantages in terms of its straightforward applicability, its computational effectiveness and its powerful. This method has more advantages: it is direct and concise. Therefore, we deduce that the proposed method can be extended to solve many systems of nonlinear fractional differential equations. This is our task in the future.

References

[1] O. Abdulaziz, I. Hashim and E.S. Ismail, *Approximate analytical solution to fractional modified KdV equations,* Mathematical and Computer Modelling, 49(2009), 136–145.

[2] A. Bekir, *On traveling wave solutions to combined KdV–mKdV equation and modified Burgers–KdV equation,* Commun Nonlinear Sci Numer Simulat, 14(2009), 1038–1042.

[3] A. Bekir, *Application of the (G'/G)-expansion method for nonlinear evolution equations*, Physics Letters A, 372(2008)(19), 3400–3406.

[4] A. Bekir and O. Guner, *Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method,* Chin. Phys. B, 22, 11(2013), 110–202.

[5] A. Bekir, O. Guner and A.C. Cevikel, *Fractional complex transform and exp-Function methods for fractional differential equations*, Abstract and Applied Analysis, 2013(2013), 426–462.

[6] A. Esen, O. Taşbozan and N. Yağmurlu, *Approximate analytical solutions of the fractional Sharma-Tasso-Olver equation using homotopy analysis method and a comparison with other methods*, Çankaya University Journal of Science and Engineering, 9(2012)(2), 139–147.

[7] T. Elghareb, S.K. Elagan, Y.S. Hamed and M. Sayed, *An exact solutions for the generalized fractional Kolmogrove-Petrovskii Piskunov equation by using the generalized (G'/G)-expansion method*, Int. Journal of Basic & Applied Sciences, 13(2013)(1), 19–22.

[8] R. Fitzhugh, *Impulse and physiological states in models of nerve membrane*, Biophys. J., 1(1961), 445–466.

[9] K.A. Gepreel and S. Omran, *The exact solutions for the nonlinear partial fractional differential equations*, Chinese Physics B , 21(2012), 110–204.

[10] K.A. Gepreel and M.S. Mohamed, *An optimal homotopy analysis method nonlinear fractional differential equation*, Journal of Advanced Research in Dynamical and Control Systems, 6, 1(2014), 1–10.

[11] K.A. Gepreel, T.A. Nofal and A.A. Al-Thobaiti, *Numerical solutions for the nonlinear partial fractional Zakharov-Kuznetsov equations with time and space fractional*, Scientific Research and Essays, 9(2014), 471–482.

[12] K.A. Gepreel, *Optimal Q. Homotopy analysis method for nonlinear fractional dynamics equations*, Jokull Journal, 68(2014), 317–326.

[13] K.A. Gepreel, *Explicit Jacobi elliptic exact solutions for nonlinear partial fractional differential equations*, Advances in Difference equations, 2014(2014), 286.

[14] K.A. Gepreel and A.A. Al-Thobaiti, *Exact solution of nonlinear partial fractional differential equations using the fractional sub-equation method*, Indian Journal of Phys. 88, 3(2014), 293-300.

[15] O. Guner and A.C. Cevikel, *A Procedure to construct exact solutions of nonlinear fractional differential equations*, The Scientific World Journal, 2014(2014), 489–495.

[16] S. Guo, L. Mei, Y. Li and Y. Sun, *The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics*, Physics Letters A, 376(2012), 407-411.

[17] J.H. He, S.K. Elagan and Z.B. Li, *Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus*, Physics Letters A, 376(2012), 257–259.

[18] R.W. Ibrahim, *Fractional complex transforms for fractional differential equations*, Advances in Difference Equations, 2012(2012), 192.

[19] M. Inc, *An approximate solitary wave solution with compact support for the modified KdV equation*, Applied Mathematics and Computation, 184(2007), 631–637.

[20] G. Jumarie, *Modified Riemann–Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., 51(2006), 1367–1376.

[21] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouvillie derivative for nondifferentiable functions*, Appl. Maths. Lett., 22(2009), 378–385.

[22] N.A. Khan, M. Ayaz, L. Jin and A. Yildirim, *On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type*, International Journal of the Physical Sciences, 6(2011)(10), 2483–2496.

[23] T. Kawahara and M. Tanaka, *Interaction of travelling fronts: An exact solution of a nonlinear diffusion equation*, Phys Lett A, 97(1983), 311–314.

[24] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[25] M. Kurulay and M. Bayram, *Approximate analytical solution for the fractional modified KdV by differential transform method*, Commun Nonlinear Sci Numer Simulat, 15(2010), 1777–1782.

[26] B. Lu, *The first integral method for some time fractional differential equations*, J. Math. Anal. Appl., 395(2012), 684–693.

[27] H. Li,and Y. Guo, *New exact solutions to the Fitzhugh-Nagumo equation*, Applied Mathematics and Computation, 180(2006), 524–528.

[28] W.X. Ma, *A refined invariant subspace method and applications to evolution equations*, Science China Mathematics, 55(2012), 1796-1778.

[29] W.X. Ma and B. Fuchssteiner, *Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation*, International Journal of Non-Linear Mechanics, 31(1996), 329–338.

[30] W.X. Ma, H.Y. Wu and J.S. He, *Partial differential equations possessing Frobenius integrable decompositions*, Physics Letters A, 364(2007), 29–32

[31] M. Merdan, *Solutions of time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative*, International Journal of Physical Sciences, 7(2012)(15), 2317–2326.

[32] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[33] J.S. Nagumo, S. Arimoto and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proc IRE., 50(1962), 61–70.

[34] Y. Pandir, Y.A. Tandoğan, *Exact solutions of the time-fractional Fitzhugh-Nagumo equation*, AIP Conference Proceedings, 1558(2013), 1919.

[35] I. Podlubny, *Fractional Differential Equations,* Academic Press, California, 1999.

[36] L.N. Song, Q. Wang and H.Q. Zhang, *Rational approximation solution of the fractional Sharma–Tasso–Olver equation*, J. Comput. Appl. Math., 224(2009), 210–218.

[37] M. Shih, E. Momoniat and F.M. Mahomed, *Approximate conditional symmetries and approximate solutions of the perturbed Fitzhugh-Nagumo equation*, J. Math. Phys., 46(2005), 023503.

[38] B. Tong, Y. He, L. Wei and X. Zhang, *A generalized fractional sub-equation method for fractional differential equations with variable coefficients*, Physics Letters A, 376(2012), 2588–2590.

[39] M. Wang, X. Li and J. Zhang, *The (G'/G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett. A, 372(2008), 417–423.

[40] Z. Yan, *The modified KdV equation with variable coefficients: Exact uni/bi-variable travelling wave-like solutions*, Applied Mathematics and Computation, 203(2008), 106–112.

[41] E.M.E. Zayed, Y.A. Amer and R.M.A. Shohib, *Exact traveling wave solutions for nonlinear fractional partial differential equations using the improved (G'/G)-expansion method*, International Journal of Engineering and Applied Science, 4(2014), 7.

[42] E.M.E. Zayed and K.A. Gepreel, *The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics*, J. Math. Phys., 50(2009)(1), 013–502.

[43] S. Zhang, J-L. Tong and W. Wang, *A generalized (G'/G)-expansion method for the mKdV equation with variable coefficients*, Physics Letters A, 372(2008), 2254–2257.

[44] S. Zhang and H-Q. Zhang, *Fractional sub-equation method and its applications to nonlinear fractional PDEs,* Physics Letters A, 375(2011), 1069–1073.

[45] B. Zheng, *Exp-function method for solving fractional partial differential equations,* The Scientific World Journal, 2013(2013), 465723.

[46] B. Zheng, *(G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics*, Commun. Theor. Phys., 58(2012), 623–630.

[47] B. Zheng and C. Wen, *Exact solutions for fractional partial differential equations by a new fractional sub-equation method*, Advances in Difference Equations, 2013(2013), 199.