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# GENERAL SOLUTION OF BASSET EQUATION WITH CAPUTO GENERALIZED HUKUHARA DERIVATIVE

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**Abstract** In this paper, the fuzzy Basset equation is introduced. This problem is related to the motion of a sphere in a viscous liquid when its parameters are fuzzy numbers. We investigate the existence and uniqueness of solution with converting the problem to a system of fuzzy fractional differential equation, and the solution is also obtained under Caputo generalized Hukuhara differentiability. Some examples show the effectiveness and efficiency our approach.

**Keywords** Caputo generalized Hukuhara differentiability, fuzzy Basset equation, system of fuzzy fractional differential equations.

**MSC(2010)** 34A07, 34A08.

## 1. Introduction

The dynamics of a sphere immersed in an incompressible viscous fluid have attracted lots of attention in mathematical mechanic researches. Basset in [10] proposed the solution for a sphere moving in a viscous liquid when the sphere is moving in a straight line under the action of a constant force, such as gravity, and also when the sphere is surrounded by viscous liquid and is set in rotation about a fixed diameter and then left to itself. Nowadays this problem is called the Basset equation which can be modeled as fractional differential equation of order $\alpha \in (0,1)$ is most frequent, but not exclusively, used with $\alpha = 1/2$. The advantage of the fractional order models in comparison with integer-order models is based on its physical considerations. Also, the fuzzy set theory is a powerful tool for modeling uncertain problems. These vagueness in fractional order models may be appearing in each part of the problem like initial condition, boundary condition or etc. Therefore, solving fractional order problem in the sense of real conditions leads to use interval or fuzzy calculations. Recently, the basic concept as a Riemann-Liouville fractional integral, RiemannLiouville H-differentiability, Caputo type fractional derivative based on Hukuhara and generalized Hukuhara difference and strongly generalized differentiability are defined in fuzzy fractional calculus (see e.g. [2–5,7,9,15,16,19]). The Caputo generalized Hukuhara differentiability are considered here. The Caputo generalized Hukuhara derivative is presented in [5], the authors introduced an ordinary fractional differential equation under the generalized Hukuhara differentiability, and studied the existence and uniqueness of the solution to this equation.

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Moreover, the existence, and uniqueness of solutions of nonlinear differential equations of fractional order are discussed in [7] by contraction mapping principle and the fixed point theorem. See also [4] for the examination of integro-differential equations involving fractional Caputo generalized Hukuhara differentiability.

The aims of present paper briefly are 1. Introducing the Basset equation under Caputo generalized Hukuhara differentiability. 2. Converting the fuzzy Basset equation to a system of fuzzy ordinary fractional differential equation. 3. Obtaining the solution of the Basset equation in two sense of generalized Hukuhara differentiability.

In Section 2,we recall some basic concepts and results and then, we study some essential properties for fuzzy Riemann-Liouville integral and Caputo generalized Hukuhara derivative in Section 3. In Section 4, the fuzzy Basset equation is introduced, its existence and uniqueness results studied and the solutions obtained in two sense of Caputo generalized Hukuhara differentiability. Section 5 illustrates the efficiency of our approach through solving two examples, and finally, conclusions are discussed in Section 6.

## 2. Preliminaries

For the topics in this section, readers are referred to previous studies, for example, to [1, 2, 8, 9, 11, 12]. Denote $\mathbb{R}_{\mathcal{F}}$ the space of fuzzy set in $\mathbb{R}$, that is, functions $u : \mathbb{R} \longrightarrow [0, 1]$ such that

(i) u is normal, i.e. $\exists t_0 \in \mathbb{R}$ with $u(t_0) = 1$,

(ii) u is a convex fuzzy set i.e. $u((1-\lambda)t_1+\lambda t_2) \geq \min\{u(t_1), u(t_2)\}, \quad \forall t_1, t_2 \in \mathbb{R}$, $\lambda \in [0, 1]$,

(iii) u is upper semi-continuous on $\mathbb{R}$,

(iv) $cl\{t \in \mathbb{R} : u(t) > 0\}$ is compact, where $cl$ denotes the closure of a subset.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. It is clear that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. Given $0 < r \leq 1$, we denote $[u]^r = \left\{t \in \mathbb{R} \middle| u(t) \geq r\right\}$ and $[u]^0 = cl\left\{t \in \mathbb{R} \middle| u(t) > 0\right\}$. The properties (i)-(iv) concludes that the $r$-level sets of $u \in \mathbb{R}_{\mathcal{F}}$, $[u]^r = [u^-_r, u^+_r]$, are nonempty compact intervals $\forall r \in [0,\ 1], \forall u \in \mathbb{R}_{\mathcal{F}}$.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $\tilde{u} = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $u^-_r = a + (b - a)r$ and $u^+_r = c - (c - b)r$ are the endpoints of $r$-level sets $\forall r \in [0, 1]$.

The metric $D$ on fuzzy numbers given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$

$$D(u, v) = \sup_{0\leq r\leq 1} \max\{|u^-_r - v^-_r|, |u^+_r - v^+_r|\}.$$

It is well known that $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

**Definition 2.1** ( [11]). Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then $w$ is called the Hukuhara difference of $u$ and $v$, and it is denoted by $u \ominus v$.

**Definition 2.2** ( [12]). The generalized Hukuhara difference of two fuzzy number $u, v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} \quad (i) & \text{u = v + w}, \\ \text{or} \quad (ii) & \text{v = u + (-1) w}. \end{cases}$$

**Definition 2.3** ( [12]). The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \longrightarrow \mathbb{R}_{\mathcal{F}}$ at $t_0$ is defined as

$$f^{'}_{gH}(t_0) = \lim_{h\to 0} \frac{1}{h}[f(t_0 + h) \ominus_{gH} f(t_0)].$$

If $f^{'}_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at $t_0$.
Also we say that f is $[(i) - gH]-$differentiable at $t_0$ if

$$[f^{'}_{gH}(t_0)]^r = [(f^-_r)^{'}(t_0), (f^+_r)^{'}(t_0)], \qquad 0 \leq r \leq 1, \tag{2.1}$$

and that f is $[(ii) - gH]-$differentiable at $t_0$ if

$$[f^{'}_{gH}(t_0)]^r = [(f^+_r)^{'}(t_0), (f^-_r)^{'}(t_0)], \qquad 0 \leq r \leq 1. \tag{2.2}$$

**Definition 2.4** ( [5]). Let $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$. We say that $f(t)$ is m-th order $gH$-differentiable at $t_0$ whenever the function $f(t)$ is $gH$-differentiable of the order $i,\ i = 0, 1, ..., m - 1$ at $t_0$, and if there exist $f^{(m)}_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$f^{(m)}_{gH}(t_0) = \lim_{h\to 0} \frac{1}{h}[f^{(m-1)}_{gH}(t_0 + h) \ominus_{gH} f^{(m-1)}_{gH}(t_0)].$$

**Definition 2.5** ( [14]). A fuzzy-valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be absolutely continuous if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each family $\{(s_k, t_k)|k = 1, 2, ..., n\}$ of disjoint open intervals in $[a, b]$ with $\sum_{k=1}^{n}(t_k - s_k) < \delta$, we have $\sum_{k=1}^{n} D(f(t_k), f(s_k)) < \epsilon$.

Let $A^m_{\mathbb{F}}[a, b]$ denote the space of fuzzy-valued functions from $[a, b]$ into $\mathbb{R}_{\mathcal{F}}$ with $m - 1$ $gH$-derivative absolutely continuous function on $[a, b]$.

**Definition 2.6** ( [6]). A fuzzy-valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $t_0 \in [a,\ b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D\Big(f(t), f(t_0)\Big) < \epsilon$, whenever $t \in [a,\ b]$ and $|t - t_0| < \delta$. We say that $f$ is fuzzy continuous on $[a, b]$ if $f$ is continuous at each $t_0 \in [a,\ b]$ such that the continuity is one-sided at endpoints $a,\ b$.

The notation $C^m_{\mathbb{F}}[a, b]$ is the space of fuzzy-valued functions that, together with their $gH$-derivatives of order less than or equal to $m$, are continuous on $[a, b]$.

**Definition 2.7** ( [6]). A function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is called Riemann integrable on $[a, b]$, if there exists $I \in \mathbb{R}_{\mathcal{F}}$, with the property: $\forall\epsilon > 0$, $\exists\delta > 0$, such that for any division of $[a, b]$, $a = t_0 < \cdots < t_n = b$ where $|t_i - t_{i-1}| < \delta$ for $i = 1, 2, \cdots, n$ and for any points $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$, we have

$$D\big(\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}), I\big) < \epsilon.$$

Then we denote $I = \int_a^b f(t)dt$ and it is called fuzzy Riemann integral. Note that if $f$ be continuous in the metric $D$, Lebesgue integral and Riemann integral yield the same value, and also

$$\left[\int_a^b f(t)dt\right]^r = \left[\int_a^b f^-_r(t)dt, \int_a^b f^+_r(t)dt\right], \qquad 0 \leq r \leq 1.$$

Let $L_{\mathbb{F}}[a, b]$ denote the set of Lebesgue integrable fuzzy-valued functions on $[a, b]$.

# 3. Caputo generalized hukuhara derivative

In this section, we introduce some properties of fuzzy Riemann-Liouville integrals and Caputo derivative under generalized Hukuhara differentiability.

**Definition 3.1** ( [18]). Consider $f : [a, b] \to \mathbb{R}$, fractional derivative of $f(t)$ in the Caputo sense is defined as

$$(D_*^\alpha f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{(m-\alpha-1)} f^{(m)}(s) ds \quad m-1 < \alpha \leq m \, , \, m \in \mathbb{N} \, , \, t > a. \tag{3.1}$$

**Definition 3.2** ( [14]). Let $f \in L_{\mathbb{F}}[a, b]$. The Riemann-Liouville integral of fuzzy-valued function f is defined as

$$(J_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \qquad a < s < t, \qquad 0 < \alpha \leq 1. \tag{3.2}$$

For $\alpha = 1$, we have $J_a^1 = I$.

**Definition 3.3.** Let $f \in A_{\mathbb{F}}^m[a, b]$. The Caputo generalized Hukuhara differentiability of fuzzy-valued function $f$( ${}^{cf}[gH]-$differentiability for short) is defined as following:

$$( {}_{gH}D_*^\alpha f)(t) = J_a^{m-\alpha} f_{gH}^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{(m-\alpha-1)} f_{gH}^{(m)}(s) ds, \tag{3.3}$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > a$. Also we say that $f$ is ${}^{cf}[(i)-gH]$-differentiable of order $\alpha$ if

$$[{}_{gH}D_*^\alpha f(t)]^r = [(D_*^\alpha f_r^-)(t), (D_*^\alpha f_r^+)(t)], \qquad 0 \leq r \leq 1 \tag{3.4}$$

and $f$ is ${}^{cf}[(ii) - gH]$-differentiable of order $\alpha$ if

$$[{}_{gH}D_*^\alpha f(t)]^r = [(D_*^\alpha f_r^+)(t), (D_*^\alpha f_r^-)(t)], \qquad 0 \leq r \leq 1 \tag{3.5}$$

where $D_*^\alpha f$ is defined in Definition 3.1.

**Lemma 3.1.** *Let $m - 1 < \alpha \leq m, m \in \mathbb{N}$, and the fuzzy valued-function $f \in C_{\mathbb{F}}^m[0, T]$. Then ${}_{gH}D_*^\alpha f \in C_{\mathbb{F}}[0, T]$ and ${}_{gH}D_*^\alpha f(0) = 0$.*

**Proof.** It is immediate by Definition 3.3. □

**Lemma 3.2** ( [9]). *Let $\alpha, \beta > 0$ and $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function such that $f \in L_{\mathbb{F}}[0, T]$, then*

$$J_0^\alpha J_0^\beta f = J_0^{\alpha+\beta} f.$$

**Lemma 3.3** ( [4]). *Let $0 < \alpha \leq 1$ and $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function such that $f \in A_{\mathbb{F}}^1[0, T]$, then*

$$J_0^\alpha ({}_{gH}D_*^\alpha f)(t) = f(t) \ominus_{gH} f(0).$$

**Lemma 3.4.** *Let $0 < \alpha \leq 1$ and $f : [0, T] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function such that $f \in C_{\mathbb{F}}[0, T]$, then*

$${}_{gH}D_*^\alpha J_0^\alpha f = f.$$

**Proof.** Using properties of ${}_{gH}D_*^{\alpha}$, Lemma 3.2 and Theorem 10 in [11], we have

$$\begin{aligned}
{}_{gH}D_*^{\alpha}(J_0^{\alpha}f)(t) &= J_0^{1-\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds\right)'_{gH} \\
&= J_0^{1-\alpha}\left(\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}f(s)ds\right) \\
&= J_0^{1-\alpha}(J_0^{\alpha-1}f)(t) = f(t).
\end{aligned}$$

**Lemma 3.5.** *Let $f \in C_{\mathbb{F}}^m[0,T]$, and let $0 < \alpha \leq 1, \beta > 0$ be such that $k-1 \leq \alpha+\beta \leq k$ where $k \in \mathbb{N}, k \leq m-1$. Then*

$$ {}_{gH}D_*^{\alpha}\ {}_{gH}D_*^{\beta}f = {}_{gH}D_*^{\alpha+\beta}f. $$

**Proof.** To prove this statement, we consider three cases:

1. $[\beta] = \alpha+\beta$: Using Definition 3.3, Lemmas 3.1, 3.3 and 3.4, find that

$$\begin{aligned}
{}_{gH}D_*^{\alpha}\ {}_{gH}D_*^{\beta}f &= {}_{gH}D_*^{\alpha}J_a^{[\beta]+1-\beta}\ {}_{gH}D^{[\beta]+1}f = {}_{gH}D_*^{\alpha}J_a^{\alpha+1}\ {}_{gH}D^{[\beta]+1}f \\
&= {}_{gH}D_*^{\alpha}J_a^{\alpha+1}\ {}_{gH}D^{\alpha+\beta+1}f = {}_{gH}D_*^{\alpha+\beta}f,
\end{aligned}$$

2. $\beta \in \mathbb{N}$: In this case, by Definition 3.3, we have

$$ {}_{gH}D_*^{\alpha}\ {}_{gH}D_*^{\beta}f = J_0^{1-\alpha}\ {}_{gH}D^{1+\beta}f = {}_{gH}D_*^{\alpha+\beta}f, $$

3. $[\beta] = [\alpha+\beta]$: Here we have, using Definition 3.3 and Lemma 3.2

$$\begin{aligned}
{}_{gH}D_*^{\alpha}\ {}_{gH}D_*^{\beta}f &= {}_{gH}D_*^{\alpha}J_a^{[\beta]+1-\beta}\ {}_{gH}D^{[\beta]+1}f = J_0^{1-\alpha}J_a^{[\beta]-\beta}\ {}_{gH}D^{[\beta]+1}f \\
&= J_0^{1+[\alpha+\beta]-(\alpha+\beta)}\ {}_{gH}D^{[\alpha+\beta]+1}f = {}_{gH}D_*^{\alpha+\beta}f.
\end{aligned}$$

**Lemma 3.6** ( [5])**.** *Let $0 < \alpha \leq 1$, the fuzzy initial value problem $({}_{gH}D_*^{\alpha}y)(t) = f(t, y(t))$, $y(0) = y_0 \in \mathbb{R}_{\mathcal{F}}$ where $f$ is supposed be continuous fuzzy function on $[0,T]$, is equivalent to one of the following integral equations:*

$$ y(t) = y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds, $$

*if $y(t)$ be ${}^{cf}[(i)-gH]$-differentiable, and*

$$ y(t) = y(0) \ominus \frac{(-1)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds, $$

*if $y(t)$ be ${}^{cf}[(ii)-gH]$-differentiable.*

**Theorem 3.1** ( [5])**.** *Let $0 < \alpha \leq 1$ and $\lambda \in \mathbb{R}$. Then the solution of initial value problem ${}_{gH}D_*^{\alpha}y(t) = \lambda y(t) + f(t)$, $y(0) = y_0 \in \mathbb{R}_{\mathcal{F}}$ where $f \in C_{\mathbb{F}}[0,T]$ is a given function, can be expressed in the form*

$$ y(t) = y_0E_{\alpha}(\lambda t^{\alpha}) + \widehat{y}(t), $$

*if $\lambda > 0$ and $y(t)$ be ${}^{cf}[(i)-gH]-$differentiable and*

$$ y(t) = y_0E_{\alpha}(\lambda t^{\alpha}) \ominus (-1)\widehat{y}(t), $$

*if* $\lambda < 0$ *and* $y(t)$ *be* ${}^{cf}[(ii) - gH]-$*differentiable, where*

$$\widehat{y}(t) = \frac{1}{\lambda} \int_0^t \frac{d}{ds} E_\alpha(\lambda s^\alpha) f(t-s) ds,$$

*and* $E_\alpha(t) = \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+\alpha j)}, \alpha > 0,$ *is Mittag-Leffler function which the series converges for all values of* $t$.

# 4. Fractional dynamic model and the solution

Let us consider the fuzzy Basset equation in form

$$\begin{cases} y'_{gH}(t) = f(t) + C_1 y(t) + C_2 \ {}_{gH}D_*^{\frac{1}{2}} y(t), \quad t \in [0,T], \\ y(0) = A \in \mathbb{R}_{\mathcal{F}}, \end{cases} \tag{4.1}$$

where $C_1, C_2$ are real constant, and $y(t)$ is an unknown fuzzy function of crisp variable t, $f : [0,T] \to \mathbb{R}_{\mathcal{F}}$ is continuous function.

Now, we convert Eq. (4.1) to a system of fuzzy fractional differential equation, and then use the presented method in [5] for finding solutions under Caputo generalized Hukuhara differentiability concepts. To this, substitute

$$y(t) = y_0(t), \quad and \quad {}_{gH}D_*^{\frac{1}{2}} y(t) = y_1(t).$$

Then by using Lemmas 3.1 and 3.5, we have

$$\begin{aligned} &{}_{gH}D_*^{\frac{1}{2}} y_0(t) = y_1(t), \qquad (4.2) \\ &{}_{gH}D_*^{\frac{1}{2}} y_1(t) = {}_{gH} D_*^{\frac{1}{2}} ({}_{gH}D_*^{\frac{1}{2}} y_0(t)) = (y_0)'_{gH}(t) = f(t) + C_1 y_0(t) + C_2 y_1(t), \end{aligned}$$

with the initial conditions

$$y_0(0) = A, \quad y_1(0) = 0.$$

Now, we can write system (4.2) in matrix form as

$${}_{gH}D^{\frac{1}{2}} Y(t) = F(t) + \lambda Y(t), \tag{4.3}$$

where

$$Y(t) = \begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix}, \qquad \lambda = \begin{pmatrix} 0 & 1 \\ C_1 & C_2 \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

For solving system (4.3), Theorem 3.1 can be adopt such that here the parameter $\lambda$ is a matrix. Hence the ${}^{cf}[(i) - gH]$-differentiable solution of system (4.3) is

$$Y(t) = Y(0) E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) + \lambda^{-1} \int_0^t q(s) F(t-s) ds. \tag{4.4}$$

And the ${}^{cf}[(ii) - gH]$-differentiable solution of system (4.3) is

$$Y(t) = Y(0) E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) \ominus (-1) \lambda^{-1} \int_0^t q(s) F(t-s) ds, \tag{4.5}$$

where

$$q(t) = \frac{d}{dt} E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) = \frac{d}{dt} \sum_{j=0}^{\infty} \frac{(\lambda t^{\frac{1}{2}})^j}{\Gamma(1+\frac{j}{2})}$$

is convergent series (see [13]).

**Theorem 4.1.** *Assume that $y$ is a continuous function in $[0,T]$. Then $y$ is a solution of Eq.(4.1) if and only if the vector function $Y = (y_0, y_1)^t = (y, {}_{gH}D_*^{\frac{1}{2}} y)^t$ solves the system (4.2).*

**Proof.** First suppose that $Y = (y_0, y_1)^t$ is a solution of system (4.2), then we prove that $y := y_0$ solves the Eq.(4.1). By Lemma 3.5, we have

$$\begin{aligned} &{}_{gH}D_*^{\frac{1}{2}} y(t) = {}_{gH}D_*^{\frac{1}{2}} y_0(t) = y_1(t), \\ &y'_{gH}(t) = {}_{gH}D_*^{\frac{1}{2}}\ {}_{gH}D_*^{\frac{1}{2}} y(t) = {}_{gH}D_*^{\frac{1}{2}} y_1 = f(t) + C_1 y_0(t) + C_2 y_1(t), \end{aligned} \tag{4.6}$$

and it is clear that $y_0(0) = y(0)$. Thus the function $y$ satisfies the Basset equation (4.1).

Reciprocally, let us consider the fuzzy-valued function $y \in C_{\mathbb{F}}[0,T]$ satisfies in Eq.(4.1), then $y$ satisfies in system (4.6). Hence, $Y = (y, {}_{gH}D_*^{\frac{1}{2}} y)^t$ satisfies in system (4.2) and initial condition $y_0(0) = A$ and also lemma 3.1 from continuity of $y$ concludes that $y_1(0) = {}_{gH}D_*^{\frac{1}{2}} y(0) = 0$. □ □

Since instead governing equation, we consider equivalent fractional system, here investigates the existence and uniqueness of solutions for the system of fuzzy fractional differential equations involving the Caputo generalized Hukuhara derivative with initial conditions.

**Theorem 4.2.** *Let $0 < \alpha_j \leq 1$ for $j = 1, 2, \cdots, k$ and consider the following fuzzy fractional differential system*

$$\begin{cases} {}_{gH}D_*^{\alpha_j} y_j(t) = f_j(t, y_1(t), \cdots, y_k(t)), \\ y_j(0) = y_0^{(j)}, j = 1, 2, \cdots, k, \end{cases} \tag{4.7}$$

*where $f_j : [0,T] \times \mathbb{R}_{\mathcal{F}}^k$, $j = 1, 2, \cdots, k$, are continuous and satisfy Lipschitz conditions, i.e.*

$$D(f_j(t, y_1(t), \cdots, y_k(t)), f_j(t, z_1(t), \cdots, z_k(t))) \leq L_j \sum_{i=1}^{k} D(y_i, z_i), \quad j = 1, 2, \cdots, k,$$

*where $L_j$, $j = 1, 2, \cdots, k$ are real positive functions. Then the system of equation (4.7) has a unique continuous solution.*

**Proof.** Let $y_j$ for $j = 1, 2, \cdots, k$ are ${}^{cf}[(i) - gH]$-differentiable of order $\alpha_j$. Using Lemma 3.6, the system (4.7) can be written as following integral equations

$$y_j(t) = y_j(0) + \frac{1}{\Gamma(\alpha_j)} \int_0^t (t-s)^{\alpha_j - 1} f_j(s, y_1(s), \cdots, y_k(s)) ds, \quad j = 1, 2, \cdots, k. \tag{4.8}$$

Then we have

$$y_j(t) = y_j(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widehat{f}_j(s, y_1(s), \cdots, y_k(s)) ds, \quad j = 1, 2, \cdots, k, \tag{4.9}$$

where $\alpha = \min_j \alpha_j$, and $\widehat{f}_j(s, y_1(s), \cdots, y_k(s)) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_j)}(t-s)^{\alpha_j-\alpha} f_j(s, y_1(s), \cdots, y_k(s))$.

By setting $Y = (y_1, y_2, \cdots, y_N)^t$, and $\widehat{F} = (\widehat{f}_1, \widehat{f}_2, \cdots, \widehat{f}_N)^t$, the system (4.9) becomes

$$Y(t) = Y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widehat{F}(s, Y(s))ds. \tag{4.10}$$

By Lipschitz continuity of $f_j$ for $j = 1, 2, \cdots, k$ we have $\widehat{f}_j$ for $j = 1, 2, \cdots, k$ are continuous and satisfies in Lipschitz condition with respect to $y_j, j = 1, 2, \cdots, k$. Consequently $\widehat{F}$ is continuous and satisfies in a Lipschitz condition with respect to $Y$. Hence, from Theorem 3. 1 in [17], we conclude the existence and uniqueness of ${}^{cf}[(i) - gH]$-differentiable solution $Y = (y_1, y_2, \cdots, y_N)^t$.
Next let us consider the solutions of system (4.7), $y_j$ for $j = 1, 2, \cdots, k$ are ${}^{cf}[(ii) - gH]$-differentiable. By Lemma 3.6, the system (4.7) can be converted to

$$y_j(t) = y_j(0) \ominus \frac{-1}{\Gamma(\alpha_j)} \int_0^t (t-s)^{\alpha_j-1} f_j(s, y(s))ds, \quad j = 1, 2, \cdots, k. \tag{4.11}$$

Similarly, we write the system (4.11) as following integral equation

$$Y(t) = Y(0) \ominus \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widehat{F}(s, Y(s))ds. \tag{4.12}$$

The proof of Theorem 3. 1 in [17] can be easily extended for existence and uniqueness of ${}^{cf}[(ii) - gH]$-differentiable solution of Eq. (4.12) as well. $\square$ $\square$

## 5. Examples

In this section, we solve two examples of Fuzzy Basset equation under Caputo generalized Hukuhara differentiability.

**Example 5.1.** Consider the following Basset equation

$$y'_{gH}(t) = 2y(t) + {}_{gH}D_*^{\frac{1}{2}} y(t) \ominus f(t), \ t \in [0, 2], \tag{5.1}$$

where $f(t) = (0, 1, 2) \odot (2t + 2\sqrt{\frac{t}{\pi}} + 1)$ and initial condition is $y(0) = (0, 1, 2)$. The Basset equation (5.1) is equivalent to following system

$$\begin{aligned} &{}_{gH}D_*^{\frac{1}{2}} y_0(t) = y_1(t), \\ &{}_{gH}D_*^{\frac{1}{2}} y_1(t) = 2y_0(t) + y_1(t) \ominus f(t). \end{aligned} \tag{5.2}$$

The system (5.2) can be written as

$${}_{gH}D_*^{\frac{1}{2}} Y(t) = \lambda Y(t) \ominus F(t),$$

where

$$\lambda = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \qquad \lambda^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \qquad F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

By Eq.(4.4) the ${}^{cf}[(i)-gH]$-differentiable solution is equal

$$\begin{aligned} Y(t)=\begin{pmatrix} y_0(t) \\ y_1(t) \end{pmatrix} &= E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}})\begin{pmatrix} (0,1,2) \\ 0 \end{pmatrix} \ominus \lambda^{-1}\int_0^t \frac{d}{ds}(E_{\frac{1}{2}}(\lambda s^{\frac{1}{2}}))\begin{pmatrix} 0 \\ f(t-s) \end{pmatrix} ds \\ &= E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}})\begin{pmatrix} (0,1,2) \\ 0 \end{pmatrix} \ominus \int_0^t \frac{d}{ds}(E_{\frac{1}{2}}(\lambda s^{\frac{1}{2}}))\begin{pmatrix} \frac{1}{2}f(t-s) \\ 0 \end{pmatrix} ds. \end{aligned}$$

Hence, the ${}^{cf}[(i)-gH]$-differentiable solution of Eq. (5.1) is the first component of $Y(t)$, $y_0(t)$, that is

$$y(t)=(0,1,2)e_{11}(t)\ominus\frac{1}{2}\int_0^t q_{11}(s)f(t-s)ds,$$

where $e_{11}(t)$ and $q_{11}(t)$ are the top left component of the matrix $E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}})$ and $\frac{d}{dt}(E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}))$, respectively. Then, we obtain

$$y(t)=(0,1,2)\odot(t+1)$$

with $r$-level set $[y(t)]^r=[y_r^-,y_r^+]=[r(t+1),(2-r)(t+1)]$ that is plotted in Figure 1.

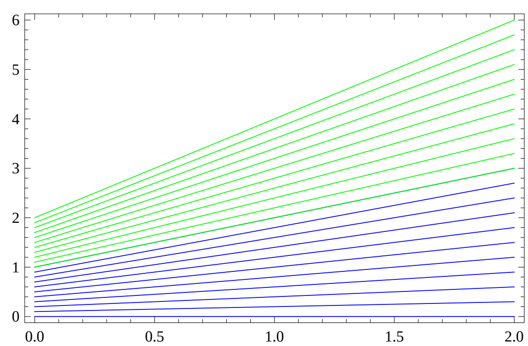


**Figure 1.** The level sets of the solution of Example 5.1

**Example 5.2.** Consider the following Basset equation

$$\begin{cases} y'_{gH}(t)=-y(t)+{}_{gH}D_*^{\frac{1}{2}}y(t)\ominus(-6,-4,-2)\odot(\sqrt{\frac{t}{\pi}}e^{-t}\ {}_1F_1(0.5,1.5;t)), t\in[0,4], \\ y(0)=\tilde{2}, \end{cases} \tag{5.3}$$

where $\tilde{2}=(1,2,3)$ is a triangular fuzzy number and ${}_pF_q(a,b;t)$ is the generalized hypergeometric function. The Basset equation (5.3) is equivalent to following system

$$\begin{aligned} {}_{gH}D_*^{\frac{1}{2}}y_0(t) &= y_1(t), \\ {}_{gH}D_*^{\frac{1}{2}}y_1(t) &= -y_0(t)+y_1(t)\ominus(-6,-4,-2)\odot(\sqrt{\frac{t}{\pi}}e^{-t}\ {}_1F_1(0.5,1.5;t)). \end{aligned}$$

i.e.

$${}_{gH}D_*^{\frac{1}{2}}Y(t)=\lambda Y(t)\ominus F(t),$$

where

$$\lambda = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \lambda^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$F(t) = \begin{pmatrix} 0 \\ (-6,-4,-2) \odot (\sqrt{\frac{t}{\pi}} e^{-t} \ {}_1F_1(0.5, 1.5; t)) \end{pmatrix}.$$

According Eq.(4.4) the ${}^{cf}[(ii) - gH]$-differentiable solution is

$$\begin{aligned} Y(t) =& E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) \begin{pmatrix} \tilde{2} \\ 0 \end{pmatrix} \\ &\oplus \int_0^t \frac{d}{ds}(E_{\frac{1}{2}}(\lambda s^{\frac{1}{2}}))\lambda^{-1} \begin{pmatrix} 0 \\ (-6,-4,-2) \odot (\sqrt{\frac{t-s}{\pi}} e^{s-t} \ {}_1F_1(0.5, 1.5; t-s)) \end{pmatrix} ds \\ =& E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}) \begin{pmatrix} \tilde{2} \\ 0 \end{pmatrix} \\ &\oplus \int_0^t \frac{d}{ds}(E_{\frac{1}{2}}(\lambda s^{\frac{1}{2}})) \begin{pmatrix} (2,4,6) \odot (\sqrt{\frac{t-s}{\pi}} e^{s-t} \ {}_1F_1(0.5, 1.5; t-s)) \\ 0 \end{pmatrix} ds. \end{aligned}$$

Hence, the ${}^{cf}[(ii) - gH]$-differentiable solution of Basset Eq.(5.3) is

$$y(t) = \tilde{2} e_{11} \oplus (\frac{2}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}) \odot \int_0^t q_{11}(\sqrt{t-s} e^{s-t} \ {}_1F_1(0.5, 1.5; t-s)) ds,$$

where $e_{11}(t)$ and $q_{11}(t)$ are the top left component of the matrix $E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}})$ and $\frac{d}{dt}(E_{\frac{1}{2}}(\lambda t^{\frac{1}{2}}))$, respectively. So, we obtain closed form of exact solution, $y(t) = \tilde{2} e^{-t}$, which is level-wise as $[y(t)]^r = [y_r^-, y_r^+] = [(1+r)e^{-t}, (3-r)e^{-t}]$ and pictured in Figure 2 .

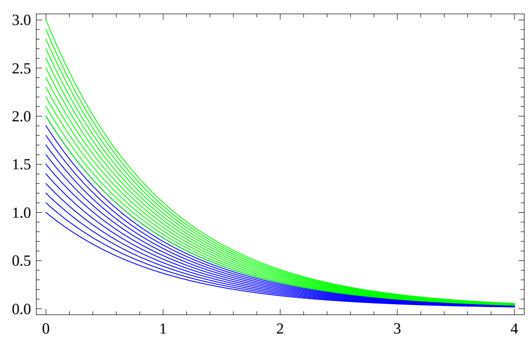


**Figure 2.** The level sets of the solution of Example 5.2

# 6. Conclusion

The Basset equation is introduced and studied under Caputo generalized Hukuhara differentiability. We converted the problem to the equivalent system of fuzzy ordinary differential equations of fractional order, for this, some properties of Caputo derivative are needed that proved with details. The solution is obtained in two concepts of differentiability by using presented method in [5] with this difference that we have used it in the matrix form.

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