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GLOBAL DYNAMICS IN A MULTI-GROUP EPIDEMIC MODEL FOR DISEASE WITH LATENCY SPREADING AND NONLINEAR TRANSMISSION RATE*

Haitao Song[1,†], Jinliang Wang[2] and Weihua Jiang[3]

Abstract In this paper, we investigate a class of multi-group epidemic models with general exposed distribution and nonlinear incidence rate. Under biologically motivated assumptions, we show that the global dynamics are completely determined by the basic production number R_0. The disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and there exists a unique endemic equilibrium which is globally asymptotically stable if $R_0 > 1$. The proofs of the main results exploit the persistence theory in dynamical system and a graph-theoretical approach to the method of Lyapunov functionals. A simpler case that assumes an identical natural death rate for all groups and a gamma distribution for exposed distribution is also considered. In addition, two numerical examples are showed to illustrate the results.

Keywords Multi-group epidemic model, Exposed distribution, Global stability, Lyapunov functional, Graph-theoretic approach.

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1. Introduction

Multi-group epidemic models have been used in the literature to describe the transmission dynamics of many different infectious diseases such as mumps, measles, gonorrhea and HIV/AIDS and vector borne diseases such as Malaria [33]. Since the difference of contact patterns, education levels, gender, age and mode of transmission, then the host population can be divided into several groups. They can also be formed geographically, such as by schools, communities and cities, or epidemiological, to incorporate differential infectivity or co-infection of multiple strains of the disease agent. So that within-group and inter-group interactions could be modeled separately.

It is well known that global dynamics of multi-group models with higher dimensions, especially the global stability of the endemic equilibrium, is a very challenging problem. There are many research papers about multi-group models, see

†the corresponding author. Email address:htsong@sxu.edu.cn(H. Song)
[1]Complex Systems Research Center, Shanxi University, Taiyuan, Shanxi 030006, P. R. China
[2]School of Mathematical Science, Heilongjiang University, Harbin, Heilongjiang 150080, P. R. China
[3]Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, P. R. China
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[1, 4, 6, 7, 9–11, 17, 21, 23, 27, 36, 37, 41, 42] and references therein. One of the earliest work on multi-group epidemic models was that by Lajmanovich and Yorke [17] where the transmission dynamics of gonorrhea for a class of SIS multi-group models was analyzed, and they proved the global stability of the unique endemic equilibrium by constructing Lyapunov functionals. Hethcote [9] investigated the global stability of the endemic equilibrium for a class of multi-group SIR model. The paper [7] proposed a graph-theoretic approach to the method of global Lyapunov functions and used it to establish the global stability of a unique endemic equilibrium for a multi-group SIR model with varying subpopulation sizes.

The global stability of an epidemic model with a gamma distribution for the latency in a heterogeneous host population was studied by Yuan and Zou [44]. Sun and Shi [32] considered a multi-group SEIR model with nonlinear incidence of infection and nonlinear removal functions between compartments and obtained global stability results. Motivated by the above two works, in this paper a more general multi-group epidemic model is proposed to describe the disease spread in a heterogeneous host population with general exposed distribution and nonlinear incidence rates. The host population is divided into n distinct groups. For $1 \leq k \leq n$, the k-th group is further partitioned into four compartments: the susceptible, exposed, infectious, and recovered, whose numbers of individuals at time t are denoted by $S_k(t), E_k(t), I_k(t)$ and $R_k(t)$, respectively. Susceptible individuals infected with the disease but not yet infective are in the exposed (latent) class. A fixed latent period can be considered as an approximation of the mean latent period, and this would be appropriate for those diseases whose latent periods vary only relatively slightly. For example, poliomyelitis has a latent period of 1-3 days (comparing to its much longer infectious period of 14-20 days), and hepatitis B has a latent period of 13-17 days (comparing to its infectious period of 19-22 days). However, disease such as tuberculosis including bovine tuberculosis (a disease spread from animal to animal mainly by direct contact) may take months to develop to the infectious stage, and also can relapse. Furthermore, since the time it takes from the moment of new infection to the moment of becoming infectious may be different from various diseases; even for the same disease, it differs in diverse individuals, it is indeed a random variable. It is thus of interest to investigate models with general exposed distribution in order to determine whether sustained oscillations can occur.

Following the method of [34], we use $P_k(t)$ (without taking death into account) to denote the probability that an exposed individual remains in the exposed class t time units after entering the exposed class. Assume that the disease does not cause deaths during the latent period, and we take the natural death rate into consideration. For $1 \leq k, j \leq n$, β_{kj} denotes the transmission coefficient between compartments S_k and I_j. We assume that β_{kj} is nonnegative and n-square matrix $(\beta_{kj})_{1\leq k,j\leq n}$ is irreducible [2], which implies that every pair of groups is joined by an infectious path so that the presence of an infectious individual in the first group can cause infection in the second group. The proportion of exposed individuals can be expressed by the integral

$$E_k(t) = \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} P_k(t-u) \mathrm{d}u, \tag{1.1}$$

where the sum takes into account cross-infections from all groups, integrals are in the Riemann-Stieltjes sense and $P_k(t)$ satisfies the following reasonable properties:

(*A*) $P_k : [0,\infty) \to [0,1]$ is non-increasing, piecewise continuous with possibly finitely many jumps and satisfy $P_k(0^+) = 1$, $\lim_{t\to\infty} P_k(t) = 0$ with $\int_0^\infty P_k(t)\mathrm{d}t$ is positive and finite.

Differentiating (1.1) leads to

$$\begin{aligned} E_k'(t) =& \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)) \\ &+ \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} P_k'(t-u)\mathrm{d}u - \delta_k E_k(t), \end{aligned} \tag{1.2}$$

where the first term on the right hand side in (1.2) is the rate at which new infected individuals come into the exposed class, and the last term explains the natural deaths. The second term denotes the rate at which the individuals move to the infectious class (noting that $P_k'(t-u) \le 0$ due to the above property) from the exposed class, hence

$$I_k'(t) = -\sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} P_k'(t-u)\mathrm{d}u - (\delta_k + \varepsilon_k + \gamma_k) I_k(t). \tag{1.3}$$

Let $g_k(t) = -P_k'(t)$, then equation (1.3) becomes

$$I_k'(t) = \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} g_k(t-u)\mathrm{d}u - (\delta_k + \varepsilon_k + \gamma_k) I_k(t). \tag{1.4}$$

In the k-th group, $\varphi_k(S_k)$ represents the intrinsic growth rate of S_k, which includes both the production and the natural death of susceptible individuals. On the basis of these assumptions, our general new multi-group epidemic model with group mixing and nonlinear incidence rates can be given using the following differential and integral equations:

$$\begin{cases} S_k'(t) = \varphi_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)), \\ E_k'(t) = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)) \\ \qquad\quad - \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} g_k(t-u)\mathrm{d}u - \delta_k E_k(t), \\ I_k'(t) = \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} g_k(t-u)\mathrm{d}u \\ \qquad\quad - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)), \\ R_k'(t) = \gamma_k h_k(I_k(t)) - \delta_k l_k(R_k(t)). \end{cases} \tag{1.5}$$

Since the variables $E_k(t)$ and $R_k(t)$, $i = 1, ..., n$, are decoupled from the $S_k(t)$ and $I_k(t)$ equations, we can consider the sub-system of (1.5) consisting of the S_k

and I_k equations:

$$\begin{cases} S_k'(t) = \varphi_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)), \\ I_k'(t) = \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} g_k(t-u)\mathrm{d}u \\ \qquad - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)), \end{cases} \tag{1.6}$$

where δ_k denotes the natural death rates of I_k compartments in the k-th group, ε_k is the death rate caused by disease in the k-th group and γ_k is the rate of recovery of infectious individuals in the k-th group. $(\delta_k + \varepsilon_k + \gamma_k)h_k(I_k)$ denote the removal of the infectious classes in the k-th group. Assume that all constants are nonnegative and δ_k, ε_k, γ_k are positive for $k = 1, 2, \ldots, n$. In what follows, we investigate the global stability of system (1.6). More precisely, we will identify the basic reproduction number R_0 of the system (1.6), and show that R_0 completely determines the global dynamics of the system (1.6), as stated in Theorem 2.1 and Theorem 2.2. Our proof for the global stability of endemic equilibrium uses global Lyapunov functionals (see [13–16,24,25,30,31,38]) and the graph-theoretic approach (see [6, 7, 20]).

The paper is organized as follows. In section 2, we obtain the global dynamics are completely determined by the basic production number R_0. If $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable and the disease dies out; if $R_0 > 1$, there exists a unique endemic equilibrium which is globally asymptotically stable and the disease persists at the endemic equilibrium. In section 3, we investigate one case for gamma distribution. In section 4, some numerical simulations are showed to support the results. A brief conclusion ends the paper.

2. Main results

On the basis of biological considerations, we make the following assumptions for the intrinsic growth rate of susceptible individuals in the k-th group $\varphi_k(S_k)$, and the transmission functions $h_k(I_k)$ and $f_{kj}(S_k, I_j)$.

(G_1) h_k are local Lipschitz on $[0, \infty)$ with $h_k(0) = 0$. h_k are continuous and positive on $(0, \infty)$, the function $\frac{\xi}{h_k(\xi)}$ is non-increasing on $(0, \infty)$, and $\lim_{\xi \to 0^+} \frac{\xi}{h_k(\xi)} = \sigma_k$ for positive constant $\sigma_k > 0$.

(G_2) φ_k are C^1 non-increasing functions on $[0, \infty)$ with $\varphi_k(0) > 0$, and there is a unique positive solution $\xi = S_k^0$ for the equation $\varphi_k(\xi) = 0$. $\varphi_k(S) > 0$ for $0 \leq S < S_k^0$, and $\varphi_k(S) < 0$ for $S > S_k^0$.

(G_3) There exist positive constant A_k, B_k and C_k satisfying

$$\xi \geq \frac{1}{\delta_k} \max_{\eta \in [0, S_k^0]} \{\varphi_k(\eta)\}, \quad \xi \geq A_k, \quad l_k(\xi) \geq \frac{1}{\delta_k} \max_{\eta \in [0, S_k^0]} \{\varphi_k(\eta)\}, \quad \xi \geq C_k,$$

$$h_k(\xi) \geq \frac{Q_k}{\delta_k + \varepsilon_k + \gamma_k} \max_{\eta \in [0, S_k^0]} \{\varphi_k(\eta)\}, \quad \xi \geq B_k,$$

where

$$J(\xi) = \int_\xi^\infty g_k(u) e^{-\delta_k u}\mathrm{d}u \quad \text{and} \quad Q_k = J(0) = \int_0^\infty g_k(u) e^{-\delta_k u}\mathrm{d}u.$$

It can be verified that $Q_j \in (0,1)$. Assume that the functions $f_{kj}(S_k, I_j)$ satisfy

(H_1) $0 < \lim_{I_j \to 0^+} \frac{f_{kj}(S_k,I_j)}{I_j} = C_{kj}(S_k) \leq +\infty$ for $0 < S_k \leq S_k^0$,

(H_2) $f_{kj}(S_k, I_j) \leq C_{kj}(S_k) I_j$ for all $I_j > 0$,

(H_3) $C_{kj}(S_k) \leq C_{kj}(S_k^0)$ for $0 < S_k \leq S_k^0$, $k, j = 1, 2, \ldots, n$.

Moreover, we assume that

(G_4) $\int_{0^+}^{1} \frac{1}{q(x)} \mathrm{d}x = +\infty$, for $q \in \{C_{kk}, h_k\}, k = 1, 2, \ldots, n$.

Observe that the common forms of $\varphi_k(S_k)$ satisfying (G_2) are

$$\varphi_k(S_k) = \Lambda_k - \delta_k S_k + r_k S_k (1 - \frac{S_k}{N_k}) \ \text{ and } \ \varphi_k(S_k) = \Lambda_k - \delta_k S_k,$$

which have been widely used in many papers, see [6,28,40,42] and references therein. The forms of $f_{kj}(S_k, I_j)$ satisfying assumptions $(H_1) - (H_3)$ include common transmission functions (see [7,28,39,42–44]) such as

$$\begin{aligned} f_{kj}(S_k, I_j) &= S_k I_j, \\ f_{kj}(S_k, I_j) &= S_k^q I_j, \\ f_{kj}(S_k, I_j) &= \frac{p S_k I_j}{1 + \alpha I_j^2}, \\ f_{kj}(S_k, I_j) &= C_k(S_k) F_j(I_j). \end{aligned}$$

From the equations of (1.5), we obtain that

$$\begin{aligned} (S_k + E_k)' = \ & \varphi_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} \int_0^t f_{kj}(S_k(u), I_j(u)) e^{-\delta_k(t-u)} g_k(t-u) \mathrm{d}u - \delta_k E_k(t) \\ \leq \ & \varphi_k(S_k(t)) - \delta_k E_k(t) \leq \max_{u \in [0, S_k^0]} \varphi(u) - \delta_k E_k(t) \leq 0, \text{ if } E_k \geq A_k, \\ (S_k + E_k + I_k)' = \ & \varphi_k(S_k(t)) - \delta_k E_k(t) - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)) \\ \leq \ & \varphi_k(S_k(t)) - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)) \\ \leq \ & \max_{u \in [0, S_k^0]} \varphi(u) - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)) \leq 0, \text{ if } I_k \geq B_k \end{aligned}$$

and

$$\begin{aligned} (S_k + E_k + I_k + R_k)' = \ & \varphi_k(S_k(t)) - \delta_k E_k(t) - (\delta_k + \varepsilon_k) h_k(I_k(t)) - \delta_k l_k(R_k(t)) \\ \leq \ & \varphi_k(S_k(t)) - \delta_k l_k(R_k(t)) \\ \leq \ & \max_{u \in [0, S_k^0]} \varphi(u) - \delta_k l_k(R_k(t)) \leq 0, \quad \text{if } R_k \geq C_k. \end{aligned}$$

Denote

$$\begin{aligned} D = \ & \big\{ (S_k, I_k) \in \mathbb{R}_+^{2n} : S_k \leq S_k^0, S_k + E_k \leq S_k^0 + A_k, S_k + E_k + I_k \leq S_k^0 + B_k, \\ & S_k + E_k + I_k + R_k \leq S_k^0 + C_k, k = 1, 2, \ldots, n \big\} \end{aligned}$$

is the feasible region for system (1.6), which is positively invariant with respect to (1.6). Int$D$ denotes the interior of D.

It is clear that system (1.6) has a disease-free equilibrium $P_0 = (S_1^0, 0, S_1^0, 0, \ldots, S_n^0, 0)$. For finite time t, system (1.6) may not have an endemic equilibrium. According to [26], if system (1.6) has an endemic equilibrium, the endemic equilibrium must satisfy the limiting system

$$\begin{cases} S_k'(t) = \varphi_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)), \\ I_k'(t) = \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k(t-u), I_j(t-u)) e^{-\delta_k u} g_k(u) \mathrm{d}u \\ \qquad - (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)). \end{cases} \tag{2.1}$$

Note that system (2.1) has an infinite delay, then its initial conditions are restricted in an appropriate fading memory space. For all $\lambda_k \in (0, \delta_k)$, define (see [8, 12])

$$O_k = \big\{\varphi \in C((-\infty, 0], \mathbb{R}) : \varphi(s)e^{\lambda_k s} \text{ is uniformly continuous in } (-\infty, 0] \text{ and } \sup_{s\le 0} |\varphi(s)|e^{\lambda_k s} < \infty\big\}$$

and $Y = \{\varphi_k \in O_k : \varphi_k(s) \ge 0 \text{ for all } s \le 0\}$ with $\| \varphi \|_k = \sup_{s\le 0} |\varphi(s)|e^{\lambda_k s}$. It can be easily verified that O_k is positively invariant with respect to system (2.1). Let $\mathrm{Int}O_k$ be the interior of O_k. Let $\varphi \in O_k$ be such that $\varphi_t(s) = \varphi(t+s), s \in (-\infty, 0]$. Assume $\varphi_k \in O_k$ and $S_k^0 \in \mathbb{R}_+$ satisfy that $\varphi_k(s) \ge 0, s \in (-\infty, 0]$. We consider the solutions of system (2.1) with initial conditions

$$S_k^0 = S_k(0), I_k^0 = \varphi_k, k = 1, 2, \ldots, n. \tag{2.2}$$

From the standard theory of functional differential equations [8], we have $I_k(t) \in O_k$ for $t > 0$. Thus, we will consider system (2.1) in $\Omega = \Pi_{k=1}^n (\mathbb{R} \times O_k)$. It can be verified that the solutions of system (2.1) with initial conditions (2.2) are nonnegative.

An equilibrium $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ in $\mathrm{Int}O_k$ is an endemic equilibrium of system (2.1), where $S_k^*, I_k^* > 0$ satisfy the equilibrium equations

$$\begin{aligned} &\varphi_k(S_k^*) = \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*), \\ &\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) Q_k = (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k^*). \end{aligned} \tag{2.3}$$

Under the biologically reasonable conditions, we will show that the endemic equilibrium P^* is unique. Let $R_0 = \rho(M^0)$ denote the special radius of the matrix M^0, where

$$M^0 = \left(\frac{\beta_{kj} \sigma_j C_{kj}(S_k^0) Q_k}{\delta_k + \varepsilon_k + \gamma_k} \right)_{n\times n}.$$

If $C_{kj}(S_k^0) = +\infty$ for some k and j, we let $R_0 = +\infty$. The parameter R_0 is referred to as the basic reproduction number, which is defined as the expected number of infected but non-infectious individuals produced in an entirely susceptible population by a typical infected individual during its entire infectious period [4]. We have the following result:

Theorem 2.1. *Assume that the functions φ_k, h_k and f_{kj} satisfy assumptions $(G_1)-(G_4)$ and $(H_1)-(H_3)$, and the matrix $B=(\beta_{kj})_{n\times n}$ is irreducible.*

(i) *If $R_0\le 1$, then P_0 is the unique equilibrium of system (1.6), and P_0 is globally asymptotically stable in D.*

(ii) *If $R_0>1$, then P_0 is unstable and system (1.6) is uniformly persistent in $IntD$.*

Proof. Let $Z(S,I)=\left(\frac{\beta_{kj}C_{kj}(S_k)I_jQ_k}{(\delta_k+\varepsilon_k+\gamma_k)h_j(I_j)}\right)_{n\times n}$, where $S=(S_1,S_2,\dots,S_n)$ and $I=(I_1,I_2,\dots,I_n)$. Since $B=(\beta_{kj})_{n\times n}$ is irreducible, then the matrix M^0 is also irreducible. By the theory of nonnegative matrices [3], $\rho(M^0)$ is an eigenvalue of M^0, then there exists a positive eigenvector $(\omega_1,\omega_2,\dots,\omega_n)$ such that $(\omega_1,\omega_2,\dots,\omega_n)\rho(M^0)=(\omega_1,\omega_2,\dots,\omega_n)M^0$. Now we construct a Lyapunov functional

$$V_0=\sum_{k=1}^{n}\frac{\omega_k}{\delta_k+\varepsilon_k+\gamma_k}I_k.$$

Differentiating V_0 along the solutions of system (1.6), we have

$$\begin{aligned}
V_0' &= \sum_{k=1}^{n}\omega_k\Big[\frac{1}{\delta_k+\varepsilon_k+\gamma_k}\sum_{j=1}^{n}\beta_{kj}\int_0^t f_{kj}(S_k(u),I_j(u))e^{-\delta_k(t-u)}g_k(t-u)\mathrm{d}u\\
&\quad -h_k(I_k(t))\Big]\\
&\le \sum_{k=1}^{n}\omega_k\Big[\frac{1}{\delta_k+\varepsilon_k+\gamma_k}\sum_{j=1}^{n}\beta_{kj}C_{kj}(S_k)I_jQ_k-h_k(I_k(t))\Big]\\
&= (\omega_1,\omega_2,\dots,\omega_n)\big[Z(S,I)h(I)-h(I)\big]\\
&\le (\omega_1,\omega_2,\dots,\omega_n)\left[M^0h(I)-h(I)\right]\\
&= [\rho(M^0)-1](\omega_1,\omega_2,\dots,\omega_n)h(I),
\end{aligned}$$

where $h(I)=(h_1(I_1),h_2(I_2),\dots,h_n(I_n))$. Thus, if $R_0=\rho(M^0)\le 1$, then $V_0'\le 0$, and $V_0'=0$ if and only if $I=0$ and $S=S^0=(S_1^0,S_2^0,\dots,S_n^0)$. It can be verified that for system (1.6), the only compact invariant subset of the set where $V_0'=0$ is the singleton $\{P_0\}$. By LaSalle's Invariance Principle [18], if $R_0=\rho(M^0)\le 1$, P_0 is globally asymptotically stable in D.

Note that

$$M^0=\left(\frac{\beta_{kj}Q_k}{\delta_k+\varepsilon_k+\gamma_k}\lim_{I_j\to 0^+}\frac{f_{kj}(S_k^0,I_j)}{I_j}\lim_{I_k\to 0^+}\frac{I_k}{h_k(I_k)}\right)_{n\times n}.$$

If $R_0=\rho(M^0)>1$, then

$$(\omega_1,\omega_2,\dots,\omega_n)M^0-(\omega_1,\omega_2,\dots,\omega_n)=[\rho(M^0)-1](\omega_1,\omega_2,\dots,\omega_n)>0,$$

and then, by continuity, we can obtain

$$\begin{aligned}
V_0' =& \sum_{k=1}^{n}\omega_k\Big[\frac{1}{\delta_k+\varepsilon_k+\gamma_k}\sum_{j=1}^{n}\beta_{kj}\int_0^t f_{kj}(S_k(u),I_j(u))e^{-\delta_k(t-u)}g_k(t-u)\mathrm{d}u\\
&-h_k(I_k(t))\Big]>0
\end{aligned}$$

in a neighborhood of P_0 in $IntD$, then P_0 is unstable.

Assume $R_0 = \rho(M^0) > 1$. By the uniform persistence result from [5] and a similar argument as in the proof of [19, Proposition 3.3], then P_0 is unstable implying that system (1.6) is uniformly persistent when $R_0 > 1$. The proof is completed. □

The uniform persistence of system (1.6) and the uniform boundness of solutions of system (1.6) in Int$D$, imply that system (1.6) has at least an endemic equilibrium $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$, S_k^* *and* $I_k^* > 0$ *for* $1 \le k \le n$. In what follows we prove that the endemic equilibrium P^* of system (1.6) is globally asymptotically stable when $R_0 > 1$.

Throughout the paper, we denote $H(z) = z - 1 - \ln z$, then $H(z) \ge 0$ for $z > 0$, and has global minimum at $z = 1$. In order to establish the global stability of P^*, we make the following assumptions

(G_5) $(h_k(I_k) - h_k(I_k^*))(I_k - I_k^*) > 0$ for $I_k \ge 0$ and $I_k \ne I_k^*$,

(G_6) $(\varphi_k(S_k) - \varphi_k(S_k^*))(S_k - S_k^*) < 0$ for $S_k \ge 0$ and $S_k \ne S_k^*$.

Remark 2.1. Condition (G_5) holds if $h_k(I_k)$ is strictly monotonically increasing with respect to I_k, and condition (G_6) holds if $\varphi_k(S_k)$ is strictly monotonically decreasing with respect to S_k. However, the monotonicity of functions h_k and φ_k are not necessary for $(G_5)-(G_6)$ hold. For example, $\varphi_k(S_k) = \Lambda_k - \delta_k S_k + r_k S_k(1 - \frac{S_k}{N_k})$ is not strictly monotone but (G_6) is satisfied.

Two difficult mathematical questions for system (1.6) are that whether the endemic equilibrium P^* is unique when $R_0 > 1$, and whether P^* is globally asymptotically stable when it is unique. Now the following theorem can answer the questions.

Theorem 2.2. *Consider system (2.1). Assume that* $(G_1) - (G_6)$ *and* $(H_1) - (H_3)$ *hold, and the matrix* $B = (\beta_{ij})_{n\times n}$ *is irreducible. If* $R_0 > 1$ *and*

(H_4) $(\varphi_k(S_k) - \varphi_k(S_k^*))[f_{kk}(S_k, I_k^*) - f_{kk}(S_k^*, I_k^*)] < 0$ *for* $S_k \ne S_k^*$,

(H_5) $\left(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}{f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)} - 1\right)\left(1 - \frac{h_j(I_j) f_{kk}(S_k, I_k^*) f_{kj}(S_k^*, I_j^*)}{h_j(I_j^*) f_{kk}(S_k^*, I_k^*) f_{kj}(S_k, I_j)}\right) \le 0$ *for* $S_k, I_j > 0$

hold, then there is a unique endemic equilibrium P^* *for system (2.1), and* P^* *is globally asymptotically stable in* $IntO_k$. *Consequently, all solutions of the system (1.6) in* $IntD$ *approach to the endemic equilibrium of the limiting system (2.1) when* $R_0 > 1$.

Proof. We show that P^* is globally asymptotically stable in IntO_k, which implies that there exists a unique endemic equilibrium. Consider the Lyapunov functional as

$$V_k = Q_k \int_{S_k^*}^{S_k(t)} \frac{f_{kk}(\xi, I_k^*) - f_{kk}(S_k^*, I_k^*)}{f_{kk}(\xi, I_k^*)} \mathrm{d}\xi + \int_{I_k^*}^{I_k(t)} \frac{h_k(\tau) - h_k(I_k^*)}{h_k(\tau)} \mathrm{d}\tau + V_+,$$

where

$$V_+ = \sum_{j=1}^{n} \beta_{kj} \int_0^\infty f_{kj}(S_k^*, I_j^*) J(u) H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)}\right) \mathrm{d}u.$$

First, calculating the derivatives of V_+, we obtain

$$\begin{aligned}
V_+' =& \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) J(u) \frac{\mathrm{d}}{\mathrm{d}t} H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)}\right) \mathrm{d}u \\
=& -\sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) J(u) \frac{\mathrm{d}}{\mathrm{d}u} H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)}\right) \mathrm{d}u \\
=& -\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) J(u) H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)}\right) |_{u=0}^{\infty} \\
&+ \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) H\left(\frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)}\right) \mathrm{d}J(u) \\
=& \sum_{j=1}^{n} Q_k \beta_{kj} \left[f_{kj}(S_k(t), I_j(t)) - f_{kj}(S_k^*, I_j^*) \ln \frac{f_{kj}(S_k(t), I_j(t))}{f_{kj}(S_k^*, I_j^*)} \right] \\
&- \sum_{j=1}^{n} \beta_{kj} g_k(u) e^{-\delta_k u} f_{kj}(S_k(t-u), I_j(t-u)) \mathrm{d}u \\
&+ \sum_{j=1}^{n} \beta_{kj} g_k(u) e^{-\delta_k u} f_{kj}(S_k^*, I_j^*) \ln \frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k^*, I_j^*)} \mathrm{d}u.
\end{aligned}$$

Calculating the time derivative of V_k along the solutions of system (2.1) and using equilibrium equations (2.3), we obtain

$$\begin{aligned}
V_k' =& Q_k \Big(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)}\Big) \Big[\varphi_k(S_k(t)) - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t))\Big] \\
&+ \Big(1 - \frac{h_k(I_k^*)}{h_k(I_k(t))}\Big) \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k(u), I_j(u)) e^{-\delta_k (t-u)} g_k(t-u) \mathrm{d}u \\
&- \Big(1 - \frac{h_k(I_k^*)}{h_k(I_k(t))}\Big) (\delta_k + \varepsilon_k + \gamma_k) h_k(I_k(t)) + V_+' \\
=& Q_k (\varphi_k(S_k(t)) - \varphi_k(S_k^*)) \Big(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)}\Big) \\
&+ \sum_{j=1}^{n} Q_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[2 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)} \\
&+ \frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k(t), I_j(t))}{f_{kk}(S_k(t), I_k^*) f_{kj}(S_k^*, I_j^*)} - \frac{h_k(I_k(t))}{h_k(I_k^*)}\Big] \\
&- \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) g_k(u) e^{-\delta_k u} \frac{h_k(I_k^*) f_{kj}(S_k(t-u), I_j(t-u))}{h_k(I_k(t)) f_{kj}(S_k^*, I_j^*)} \mathrm{d}u \\
&+ \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) g_k(u) e^{-\delta_k u} \ln \frac{f_{kj}(S_k(t-u), I_j(t-u))}{f_{kj}(S_k(t), I_j(t))} \mathrm{d}u
\end{aligned}$$

Therefore, we have

$$V_k' = Q_k (\varphi_k(S_k(t)) - \varphi_k(S_k^*)) \Big(1 - \frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)}\Big)$$

$$
\begin{aligned}
&- Q_k \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[H\Big(\frac{f_{kk}(S_k^*, I_k^*)}{f_{kk}(S_k(t), I_k^*)} \Big) \\
&+ H\Big(\frac{f_{kk}(S_k(t), I_k^*) f_{kj}(S_k^*, I_j^*) h_j(I_j(t))}{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k(t), I_j(t)) h_j(I_j^*)} \Big) \Big] \\
&+ Q_k \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big(\frac{f_{kk}(S_k^*, I_k^*) f_{kj}(S_k(t), I_j(t))}{f_{kk}(S_k(t), I_k^*) f_{kj}(S_k^*, I_j^*)} - 1 \Big) \\
&\Big(1 - \frac{h_j(I_j(t)) f_{kk}(S_k(t), I_k^*) f_{kj}(S_k^*, I_j^*)}{h_j(I_j^*) f_{kk}(S_k^*, I_k^*) f_{kj}(S_k(t), I_j(t))} \Big) \\
&- \sum_{j=1}^{n} \beta_{kj} \int_0^{\infty} f_{kj}(S_k^*, I_j^*) g_k(u) e^{-\delta_k u} H\Big(\frac{h_k(I_k^*) f_{kj}(S_k(t-u), I_j(t-u))}{h_k(I_k) f_{kj}(S_k^*, I_j^*)} \Big) \mathrm{d}u \\
&+ Q_k \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[\frac{h_j(I_j)}{h_j(I_j^*)} - \frac{h_k(I_k)}{h_k(I_k^*)} - \ln \frac{h_j(I_j)}{h_j(I_j^*)} + \ln \frac{h_k(I_k)}{h_k(I_k^*)} \Big].
\end{aligned}
$$

Thus, by (H_4) and (H_5), we have

$$
V_k' \le \sum_{j=1}^{n} Q_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \left[\frac{h_j(I_j)}{h_j(I_j^*)} - \frac{h_k(I_k)}{h_k(I_k^*)} - \ln \frac{h_j(I_j)}{h_j(I_j^*)} + \ln \frac{h_k(I_k)}{h_k(I_k^*)} \right].
$$

Set $\bar{\beta}_{kj} = \beta_{kj} f_{kj}(S_k^*, I_j^*), 1 \le k, j \le n$, and

$$
\bar{B} = \begin{pmatrix}
\sum_{l \ne 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{nl} \\
-\bar{\beta}_{12} & \sum_{l \ne 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l \ne n} \bar{\beta}_{nl}
\end{pmatrix}.
$$

Since $B = (\beta_{kj})_{n \times n}$ is irreducible, then $\bar{B}$ is also irreducible. By [7, Lemma 2.1], we know that the linear system

$$
\bar{B} \upsilon = 0
$$

has a positive solution $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ and $(\upsilon_1, \upsilon_2, \ldots, \upsilon_n) = (C_{11}, C_{22}, \ldots, C_{nn})$, where $C_{kk} > 0 (k = 1, 2, \ldots, n)$ is the cofactor of the k-th diagonal entry of $\bar{B}$.

Set $V = \sum_{k=1}^{n} \upsilon_k V_k$, then

$$
\begin{aligned}
&\sum_{k=1}^{n} \upsilon_k V_k' \le \sum_{j=1}^{n} Q_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[\frac{h_j(I_j)}{h_j(I_j^*)} - \frac{h_k(I_k)}{h_k(I_k^*)} - \ln \frac{h_j(I_j)}{h_j(I_j^*)} + \ln \frac{h_k(I_k)}{h_k(I_k^*)} \Big] \\
&= \sum_{j=1}^{n} Q_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[\frac{h_j(I_j)}{h_j(I_j^*)} - \frac{h_k(I_k)}{h_k(I_k^*)} \Big] - \sum_{j=1}^{n} Q_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \Big[\ln \frac{h_j(I_j) h_k(I_k^*)}{h_j(I_j^*) h_k(I_k)} \Big] \\
&=: H_1 - H_2.
\end{aligned}
$$

First, we show that $H_1 = 0$ for $I_1, I_2, \ldots, I_n > 0$. From $\bar{B} \upsilon = 0$, we have

$$
\sum_{j=1}^{n} \bar{\beta}_{jk} \upsilon_j = \sum_{i=1}^{n} \bar{\beta}_{ki} \upsilon_k
$$

and using $\bar{\beta}_{jk} = \beta_{jk} f_{jk}(S_i^*, I_j^*)$, we obtain

$$\sum_{j=1}^{n} \beta_{jk} f_{jk}(S_j^*, I_k^*) \upsilon_j = \sum_{i=1}^{n} \beta_{ki} f_{ki}(S_k^*, I_i^*) \upsilon_k, k = 1, 2, \ldots, n.$$

This implies that

$$\begin{aligned}
\sum_{k,j=1}^{n} \upsilon_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \frac{h_j(I_j)}{h_j(I_j^*)} &= \sum_{k=1}^{n} \frac{h_k(I_k)}{h_k(I_k^*)} \sum_{j=1}^{n} \beta_{jk} f_{jk}(S_j^*, I_k^*) \upsilon_j \\
&= \sum_{k=1}^{n} \frac{h_k(I_k)}{h_k(I_k^*)} \sum_{i=1}^{n} \beta_{ki} f_{ki}(S_k^*, I_i^*) \upsilon_k \\
&= \sum_{k,j=1}^{n} \upsilon_k \beta_{kj} f_{kj}(S_k^*, I_j^*) \frac{h_k(I_k)}{h_k(I_k^*)},
\end{aligned}$$

then $H_1 = 0$ for $I_1, I_2, \ldots, I_n > 0$. Next we prove $H_2 = 0$ for $I_1, I_2, \ldots, I_n > 0$.

Let G represent the directed graph associated with matrix $(\bar{\beta}_{kj})_{n\times n}$. Then G has vertices $1, 2, \ldots, n$ with a directed arc (k, j) from k to j iff $\bar{\beta}_{kj} \neq 0$. Then $E(G)$ is the set of all directed arcs of G. By Kirchhoff's Matrix-Tree Theorem (see [7, Lemma 2.1]) we know that $\upsilon_k = C_{kk}$ can be expressed as a sum of weights of all directed spanning subtrees T of G that are rooted at vertex k. Thus, each term in $\upsilon_k \bar{\beta}_{kj}$ is the weight $\omega(Q)$ of a unicyclic subgraph Q of G obtained from such a tree T by adding a directed arc (k, j) from the root k to vertex j. Since the arc (k, j) is a part of the unique cycle CQ of Q, and that the same unicyclic graph Q can be formed when each arc of CQ is added to a corresponding rooted tree T. Then the double sum in H_2 can be expressed as a sum over all unicyclic subgraphs Q containing vertices $1, 2, \ldots, n$. Thus, $H_2 = \sum_Q H_Q$, where

$$H_Q = \omega_Q \sum_{(k,j)\in E(CQ)} \ln \frac{h_j(I_j) h_k(I_k^*)}{h_j(I_j^*) h_k(I_k)} = \omega_Q \ln \left(\prod_{(k,j)\in E(CQ)} \frac{h_j(I_j) h_k(I_k^*)}{h_j(I_j^*) h_k(I_k)} \right).$$

Note that $E(CQ)$ is the set of arcs of a cycle CQ, we can obtain

$$\prod_{(k,j)\in E(CQ)} \frac{h_j(I_j) h_k(I_k^*)}{h_j(I_j^*) h_k(I_k)} = 1, \text{ and then } \ln \left(\prod_{(k,j)\in E(CQ)} \frac{h_j(I_j) h_k(I_k^*)}{h_j(I_j^*) h_k(I_k)} \right) = 0.$$

Thus, $H_Q = 0$ for each Q, then $H_2 = 0$ for $I_1, I_2, \ldots, I_n > 0$. The conditions (H_4) and (H_5) imply that $V' \leq 0$ for all $(S_1, I_1(\cdot), S_2, I_2(\cdot), \ldots, S_n, I_n(\cdot)) \in IntD$. Then we know that the largest invariant subset of the set where $V' = 0$ is the singleton $\{P^*\}$(see [7]). Using LaSalle's Invariance Principle [18], we can show that P^* is globally asymptotically stable in $\mathrm{Int}O_k$ when $R_0 > 1$. An immediate consequence of Theorem 7.2 in [26] is that P^* attracts all solutions of the system (1.6) in $IntD$. The proof is completed. □

Remark 2.2. Condition (H_4) holds if $f_{kk}(S_k, I_k^*)$ is strictly increasing with respect to $S_k > 0$.

Remark 2.3. In the special case $f_{kj}(S_k, I_j) = p_k(S_k)q_j(I_j)$ and $h_j(I_j) = a_j I_j$, condition (H_5) becomes

$$\left(\frac{q_j(I_j)}{q_j(I_j^*)} - 1\right)\left(1 - \frac{I_j q_j(I_j^*)}{I_j^* q_j(I_j)}\right) \leq 0, \quad \text{for} \quad I_j > 0. \tag{2.4}$$

Therefore, (2.4) holds if $q_j(I_j)$ is monotonically increasing and concave down, which is satisfied by nonlinear functions with the forms of $q_j(I_j) = I_j^{p_j}$ and $\frac{I_j}{1+\alpha I_j^2}$ for $0 < p_j \leq 1$ and $\alpha > 0$ widely used in many papers of epidemic modeling. In this case, conditions like (2.4) are subjected only on $q_j(I_j)$.

Remark 2.4. In the case of $f_{kj}(S_k, I_j) = S_k I_j$, system (1.6) will reduce to the system studied in [44]. Theorem 2.2 generalizes the global stability results in [44] to nonlinear incidence rates.

Remark 2.5. In the special case $f_{kj}(S_k, I_j) = \beta_{kj} S_k I_j$ and $h_k(I_k) = I_k$, system (1.6) becomes the multi-group SEIR model studied in [28], Theorem 2.2 generalizes Theorem 3.3 in [28].

3. One case for Gamma distribution

To get the main result more conveniently, we suppose that all groups have the same natural death rate $\delta_k = \delta$ for $k = 1, 2, \ldots, n$. We assume $g_k(s) = g(s)$ for $k = 1, 2, \ldots, n$. Furthermore, we choose the gamma distribution

$$g(s) = g_{n,b}(s) = \frac{s^{n-1}}{(n-1)!b^n} e^{-\frac{s}{b}},$$

which is widely used and can approximate several frequently used distributions. For example, when $b \to 0^+$, $g_{n,b}(s)$ will approach the Dirac delta function; when $n = 1$, $g_{n,b}(s)$ is an exponentially decaying function. Moreover, we define $\hat{b} = b/(1 + \delta b)$ such that we can absorb the exponential term $e^{-\delta s}$ into the delay kernel. Then system (1.6) becomes

$$\begin{cases} S_k'(t) = \varphi_k(S_k(t)) - \sum\limits_{j=1}^{n} \beta_{kj} f_{kj}(S_k(t), I_j(t)), \\ I_k'(t) = \sum\limits_{j=1}^{n} \dfrac{\beta_{kj}}{(1+\delta b)^n} \displaystyle\int_0^t f_{kj}(S_k(u), I_j(u)) g_{n,\hat{b}}(t-u)\mathrm{d}u - (\delta + \varepsilon_k + \gamma_k) h_k(I_k(t)). \end{cases} \tag{3.1}$$

For $l = 1, 2, \ldots, n$, let

$$y_{k,l}(t) = \sum_{j=1}^{n} \frac{\beta_{kj}\hat{b}}{(1+\delta b)^n} \int_0^t f_{kj}(S_k(u), I_j(u)) g_{l,\hat{b}}(t-u)\mathrm{d}u, k = 1, 2, \ldots, n.$$

Thus, for $l \in 2, \ldots, n$, we have

$$y_{k,l}'(t) = g_{l,\hat{b}}(0) \sum_{j=1}^{n} \frac{\beta_{kj}\hat{b}}{(1+\delta b)^n} f_{kj}(S_k(t), I_j(t))$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{\beta_{kj}\hat{b}}{(1+\delta b)^n}\int_{-\infty}^{t}\frac{(l-1)(t-u)^{l-2}}{(l-1)!\hat{b}^l}e^{-(t-u)/\hat{b}}f_{kj}(S_k(u),I_j(u))\mathrm{d}u\\
&-\sum_{j=1}^{n}\frac{\beta_{kj}\hat{b}}{(1+\delta b)^n}\int_{-\infty}^{t}\frac{(t-u)^{l-1}}{(l-1)!\hat{b^{l+1}}}e^{-(t-u)/\hat{b}}f_{kj}(S_k(u),I_j(u))\mathrm{d}u\\
=&[y_{k,l-1}(t)-y_{k,l}(t)]/\hat{b}.
\end{aligned}
$$

For $l=1$, we obtain

$$
y_{k,1}(t)=\sum_{j=1}^{n}\frac{\beta_{kj}\hat{b}}{(1+\delta b)^n}\int_{-\infty}^{t}\frac{e^{-(t-u)/\hat{b}}}{\hat{b}}f_{kj}(S_k(u),I_j(u))\mathrm{d}u, k=1,2,\ldots,n,
$$

then

$$
y'_{k,1}(t)=\sum_{j=1}^{n}\frac{\beta_{kj}}{(1+\delta b)^n}f_{kj}(S_k(t),I_j(t))-\frac{1}{\hat{b}}y_{k,1}(t), k=1,2,\ldots,n.
$$

Then the integro-differential system (3.1) is equivalent to the ordinary differential equations

$$
\begin{cases}
S'_k(t)=\varphi_k(S_k(t))-\sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k(t),I_j(t)),\\
y'_{k,1}(t)=\frac{1}{(1+\delta b)^n}\sum_{j=1}^{n}\beta_{kj}f_{kj}(S_k(t),I_j(t))-\frac{1}{\hat{b}}y_{k,1}(t),\\
y'_{k,2}(t)=\frac{1}{\hat{b}}(y_{k,1}(t)-y_{k,2}(t)), \qquad k=1,2,\ldots,n.\\
\vdots\\
y'_{k,n}(t)=\frac{1}{\hat{b}}(y_{k,n-1}(t)-y_{k,n}(t)),\\
I'_k(t)=\frac{1}{\hat{b}}y_{k,n}(t)-(\delta+\varepsilon_k+\gamma_k)h_k(I_k(t)).
\end{cases}
\tag{3.2}
$$

By the Theorem 2.1 in [29], we can get that the corresponding solution remains nonnegative for a set of nonnegative initial conditions. Let the maximum of φ_k be N_{φ_k} on $\mathbb{R}_+$, and let $q>\hat{b}N_{\varphi_k}$. Denote by $Y_k=(S_k,y_{k,1},y_{k,2},\ldots,y_{k,n},I_k)$ is the k-th tube for the system (3.2). Define

$$
\begin{aligned}
D_0=\ &\Big\{(Y_1,Y_2,\ldots,Y_n)\in\mathbb{R}_+^{n(n+2)}: S_k\le S_k^0, S_k+(1+\delta b)^n y_{k,1}\le q+S_k^0,\\
&y_{k,1}\le\frac{q+S_k^0}{(1+\delta b)^n}, I_k\le\frac{q+S_k^0}{\hat{b}(1+\delta)^n(\delta+\varepsilon_k+\gamma_k)}, k=1,2,\ldots,n\Big\},
\end{aligned}
$$

which is the positively invariant set.

Thus, we can obtain that system (3.2) always has the disease-free equilibrium

$$
\bar{P^0}=(S_1^0,0,\ldots,0,S_2^0,0,\ldots,0,\ldots,0,S_n^0,0,\ldots,0)\in\mathbb{R}_+^{n(n+2)} \tag{3.3}
$$

and $\bar{P}^*$ corresponding to the equilibrium for (3.2)

$$
\bar{P^*}=(S_1^*,y_{1,1}^*,\ldots,y_{1,n}^*,I_1^*,S_2^*,y_{2,1}^*,\ldots,y_{2,n}^*,I_2^*,\ldots,S_n^*,y_{n,1}^*,\ldots,y_{n,n}^*,I_n^*)\in\mathbb{R}_+^{n(n+2)}. \tag{3.4}
$$

From the theory of [35], we can define the basic reproduction number as $\bar{R}_0 = \rho(\bar{M}^0)$ which is the spectral radius of the matrix $\bar{M}^0 = (\bar{M}^0_{ij})_{n\times n}$, where

$$\bar{M}^0_{ij} = \frac{\beta_{ij} C_{ij}(S_i^0)\sigma_j}{(1+\delta b)^n(\delta_i+\varepsilon_i+\gamma_i)}, \quad i,j = 1,2,\ldots,n.$$

By the similar to the proof of Section 2 and [44, Theorem 2.2], we can get the following result.

Theorem 3.1. *Assume that $B = (\beta_{ij})_{n\times n}$ is irreducible and $(G_1)-(G_4)$ and $(H_1)-(H_3)$ hold.*

(i) *The disease-free equilibrium $\bar{P}^0$ of system (3.2) is globally asymptotically stable on $\mathbb{R}_+^{n(n+2)}$ if $\bar{R}_0 \le 1$, and $\bar{P}^0$ is unstable if $\bar{R}_0 > 1$.*

(ii) *If $\bar{R}_0 > 1$ and assumptions $(G_5)-(G_6)$ and $(H_4)-(H_5)$ hold, there exists a unique endemic equilibrium $\bar{P}^*$ of system (3.2) which is globally asymptotically stable on $\mathbb{R}_+^{n(n+2)}$ excepted for initial conditions satisfying $I_k(0) = 0$ for $k = 1,2,\ldots,n$.*

4. Numerical examples

Consider the system (1.6) when $k = 2$, then we have a two-group model as follows:

$$\begin{cases} S_1' = \varphi_1(S_1) - [\beta_{11}f_{11}(S_1,I_1) + \beta_{12}f_{12}(S_1,I_2)], \\ I_1' = [Q_1\beta_{11}f_{11}(S_1,I_1) + Q_1\beta_{12}f_{12}(S_1,I_2)] - (\delta_1+\varepsilon_1+\gamma_1)h_1(I_1), \\ S_2' = \varphi_2(S_2) - [\beta_{21}f_{21}(S_2,I_1) + \beta_{22}f_{22}(S_2,I_2)], \\ I_2' = [Q_2\beta_{21}f_{21}(S_2,I_1) + Q_2\beta_{22}f_{22}(S_2,I_2)] - (\delta_2+\varepsilon_2+\gamma_2)h_2(I_2), \end{cases} \tag{4.1}$$

where $f_{kj}(S_k,I_j) = \frac{S_kI_j}{1+I_j}$ (see [22]), k, j=1, 2,

$$\varphi_1(u) = 1-u, \quad \varphi_2(u) = 2-u, \quad h_1(u) = h_2(u) = u \tag{4.2}$$

and

$$Q_1 = \frac{2}{3}, \quad \delta_1 = \frac{1}{4}, \quad \varepsilon_1 = \frac{1}{6}, \quad \gamma_1 = \frac{1}{2}, \quad Q_2 = \frac{2}{3}, \quad \delta_2 = \frac{1}{4}, \quad \varepsilon_2 = \frac{1}{5}, \quad \gamma_2 = \frac{2}{3}. \tag{4.3}$$

If β_{kj} are chosen as

$$\beta_{11} = \frac{5}{24}, \quad \beta_{12} = \frac{1}{24}, \quad \beta_{21} = \frac{5}{36}, \quad \beta_{22} = \frac{5}{36}. \tag{4.4}$$

Correspondingly, we have $R_0 \approx 0.2299 < 1$, then $P_0 = (1,0,2,0)$ is the unique equilibrium of the system (4.1) and it is globally asymptotically stable in D from Theorem 2.1 (see Fig. 1 left panel). In addition, if β_{kj} are chosen as

$$\beta_{11} = 4, \quad \beta_{12} = 1, \quad \beta_{21} = \frac{1}{2}, \quad \beta_{22} = 4. \tag{4.5}$$

Correspondingly, we have $R_0 \approx 3.0744 > 1$, then $P^* = (0.3710, 0.4574, 1.0000, 2.7844)$ is the unique endemic equilibrium of the system (4.1) and it is globally asymptotically stable in $IntD$ from Theorem 2.2 (see Fig. 1 right panel).

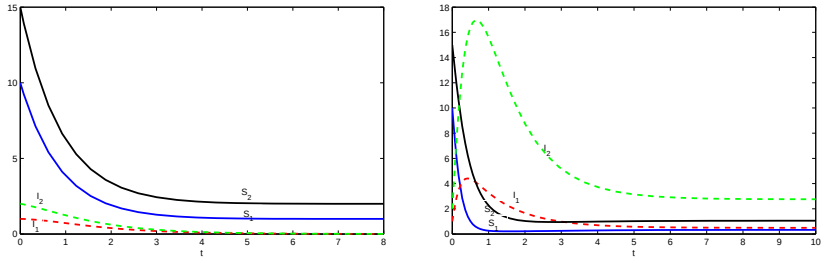

Figure 1. Numerical simulations of system (4.1) with functions (4.2) and parameters (4.3). (Left) β_{kj} as in (4.4), and P_0 is globally asymptotically stable; (Right) β_{kj} as in (4.5), and P^* is globally asymptotically stable. Initial conditions with $S_1(0) = 10$, $I_1(0) = 1$, $S_2(0) = 15$, $I_2(0) = 2$.

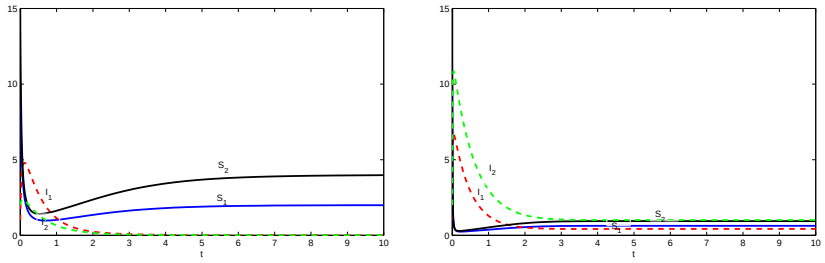

Figure 2. Numerical simulations of system (4.1) with functions (4.6) and parameters (4.7). (Left) β_{kj} as in (4.8), and P_0 is globally asymptotically stable; (Right) β_{kj} as in (4.9), and P^* is globally asymptotically stable. Initial conditions with $S_1(0) = 10$, $I_1(0) = 1$, $S_2(0) = 15$, $I_2(0) = 2$.

A second example, we consider system (4.1) with $f_{kj}(S_k, I_j) = S_k^2 I_j$(see [39]), k, j=1, 2,

$$\varphi_1(u) = 2 - u, \quad \varphi_2(u) = 4 - u, \quad h_1(u) = h_2(u) = u \tag{4.6}$$

and

$$Q_1 = \frac{3}{4}, \quad \delta_1 = 1, \varepsilon_1 = \frac{1}{6}, \quad \gamma_1 = 1, \quad Q_2 = \frac{3}{4}, \quad \delta_2 = \frac{1}{2}, \quad \varepsilon_2 = \frac{1}{3}, \quad \gamma_2 = 1. \tag{4.7}$$

If β_{kj} are chosen as

$$\beta_{11} = \frac{1}{24}, \quad \beta_{12} = 1, \quad \beta_{21} = 0, \quad \beta_{22} = \frac{1}{24}. \tag{4.8}$$

Correspondingly, we have $R_0 \approx 0.0682 < 1$, then $P_0 = (2, 0, 4, 0)$ is the unique equilibrium of the system (4.1) and it is globally asymptotically stable in D from Theorem 2.1 (see Fig. 2 left panel). In addition, if β_{kj} are chosen as

$$\beta_{11} = 1, \quad \beta_{12} = 3, \quad \beta_{21} = 1, \quad \beta_{22} = 2. \tag{4.9}$$

Correspondingly, we have $R_0 \approx 3.9650 > 1$, then $P^* = (0.6494, 0.4903, 0.7061, 1.0002)$ is the unique endemic equilibrium of the system (4.1) and it is globally asymptotically stable in $IntD$ from Theorem 2.2 (see Fig. 2 right panel).

5. Conclusions

In this paper, a class of multi-group epidemic models with exposed distribution and nonlinear incidence rates is investigated. Under biologically motivated conditions,

we show that the global dynamics are completely determined by the basic production number R_0. If $R_0 \leq 1$, the disease-free equilibrium is global asymptotically stable, and the disease will die out; if $R_0 > 1$, there exists a unique endemic equilibrium which is global asymptotically stable, and the disease will survive in each group. When $g_k(s), 1 \leq k \leq n$, are gamma distributions, we establish the global stability of the disease-free equilibrium and endemic equilibrium.

In the special case $g_k(s), 1 \leq k \leq n$, are gamma distributions and $f_{kj}(S_k, I_j)$ and $h_k(I_k)$ are linear functions, the global stability of the disease-free equilibrium and endemic equilibrium has been established in [44]. When $g_k(s), 1 \leq k \leq n$, are exponentially distributions, system (1.5) becomes the multi-group SEIR model, and the global dynamics have been established in [6, 7]. Theorem 2.1 and Theorem 2.2 extend these stability results to the case with general distributions $g_k(s)$ and nonlinear functions $f_{kj}(S_k, I_j)$ and $h_k(I_k)$. In the special case $f_{kj}(S_k, I_j) = \beta_{kj} S_k I_j$ and $h_k(I_k) = I_k$, system (1.5) becomes the multi-group SEIR model studied in [28], Theorem 2.2 generalizes Theorem 3.1 and Theorem 3.3 in [28].

Our methods can be used to consider the global dynamics of epidemic models and then can help Disease Control and Prevention Center to prevent and control diseases. When some of the assumptions $(G_1)-(G_6)$ and $(H_1)-(H_5)$ are invalid, it would be interesting to consider the multi-group model with nonlinear transmission rates.

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