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EXTREMAL SOLUTIONS FOR A NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH MULTI-ORDERS FRACTIONAL DERIVATIVES*

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Abstract In this paper, by employing the lower and upper solutions method, we give an existence theorem for the extremal solutions for a nonlinear impulsive differential equations with multi-orders fractional derivatives and integral boundary conditions. A new comparison result is also established.

Keywords Multi-orders fractional derivatives, Impulse, Multi-orders fractional integral boundary conditions, Nonlinear fractional differential equations, Extremal solutions.

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1. Introduction

In this paper, we study an integral boundary problem of nonlinear impulsive differential equations with fractional derivatives involving several orders and given by

$$\begin{cases} {}^CD^{\alpha_k}_{t_k^+}u(t) = f(t,u(t)), \qquad 1<\alpha_k\le 2, \quad k=0,1,2,\cdots,p, \quad t\in J', \\ \triangle u(t_k) = I_k(u(t_k)), \quad \triangle u'(t_k) = I_k^*(u(t_k)), \qquad k=1,2,\cdots,p, \\ u(0) = \sum_{k=0}^{p}\lambda_k \mathcal{J}^{\beta_k}_{t_k^+}u(\eta_k)+\kappa, \quad u'(0)=0, \qquad t_k<\eta_k<t_{k+1}, \end{cases} \tag{1.1}$$

where ${}^CD^{\alpha_k}_{t_k^+}$ is the Caputo fractional derivative of order α_k and $\mathcal{J}^{\beta_k}_{t_k^+}$ is fractional Riemann-Liouville integral of order $\beta_k>0$. $f\in C(J\times\mathbb{R},\mathbb{R})$, $I_k, I_k^*\in C(\mathbb{R},\mathbb{R})$. λ_k,η_k are positive constants. $J=[0,T](T>0)$, $\kappa\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_k<\cdots<t_p<t_{p+1}=T$, $J'=J\backslash\{t_1,t_2,\cdots,t_p\}$, $\triangle u(t_k)=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t=t_k(k=1,2,\cdots,p)$, respectively. $\triangle u'(t_k)$ have a similar meaning for $u'(t)$.

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Boundary value problems for fractional differential equations have considered by many authors recently. See, for example, [1–6,10,11,13,16–18,22] and the references therein. The concept of solution is based on [15] and we show the existence of extremal solutions by using the classical Monotone Iterative Technique [7,9,12,14, 19–21].

2. Preliminaries

Let us fix $J_0 = [0, t_1], J_{k-1} = (t_{k-1}, t_k],\ k = 2, \cdots, p+1$ with $t_{p+1} = T$ and introduce the Banach space:

$PC(J,\mathbb{R}) = \{u : J \to \mathbb{R}\ |u \in C(J_k),\ k = 0, 1, \cdots, p \text{ and } u(t_k^+) \text{ exist},\ k = 1, 2, \cdots, p\}$ with the norm $\|u\| = \sup_{t\in J} |u(t)|$.

$AC^n(J) = \{h : J \to \mathbb{R} : h, h', \cdots, h^{(n-1)} \in C(J,\mathbb{R}) \text{ and } h^{(n-1)} \text{ is absolutely continuous}\}$.

For the reader's convenience, we present some necessary definitions from fractional calculus theory and several important Lemmas.

Definition 2.1 ([8]). The Riemann-Liouville fractional integral of order α for a function $f \in L^1([d,\infty),\mathbb{R})$ is defined as

$$\mathcal{J}_{d^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_d^t (t-s)^{\alpha-1} f(s)ds,\ \alpha > 0,$$

provided the integral exists.

Definition 2.2 ([8]). The Caputo fractional derivative of order α for a function $f \in AC^n[d,\infty)$ is defined by

$${}^C D_{d^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_d^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α.

Lemma 2.1. *For a given $y \in C[0,T]$ and constants $\mathcal{I}_k, \mathcal{I}_k^* (k = 1, 2, \cdots, p.)$, the impulsive integral boundary value problem*

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} u(t) = y(t), \quad 1 < \alpha_k \le 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle u(t_k) = \mathcal{I}_k, \quad \triangle u'(t_k) = \mathcal{I}_k^*, \quad k = 1, 2, \cdots, p, \\ u(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u(\eta_k) + \kappa, \quad u'(0) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0-1} y(s)ds + \mathcal{A},\ t \in J_0; \\ \int_{t_k}^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} y(s)ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-1}}{\Gamma(\alpha_{i-1})} y(s)ds + \mathcal{I}_i \Big] \\ + \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_i^* \Big] \\ + \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_i^* \Big] + \mathcal{A}, \\ t \in J_k,\ k = 1, 2, \cdots, p, \end{cases} \tag{2.2}$$

where

$$\begin{aligned}\mathcal{A} =&\Big(1-\sum_{k=0}^{p}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\Big)^{-1}\times\Big\{\sum_{k=0}^{p}\lambda_k\int_{t_k}^{\eta_k}\frac{(\eta_k-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)}y(s)ds\\&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha_{i-1}-1}}{\Gamma(\alpha_{i-1})}y(s)ds+\mathcal{I}_i\Big]\\&+\sum_{k=1}^{p}\sum_{i=1}^{k-1}\frac{\lambda_k(\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)}y(s)ds+\mathcal{I}_i^*\Big]\\&+\sum_{k=1}^{p}\sum_{i=1}^{k}\frac{\lambda_k(\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)}\Big[\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)}y(s)ds+\mathcal{I}_i^*\Big]+\kappa\Big\}.\end{aligned}$$

Proof. Let u be a solution of (2.1). Then, by [8, Lemma 2.22, pp 96], for any $t\in J_0$, we have

$$u(t)=\mathcal{J}_0^{\alpha_0}y(t)-c_1-c_2t=\frac{1}{\Gamma(\alpha_0)}\int_0^t(t-s)^{\alpha_0-1}y(s)ds-c_1-c_2t,\quad t\in J_0,\tag{2.3}$$

for some $c_1,c_2\in\mathbb{R}$. Differentiating (2.3), we get

$$u'(t)=\frac{1}{\Gamma(\alpha_0-1)}\int_0^t(t-s)^{\alpha_0-2}y(s)ds-c_2,\ t\in J_0.$$

If $t\in J_1$, then

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t(t-s)^{\alpha_1-1}y(s)ds-d_1-d_2(t-t_1),\\u'(t)&=\frac{1}{\Gamma(\alpha_1-1)}\int_{t_1}^t(t-s)^{\alpha_1-2}y(s)ds-d_2,\end{aligned}$$

for some $d_1,d_2\in\mathbb{R}$. Thus,

$$\begin{aligned}u(t_1^-)&=\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1}(t_1-s)^{\alpha_0-1}y(s)ds-c_1-c_2t_1,\qquad u(t_1^+)=-d_1,\\u'(t_1^-)&=\frac{1}{\Gamma(\alpha_0-1)}\int_0^{t_1}(t_1-s)^{\alpha_0-2}y(s)ds-c_2,\qquad u'(t_1^+)=-d_2.\end{aligned}$$

Using the impulse conditions

$$\triangle u(t_1)=u(t_1^+)-u(t_1^-)=\mathcal{I}_1,\quad \triangle u'(t_1)=u'(t_1^+)-u'(t_1^-)=\mathcal{I}_1^*,$$

we find that

$$\begin{aligned}-d_1&=\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1}(t_1-s)^{\alpha_0-1}y(s)ds-c_1-c_2t_1+\mathcal{I}_1,\\-d_2&=\frac{1}{\Gamma(\alpha_0-1)}\int_0^{t_1}(t_1-s)^{\alpha_0-2}y(s)ds-c_2+\mathcal{I}_1^*.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t(t-s)^{\alpha_1-1}y(s)ds+\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1}(t_1-s)^{\alpha_0-1}y(s)ds\\&+\frac{t-t_1}{\Gamma(\alpha_0-1)}\int_0^{t_1}(t_1-s)^{\alpha_0-2}y(s)ds+\mathcal{I}_1+(t-t_1)\mathcal{I}_1^*-c_1-c_2t,\ t\in J_1.\end{aligned}$$

By a similar process, we get

$$\begin{aligned}u(t) =& \int_{t_k}^{t} \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} y(s)ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-1}}{\Gamma(\alpha_{i-1})} y(s)ds + \mathcal{I}_1 \Big] \\ &+ \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_1^* \Big] \\ &+ \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_1^* \Big] - c_1 - c_2 t, \\ & t \in J_k,\ k = 1, 2, \cdots, p. \end{aligned} \tag{2.4}$$

The boundary condition $u'(0) = 0$ implies $c_2 = 0$. For $t \in J_k$, we have

$$\begin{aligned}&\mathcal{J}_{t_k^+}^{\beta_k} u(t) \\ =& \int_{t_k}^{t} \frac{(t-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)} y(s)ds + \sum_{i=1}^{k} \frac{(t-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-1}}{\Gamma(\alpha_{i-1})} y(s)ds + \mathcal{I}_1 \Big] \\ &+ \sum_{i=1}^{k-1} \frac{(t-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_1^* \Big] \\ &+ \sum_{i=1}^{k} \frac{(t-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_1^* \Big] - \frac{c_1 (t-t_k)^{\beta_k}}{\Gamma(\beta_k+1)}. \end{aligned} \tag{2.5}$$

Applying the boundary condition $u(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u(\eta_k) + \kappa$, we find

$$\begin{aligned}-c_1 =& \Big(1 - \sum_{k=0}^{p} \frac{\lambda_k (\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \Big)^{-1} \times \Big\{ \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(\eta_k-s)^{\alpha_k+\beta_k-1}}{\Gamma(\alpha_k+\beta_k)} y(s)ds \\ &+ \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k-t_k)^{\beta_k}}{\Gamma(\beta_k+1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-1}}{\Gamma(\alpha_{i-1})} y(s)ds + \mathcal{I}_i \Big] \\ &+ \sum_{k=1}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\eta_k-t_k)^{\beta_k}(t_k-t_i)}{\Gamma(\beta_k+1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_i^* \Big] \\ &+ \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k-t_k)^{\beta_k+1}}{\Gamma(\beta_k+2)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{\alpha_{i-1}-2}}{\Gamma(\alpha_{i-1}-1)} y(s)ds + \mathcal{I}_i^* \Big] + \kappa \Big\}. \end{aligned}$$

Substituting the value of $c_i (i = 1, 2)$ in (2.3) and (2.4), we obtain the unique solution $u(t)$ of impulsive fractional integral boundary value problem (2.1), which is given by expression (2.2), This completes the proof. □

Lemma 2.2 (Comparison theorem). *If* $\sum_{k=0}^{p} \frac{\lambda_k (\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k+1)} < 1$ *and* $u(t) \in PC(J, \mathbb{R}) \cap$

$AC^2(J_k)$ satisfies

$$\begin{cases} {}^C D^{\alpha_k}_{t_k^+} u(t) \geq 0, \quad 1 < \alpha_k \leq 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle u(t_k) \geq 0, \quad \triangle u'(t_k) \geq 0, \quad k = 1, 2, \cdots, p, \\ u(0) \geq \sum_{k=0}^{p} \lambda_k \mathcal{J}^{\beta_k}_{t_k^+} u(\eta_k), \quad u'(0) = 0. \end{cases} \tag{2.6}$$

Then $u(t) \geq 0$, $\forall t \in J$.

Proof. Consider a modified version of problem (2.1):

$$\begin{cases} {}^C D^{\alpha_k}_{t_k^+} u(t) = y(t), \quad 1 < \alpha_k \leq 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle u(t_k) = \mathcal{I}_k, \quad \triangle u'(t_k) = \mathcal{I}_k^*, \quad k = 1, 2, \cdots, p, \\ u(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}^{\beta_k}_{t_k^+} u(\eta_k) + \kappa, \quad u'(0) = 0, \end{cases} \tag{2.7}$$

where $y(t) \in C(J, \mathbb{R}^+)$ and $\mathcal{I}_k, \mathcal{I}_k^* (k = 1, 2, \cdots, p.), \kappa$ are nonnegative constants.

Then, the problem (2.7) has a unique solution

$$u(t) = \begin{cases} \frac{1}{\Gamma(\alpha_0)} \int_0^t (t-s)^{\alpha_0 - 1} y(s) ds + \mathfrak{B}, \ t \in J_0; \\ \int_{t_k}^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} y(s) ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 1}}{\Gamma(\alpha_{i-1})} y(s) ds + \mathcal{I}_i \Big] \\ + \sum_{i=1}^{k-1} (t_k - t_i) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} y(s) ds + \mathcal{I}_i^* \Big] \\ + \sum_{i=1}^{k} (t - t_k) \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} y(s) ds + \mathcal{I}_i^* \Big] + \mathfrak{B}, \\ t \in J_k, \ k = 1, 2, \cdots, p, \end{cases} \tag{2.8}$$

where

$$\begin{aligned} \mathfrak{B} = & \Big(1 - \sum_{k=0}^{p} \frac{\lambda_k (\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k + 1)} \Big)^{-1} \times \Big\{ \sum_{k=0}^{p} \lambda_k \int_{t_k}^{\eta_k} \frac{(\eta_k - s)^{\alpha_k + \beta_k - 1}}{\Gamma(\alpha_k + \beta_k)} y(s) ds \\ & + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k + 1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 1}}{\Gamma(\alpha_{i-1})} y(s) ds + \mathcal{I}_i \Big] \\ & + \sum_{k=1}^{p} \sum_{i=1}^{k-1} \frac{\lambda_k (\eta_k - t_k)^{\beta_k} (t_k - t_i)}{\Gamma(\beta_k + 1)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} y(s) ds + \mathcal{I}_i^* \Big] \\ & + \sum_{k=1}^{p} \sum_{i=1}^{k} \frac{\lambda_k (\eta_k - t_k)^{\beta_k + 1}}{\Gamma(\beta_k + 2)} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} y(s) ds + \mathcal{I}_i^* \Big] + \kappa \Big\}. \end{aligned}$$

On account of the nonnegative nature of function $y(t)$ and constants $\mathcal{I}_k, \mathcal{I}_k^*, \kappa$, then by (2.8), the conclusion of Lemma 2.2 holds. □

3. Main results

Definition 3.1. We say that $u(t)$ is called a lower solution of (1.1) if

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} u(t) \le f(t, u(t)), \quad 1 < \alpha_k \le 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle u(t_k) \le I_k(u(t_k)), \quad \triangle u'(t_k) \le I_k^*(u(t_k)), \quad k = 1, 2, \cdots, p, \\ u(0) \le \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u(\eta_k) + \kappa, \quad u'(0) = 0, \end{cases}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

To prove the existence of extremal solutions of problem (1.1), we need the following fixed point theorem in the sequel.

Theorem 3.1 ([7]). *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subset Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_* = \min\{y \in [a, b] : \ y \ge Qy\} \text{ and } x^* = \max\{y \in [a, b] : \ y \le Qy\}$$

Theorem 3.2. *Assume that*

($\mathbf{H_1}$) *The functions $f(t, u), I_k(u), I_k^*(u)(k = 1, \cdots, p.)$ are continuous and nondecreasing on u.*

($\mathbf{H_2}$) *There exist u_0 and $v_0 \in PC(J, \mathbb{R}) \cap AC^2(J_k)$, lower and upper solutions, respectively, for the problem (1.1), such that $u_0 \le v_0$.*

($\mathbf{H_3}$) $\sum_{k=0}^{p} \dfrac{\lambda_k(\eta_k - t_k)^{\beta_k}}{\Gamma(\beta_k + 1)} < 1.$

are satisfied. Then problem (1.1) *has extremal solutions in the sector* $[u_0, v_0]$.

Proof. Consider the problem (2.1) with $y(t) = f(t, \omega(t))$, $\mathcal{I}_k = I_k(\omega(t_k))$ and $\mathcal{I}_k^* = I_k^*(\omega(t_k))(k = 1, 2, \cdots, p)$. By Lemma 2.1, we know problem (2.1) has a unique solution. Define $u(t) = \mathcal{G}\omega(t)$, then $\mathcal{G}$ is an operator from $[u_0, v_0]$ to $PC(J, \mathbb{R}) \cap AC^2(J_k)$. Now we shall prove that $\mathcal{G}$ maps $[u_0, v_0]$ into $[u_0, v_0]$.

Let $u_1 = \mathcal{G}u_0, \ v_1 = \mathcal{G}v_0$. Then u_1, v_1 are well defined and satisfy

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} u_1(t) = f(t, u_0(t)), \quad 1 < \alpha_k \le 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle u_1(t_k) = I_k(u_0(t_k)), \quad \triangle u_1'(t_k) = I_k^*(u_0(t_k)), \quad k = 1, 2, \cdots, p, \\ u_1(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} u_1(\eta_k) + \kappa, \quad u_1'(0) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} v_1(t) = f(t, v_0(t)), \quad 1 < \alpha_k \le 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle v_1(t_k) = I_k(v_0(t_k)), \quad \triangle v_1'(t_k) = I_k^*(v_0(t_k)), \quad k = 1, 2, \cdots, p, \\ v_1(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} v_1(\eta_k) + \kappa, \quad v_1'(0) = 0. \end{cases} \tag{3.2}$$

Now, put $r = u_1 - u_0$. Combining with the definition of lower solution, we have

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} r(t) \ge 0, \quad 1 < \alpha_k \le 2, \quad k = 0, 1, 2, \cdots, p, \quad t \in J', \\ \triangle r(t_k) \ge 0, \quad \triangle r'(t_k) \ge 0, \quad k = 1, 2, \cdots, p, \\ r(0) \ge \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} r(\eta_k), \quad r'(0) = 0. \end{cases}$$

It follows from Lemma 2.2 that $p(t) \ge 0, \forall t \in J$. That is $\mathcal{G}u_0 \ge u_0$. Similarly, together with the definition of upper solution, we can show $\mathcal{G}v_0 \le v_0$.

Denote $q = v_1 - u_1$, by (3.1), (3.2) and $(\mathbf{H_1})$, we can get

$$\begin{cases} {}^C D_{t_k^+}^{\alpha_k} q(t) = f(t, v_0(t)) - f(t, u_0(t)) \ge 0, \quad 1 < \alpha_k \le 2, \ k = 0, 1, 2, \cdots, p, \ t \in J', \\ \triangle q(t_k) = I_k(v_0(t_k)) - I_k(u_0(t_k)) \ge 0, \quad k = 1, 2, \cdots, p, \\ \triangle q'(t_k) = I_k^*(v_0(t_k)) - I_k^*(u_0(t_k)) \ge 0, \quad k = 1, 2, \cdots, p, \\ q(0) = \sum_{k=0}^{p} \lambda_k \mathcal{J}_{t_k^+}^{\beta_k} q(\eta_k) \ge 0, \quad q'(0) = 0. \end{cases}$$

Lemma 2.2 ensures that $q(t) \ge 0$, i.e. $\mathcal{G}v_0 \ge \mathcal{G}u_0$. It means $\mathcal{G}$ is nondecreasing and $u_0 \le \mathcal{G}u \le v_0$ for any $u \in [u_0, v_0]$. Hence, $\mathcal{G} : [u_0, v_0] \to [u_0, v_0]$ and $\|\mathcal{G}u\| \le \max\{\|u_0\|, \|v_0\|\} := \text{II}$.

Let $\{u_n\}$ be a monotone sequence in $[u_0, v_0]$, then $u_0 \le \mathcal{G}u_n \le v_0$ and $\|\mathcal{G}u_n\| \le \text{II}$. Next, we shall show that the sequence $\{\mathcal{G}u_n\}$ is an equicontinuous set. For any $(t, u) \in J \times [-\text{II}, \text{II}]$, there exist nonnegative constants $L_i > 0$ $(i = 1, 2, 3)$ such that $|f(t, u)| \le L_1, |I_k(u)| \le L_2$ and $|I_k^*(u)| \le L_3$. Thus, for any $t \in J_k, \ 0 \le k \le p$, we have

$$\begin{aligned} &|(\mathcal{G}u)'(t)| \\ \le& \int_{t_k}^{t} \frac{(t-s)^{\alpha_k - 2}}{\Gamma(\alpha_k - 1)} |f(s, u(s))| ds + \sum_{i=1}^{k} \Big[\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} |f(s, u(s))| ds + |I_i^*(u(t_i))| \Big] \\ \le& L_1 \int_{t_k}^{t} \frac{(t-s)^{\alpha_k - 2}}{\Gamma(\alpha_k - 1)} ds + \sum_{i=1}^{p} \Big[L_1 \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{\alpha_{i-1} - 2}}{\Gamma(\alpha_{i-1} - 1)} ds + L_3 \Big] \\ \le& \frac{T^{\alpha_k - 1} L_1}{\Gamma(\alpha_k)} + p \Big[\frac{L_1 \max\limits_{0 \le i \le p} T^{\alpha_i - 1}}{\min\limits_{0 \le i \le p} \Gamma(\alpha_i)} + L_3 \Big] \\ \le& (p+1) \frac{L_1 \max\limits_{0 \le i \le p} T^{\alpha_i - 1}}{\min\limits_{0 \le i \le p} \Gamma(\alpha_i)} + p L_3 := \mathfrak{L}(constant). \end{aligned}$$

Hence, for $\tau_1, \tau_2 \in J_k$ with $\tau_1 \le \tau_2, \ 0 \le k \le p$, we have

$$|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)| \le \int_{\tau_1}^{\tau_2} |(\mathcal{G}u)'(s)| ds \le \mathfrak{L}(t_2 - t_1).$$

This implies that $\{\mathcal{G}u_n\}$ is an equicontinuous set on all $J_k, k = 0, 1, 2, \cdots, p$ and hence, by the Arzela-Ascoli Theorem, $\{\mathcal{G}u_n\}$ is relatively compact. In consequence, $\{\mathcal{G}u_n\}$ converges in $\mathcal{G}([u_0, v_0])$.

Theorem 3.1 ensures that $\mathcal{G}$ has a least and a greatest fixed point in $[u_0, v_0]$. This further implies that the problem (1.1) has extremal solutions on $[u_0, v_0]$. □

4. Example

Example 4.1. For $\alpha_0 = \frac{3}{2}$, $\alpha_1 = \frac{5}{4}$, $\beta_0 = \frac{3}{2}$, $\beta_1 = \frac{5}{2}$, $\lambda_0 = \frac{1}{5}$, $\lambda_1 = \frac{1}{7}$, $\eta_0 = \frac{1}{5}$, $\eta_1 = \frac{1}{3}$, and $t_1 = \frac{1}{4}$, we consider

$$\begin{cases} {}^C D^{\alpha_k}_{t_k^+} u(t) = \dfrac{t}{5}(u(t) + e^{u(t)}), \quad t \in [-0,1], \ t \neq \dfrac{1}{4}, k = 0,1, \\ \triangle u(t_1) = \dfrac{1}{32} \arctan u(t_1), \quad \triangle u'(t_1) = \dfrac{1}{5} u^3(t_1), \\ u(0) = \displaystyle\sum_{k=0}^{1} \lambda_k \mathcal{J}^{\beta_k}_{t_k^+} u(\eta_k) + \dfrac{1}{2}, \quad u'(0) = 0, \end{cases} \tag{4.1}$$

here $f(t,u) = \frac{t}{5}(u + e^u)$, $I_1(u) = \frac{1}{32} \arctan u$, $I_1^*(u) = \frac{1}{5} u^3$.

Take $u_0(t) = 0$, $v_0(t) = \begin{cases} 1 + \frac{t^2}{2}, & 0 \le t \le \frac{1}{4} \\ 1 + t^2, & \frac{1}{4} < t \le 1 \end{cases}$. Then u_0, v_0 are lower and upper solutions of problem (4.1), respectively. Moreover, by a simple calculation, $\frac{\lambda_0(\eta_0 - t_0)^{\beta_0}}{\Gamma(\beta_0+1)} + \frac{\lambda_1(\eta_1 - t_1)^{\beta_1}}{\Gamma(\beta_1+1)} = 0.013695 < 1$.

At last, it's obvious that (H_1) is satisfied. The conclusion of Theorem 3.2 applies and the problem (4.1) has extremal solutions in the sector $[u_0, v_0]$.

5. Concluding remarks

It is noted that, the problem investigated in this paper has a very general form. Some of special cases are listed below:

By setting $\alpha_k = \alpha (1 < \alpha \le 2)$ in (1.1), we obtain the special case of an integral boundary problem for nonlinear impulsive differential equations with fractional derivatives involving single order

$$\begin{cases} {}^C D^{\alpha}_{t_k^+} u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \quad k = 0,1,2,\cdots,p, \quad t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), \quad \triangle u'(t_k) = I_k^*(u(t_k)), \quad k = 1,2,\cdots,p, \\ u(0) = \displaystyle\sum_{k=0}^{p} \lambda_k \mathcal{J}^{\beta_k}_{t_k^+} u(\eta_k) + \kappa, \quad u'(0) = 0, \quad t_k < \eta_k < t_{k+1}. \end{cases} \tag{5.1}$$

Again, by setting $I_k = I_k^* = 0, \alpha_k = \alpha, \beta_k = \beta, \lambda_k = \lambda, \eta_k = \eta$ in (1.1), we can obtain a very special case of (1.1). That is, we get a nonlinear fractional differential equations without impulse effect

$$\begin{cases} {}^C D^{\alpha}_{0} u(t) = f(t, u(t)), \quad 1 < \alpha \le 2, \quad t \in J, \\ u(0) = \dfrac{\lambda}{\Gamma(\beta)} \displaystyle\int_0^{\eta} (\eta - s)^{\beta - 1} u(s) ds + \kappa, \quad u'(0) = 0. \end{cases} \tag{5.2}$$

Finally, taking $\alpha = 2, \beta = 1$ in (5.2), we have the classical second order problem with integral boundary condition

$$\begin{cases} u''(t) = f(t, u(t)), \quad t \in J, \\ u(0) = \lambda \displaystyle\int_0^{\eta} u(s) ds + \kappa, \quad u'(0) = 0. \end{cases} \tag{5.3}$$

Accordingly, our results also give rise to various interesting situations. It is a contribution to the theory of integral boundary value problem and fractional differential equation.

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