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REFINED A PRIORI ESTIMATES FOR THE AXISYMMETRIC NAVIER-STOKES EQUATIONS*

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Abstract In this paper, we consider the axisymmetric Navier-Stokes equations, and provide a refined a priori estimate for the swirl component of the vorticity. This extends Theorem 2 of [D. Chae, J. Lee, On the regularity of the axisymmetric solutions of the Navier-Stokes equations, Math. Z., 239 (2002), 645–671].

Keywords Axisymmetric Navier-Stokes equations, a priori estimate.

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1. Introduction

This paper concerns the following Navier-Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u^1, u^2, u^3)$ is the fluid velocity field, π is a scalar pressure, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0 = 0$ in the sense of distributions.

It is well-known that (1.1) possesses a global weak solution

$$\boldsymbol{u} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \tag{1.2}$$

for initial data of finite energy, see [5, 9]. However, the issue of its regularity and uniqueness is an outstanding open problem in mathematical fluid dynamics.

An interesting result on (1.1) is that the axially symmetric solutions without swirl component exists globally (see [8, 10, 14]). However, if the swirl component is non-zero, then it is still open for its global regularity. And many interesting sufficient conditions to ensure the smoothness of the solution were established, see [1–4, 6, 7, 11, 13, 15, 16] for example.

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In this paper, we concern ourselves with the a priori estimates for axisymmetric solution of (1.1). By this, we mean a solution of the form

$$\boldsymbol{u} = u^r(r,z,t)\boldsymbol{e}_r + u^\theta(r,z,t)\boldsymbol{e}_\theta + u^z(r,z,t)\boldsymbol{e}_z, \tag{1.3}$$

where

$$\begin{aligned}
\boldsymbol{e}_r &= \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0)\,, \\
\boldsymbol{e}_\theta &= \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0)\,, \\
\boldsymbol{e}_z &= (0,0,1),
\end{aligned}$$

are the standard bases in the cylindrical coordinate system. In (1.3), u^r, u^θ, u^z are called the angular, swirl and axial components of the velocity field $\boldsymbol{u}$. For the axisymmetric solutions, we can reformulate (1.1) as

$$\begin{cases}
\frac{\tilde{D}}{Dt}u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r - \frac{(u^\theta)^2}{r} + \partial_r \pi = 0, \\
\frac{\tilde{D}}{Dt}u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta + \frac{u^r u^\theta}{r} = 0, \\
\frac{\tilde{D}}{Dt}u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)u^z + \partial_z \pi = 0, \\
\partial_r(ru^r) + \partial_z(ru^z) = 0, \\
u^r(0) = u_0^r,\ u^\theta(0) = u_0^\theta,\ u^z(0) = u_0^z,
\end{cases} \tag{1.4}$$

where

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u_z\partial_z \tag{1.5}$$

is the convection derivative (or material derivative). For the axisymmetric vector field $\boldsymbol{u}$, we can compute the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ as

$$\boldsymbol{\omega} = \omega^r\boldsymbol{e}_r + \omega^\theta\boldsymbol{e}_\theta + \omega^z\boldsymbol{e}_z, \tag{1.6}$$

where

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}. \tag{1.7}$$

By taking curl of (1.1) or by applying suitable derivatives to (1.4), we may deduce the governing equations of the ω^r, ω^θ and ω^z as

$$\begin{cases}
\frac{\tilde{D}}{Dt}\omega^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\
\frac{\tilde{D}}{Dt}\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^\theta - \frac{2u^\theta\partial_z u^\theta}{r} - \frac{u^r\omega^\theta}{r} = 0, \\
\frac{\tilde{D}}{Dt}\omega^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^z - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0.
\end{cases} \tag{1.8}$$

It is well-known that for any $2 \le p < \infty$,

$$|ru^\theta|^{\frac{p}{2}} \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \tag{1.9}$$

if $ru_0^\theta \in L^p(\mathbb{R}^3)$, see [1, Proposition 1] for instance. In the same paper, Chae-Lee established another a priori bounds

$$r^3\omega^\theta \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)), \tag{1.10}$$

if $ru_0^\theta \in L^4(\mathbb{R}^3)$ and $r^3\omega_0^\theta \in L^2(\mathbb{R}^3)$.

The purpose of the present paper is to extend (1.10). Precisely, we have

Theorem 1.1. *If $\boldsymbol{u}$ is an axisymmetric smooth solution of the Navier-Stokes equations with divergence-free initial data $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ satisfying $r^d\omega_0^\theta \in L^2(\mathbb{R}^3)$ for some $2 \le d \le 3$ and $ru_0^\theta \in L^4(\mathbb{R}^3)$, then $r^d\omega^\theta \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$.*

Remark 1.1. The case $d = 3$ in Theorem 1.1 was exactly the result of [1, Theorem 2]. Moreover, if Theorem 1.1 holds for $d = 0$, then [1, Theorem 1] tells us that the solution can be extended smoothly beyond T. Thus our theorem is better than [1, Theorem 2] in this sense.

Before proving Theorem 1.1 in Section 2, we recall the well-known Gagliardo-Nirenberg inequality.

Lemma 1.1 (Gagliardo-Nirenberg inequality, see [12])**.** *Let $1 \le p, q, r \le \infty$, and j,m are arbitrary integers satisfying $0 \le j < m$. Assume $f \in C_c^\infty(\mathbb{R}^n)$. Then*

$$\left\|D^j f\right\|_{L^p} \le C \left\|f\right\|_{L^q}^{1-a} \left\|D^m f\right\|_{L^r}^{a},$$

where

$$-j + \frac{n}{p} = (1-a)\frac{n}{q} + a\left(-m + \frac{n}{r}\right),$$

and

$$a \in \begin{cases} [j/m, 1), \text{ if } m - j - n/r \text{ is an nonnegative integer}, \\ [j/m, 1]\,, \text{ otherwise.} \end{cases}$$

The constant C depends only on n, m, j, q, r, a.

Choosing $n = 3$, $j = 0$, $m = 1$ and $q = r = 2$ in Lemma 1.1 yields

$$\|f\|_{L^p} \le C \|f\|_{L^2}^{1-a} \|\nabla f\|_{L^2}^{a}, \quad \text{with } a = \frac{3}{2} - \frac{3}{p}, \quad \forall\, 2 \le p \le 6. \tag{1.11}$$

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Similar to [1], we multiply $(1.8)_2$ by $r^{2d}\omega^\theta$ with $2 \le d \le 3$, and integrate over $\mathbb{R}^3$ to obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^d\omega^\theta\right\|_{L^2}^2 + \left\|\nabla(r^d\omega^\theta)\right\|_{L^2}^2 \\ &= (d+1)\int_{\mathbb{R}^3} u^r r^{2d-1}|\omega^\theta|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^3} u^\theta\partial_z u^\theta \cdot r^{2d-1}\omega^\theta\,\mathrm{d}x + (d^2-1)\int_{\mathbb{R}^3}|r^{d-1}\omega^\theta|^2\,\mathrm{d}x \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.1}$$

We estimate I_i $(1 \le i \le 3)$ term by term as

$$\begin{aligned} I_1 &= (d+1)\int_{\mathbb{R}^3} u^r \cdot (r^d\omega^\theta)^{\frac{2d-1}{d}} \cdot (\omega^\theta)^{\frac{1}{d}}\,\mathrm{d}x \\ &\le (d+1)\left\|u^r\right\|_{L^2}\left\|r^d\omega^\theta\right\|_{L^{\frac{2(2d-1)}{d-1}}}^{\frac{2d-1}{d}}\left\|\omega^\theta\right\|_{L^2}^{\frac{1}{d}} \quad \text{(by Hölder inequailty)} \\ &\le C\left\|r^d\omega^\theta\right\|_{L^2}^{\frac{d-2}{2d}}\left\|\nabla(r^d\omega^\theta)\right\|_{L^2}^{\frac{3}{2}}\left\|\omega^\theta\right\|_{L^2}^{\frac{1}{d}} \quad \text{(by (1.2) and (1.11))} \\ &\le \frac{1}{3}\left\|\omega^\theta\right\|_{L^2}^2 + C\left\|r^d\omega^\theta\right\|_{L^2}^2 + \frac{1}{6}\left\|\nabla(r^d\omega^\theta)\right\|_{L^2}^2 \quad \text{(by Young inequality)}, \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} \partial_z(|ru^\theta|^2)\cdot r^{2d-3}\omega^\theta \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \partial_z(|ru^\theta|^2)\cdot (r^d\omega^\theta)^{\frac{2d-3}{d}}\cdot (\omega^\theta)^{\frac{3-d}{d}} \,\mathrm{d}x \\
&\le \left\|\partial_z(|ru^\theta|^2)\right\|_{L^2} \left\|r^d\omega^\theta\right\|_{L^2}^{\frac{2d-3}{d}} \left\|\omega^\theta\right\|_{L^2}^{\frac{3-d}{d}} \quad \text{(by Hölder inequailty)} \\
&\le \frac{1}{3}\left\|\omega^\theta\right\|_{L^2}^2 + C\left\|r^d\omega^\theta\right\|_{L^2}^2 + \left\|\partial_z(|ru^\theta|^2)\right\|_{L^2}^2 \quad \text{(by Young inequality)}\,,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
I_3 &= (d^2-1)\int_{\mathbb{R}^3} (r^d\omega^\theta)^{\frac{2(d-1)}{d}} \left(\omega^\theta\right)^{\frac{2}{d}} \,\mathrm{d}x \\
&\le (d^2-1)\left\|r^d\omega^\theta\right\|_{L^2}^{\frac{2(d-1)}{d}} \left\|\omega^\theta\right\|_{L^2}^{\frac{2}{d}} \quad \text{(by Hölder inequailty)} \\
&\le \frac{1}{3}\left\|\omega^\theta\right\|_{L^2}^2 + C\left\|r^d\omega^\theta\right\|_{L^2}^2 \quad \text{(by Young inequality)}\,.
\end{aligned} \tag{2.4}
$$

Gathering (2.2), (2.3) and (2.4) into (2.1), we find

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left\|r^d\omega^\theta\right\|_{L^2}^2 + \left\|\nabla(r^d\omega^\theta)\right\|_{L^2}^2 \le \left\|\omega^\theta\right\|_{L^2}^2 + \left\|\partial_z(|ru^\theta|^2)\right\|_{L^2}^2 + C\left\|r^d\omega^\theta\right\|_{L^2}^2\,.
$$

Applying Gronwall inequality, and utilizing (1.2) and (1.9), we deduce that

$$
r^d\omega^\theta \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)),
$$

as desired. The proof of Theorem 1.1 is completed.

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