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BIFURCATION AND CHAOS IN A DISCRETE TIME PREDATOR-PREY SYSTEM OF LESLIE TYPE WITH GENERALIZED HOLLING TYPE III FUNCTIONAL RESPONSE

Ali Atabaigi

Abstract This paper is devoted to study a discrete time predator-prey system of Leslie type with generalized Holling type III functional response obtained using the forward Euler scheme. Taking the integration step size as the bifurcation parameter and using the center manifold theory and bifurcation theory, it is shown that by varying the parameter the system undergoes flip bifurcation and Neimark-Sacker bifurcation in the interior of $\mathbb{R}^2_+$. Numerical simulations are implemented not only to illustrate our results with the theoretical analysis, but also to exhibit the complex dynamical behaviors, such as cascade of period-doubling bifurcation in period-2, 4, 8, quasi-periodic orbits and the chaotic sets. These results shows much richer dynamics of the discrete model compared with the continuous model. The maximum Lyapunov exponent is numerically computed to confirm the complexity of the dynamical behaviors. Moreover, we have stabilized the chaotic orbits at an unstable fixed point using the feedback control method.

Keywords Predator-prey system, discrete time dynamical system, period doubling bifurcation, Neimark-Sacker bifurcation, chaotic dynamic, chaos Control.

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1. Introduction

In this paper, we consider the following Lotka-Volterra predator-prey system

$$\begin{aligned} \dot{x} &= x\left[r(1-\frac{x}{K}) - \frac{\alpha x y}{a x^2 + b x + 1}\right], \\ \dot{y} &= s y\left(1 - \frac{h y}{x}\right), \end{aligned} \tag{1.1}$$

where x and y stand for population density of prey and predator, respectively, r, α, a, s, h are positive constants and b is a constant. The prey grows logistically with the carrying capacity K and intrinsic growth rate r. The predator consumes prey according to the generalized Holling type III functional response $\frac{\alpha x^2}{ax^2+bx+1}$ and grows logistically with intrinsic growth rate s and carrying capacity proportional to the population density of the prey.

In [8], the detailed bifurcation analysis of system (1.1) has been discussed. It is shown that the model has very rich and complicated dynamics such as the existence of a stable limit cycle enclosing two non-hyperbolic positive equilibria, a stable limit

Email address: aliaelmi@gmail.com(A. Atabaigi)
Department of Mathematics, Razi University, Kermanshah, 67149, Iran

cycle enclosing an unstable homoclinic loop, two limit cycles enclosing a hyperbolic positive equilibrium, or one stable limit cycle enclosing three hyperbolic positive equilibria. In particular, it is shown that the model undergoes degenerate focus type Bogdanov-Takens bifurcation of codimension 3.

Applying the forward Euler scheme to system (1.1), we obtain the following discrete time predator-prey dynamical system

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \delta x \left[r(1 - \frac{x}{K}) - \frac{\alpha x y}{a x^2 + b x + 1} \right] \\ y + \delta s y \left(1 - \frac{h y}{x} \right) \end{pmatrix}, \tag{1.2}$$

where δ is the integration step size. We mainly focus on flip bifurcation, Neimark-Sacker bifurcation and possible chaos in the closed first quadrant $\mathbb{R}^2_+$ using center manifold theorem and bifurcation theory [5, 9, 13].

The paper is organized as follows. In section 2, we study the existence and stability of the fixed points of the system (1.2). In section 3, we show that there exist some parameter values for which the system (1.2) exhibits flip and Neimark-Sacker bifurcation. In section 4, we use numerical simulations to support the theoretical analysis given in section 3. The numerical simulation also shows that the system (1.2) exhibits the complex dynamics such as cascades of period-doubling, quasi-periodic orbits and chaotic sets. The Lyapunov exponents are calculated to confirm the existence of chaos. In section 5, we have stabilized the chaotic orbits at an unstable fixed point using the feedback control method.

2. Fixed points and their local stability

In this section, we will study the existence and property of fixed points of system (1.2) in the region $\mathbb{R}^2_+$. It is clear that system (1.2) always has a boundary fixed point $(K, 0)$ for all parameters. Next, we consider the existence of positive fixed point of system (1.2).

Suppose that (x_0, y_0) is a positive fixed point of system (1.2). Then, x_0 and y_0 are positive solutions of the following equations

$$\begin{cases} r(1 - \dfrac{x_0}{K}) = \dfrac{\alpha x_0 y_0}{a x_0^2 + b x_0 + 1}, \\ x_0 = h y_0. \end{cases} \tag{2.1}$$

From (2.1), we can see that x_0 is the root in the interval $(0, K)$ of the following cubic equation

$$p_0 w^3 + 3p_1 w^2 + 3p_2 w + p_3 = 0, \quad (p_0 \neq 0), \tag{2.2}$$

with coefficients

$$p_0 = \frac{a}{K}, \quad 3p_1 = \frac{\alpha r}{h} + \frac{b}{K} - a, \quad 3p_2 = \frac{1}{K} - b, \quad p_3 = -1.$$

Using the substitution $z = p_0 w + p_1$, the equation (2.2) is converted to $z^3 + 3Hz + G = 0$, where $G = p_0^2 p_3 - 3p_0 p_1 p_2 + 2p_1^3$, $H = p_0 p_2 - p_1^2$. Using Cardano's method, we have the following result.

Lemma 2.1. *Let $G^2 + 4H^3 > 0$, then the system (1.2) has a unique positive fixed point (x_0, y_0), where*

$$x_0 = \frac{1}{p_0}(q - \frac{H}{q} - p_1), \quad y_0 = \frac{x_0}{h},$$

and q denotes one of the three values of $\left[\frac{-G+\sqrt{G^2+4H^3}}{2}\right]^{\frac{1}{3}}$.

Next, we study the local stability of the above mentioned fixed points. As we know, the local stability of the fixed points is determined by the modules of eigenvalues of the characteristic equation of the Jacobian matrix of system (1.2) at the fixed points.

The Jacobian matrix of system (1.2) at any point is given as follows

$$J(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.3}$$

where

$$a_{11} = 1 + r\delta - \frac{2r\delta x}{K} - \frac{\alpha\delta xy(2+bx)}{(ax^2+bx+1)^2}, \quad a_{12} = -\frac{\alpha\delta x^2}{ax^2+bx+1},$$
$$a_{21} = \frac{\delta hsy^2}{x^2}, \qquad a_{22} = 1 + \delta s - \frac{2\delta hsy}{x}.$$

The characteristic equation of the Jacobian matrix can be written as

$$\lambda^2 + p(x,y)\lambda + q(x,y) = 0, \tag{2.4}$$

where $p(x,y) = -(a_{11} + a_{22})$ and $q(x,y) = a_{11}a_{22} - a_{12}a_{21}$.

Before we discuss the local stability of the fixed points, we present the following lemma which discusses the relation between roots of the quadratic equation and its coefficients [7, 11].

Lemma 2.2. *Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff* $F(-1) > 0$ *and* $C < 1$;

(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *) iff* $F(-1) < 0$;

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *iff* $F(-1) > 0$ *and* $C > 1$;

(iv) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *iff* $F(-1) = 0$ *and* $B \neq 0, 2$;

(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *iff* $B^2 - 4C < 0$ *and* $C = 1$.

Suppose λ_1 and λ_2 are the roots of (2.4). We recall some definition concerning the topological types of a fixed point (x, y). A fixed point (x, y) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. A sink is locally asymptotically stable. (x, y) is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. A source is locally unstable. A fixed point (x, y) is called a a saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$). And (x, y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

By substituting the coordinates of the fixed point $(K, 0)$ for (x, y) of (2.3) and computing the eigenvalues of J we can obtain the following proposition.

Proposition 2.1. *The fixed point* $(K, 0)$ *is a saddle if* $0 < \delta < \frac{2}{r}$*, it is a source if* $\delta > \frac{2}{r}$*, and it is non-hyperbolic if* $\delta = \frac{2}{r}$*.*

When $\delta = \frac{2}{r}$, one of the eigenvalues of the fixed point $(K, 0)$ is -1 and the other is $1 + \frac{2s}{r}$. Thus, the flip (or period-doubling) bifurcation may occur when parameters vary in the neighborhood of $\delta = \frac{2}{r}$. In this case, (1.2) restricted to the center manifold $y = 0$ is the logistic model $x_{n+1} = rx_n(1 - \frac{x_n}{K})$, therefore the predator becomes extinct and the prey undergoes the period-doubling bifurcation which is a route to chaos in the sense of Li and Yorke [10] by choosing the r as the bifurcation parameter.

In the following we investigate the local dynamics of fixed point (x_0, y_0). The characteristic equation of the Jacobian matrix (2.3) evaluated at (x_0, y_0) can be written in the following form

$$F(\lambda) := \lambda^2 - (2 + \Delta\delta)\lambda + (1 + \Delta\delta + \Omega s\delta^2) = 0,$$

where

$$\Delta = r - s - \frac{2rx_0}{K} - \frac{\alpha x_0 y_0(2 + bx_0)}{(ax_0^2 + bx_0 + 1)^2}, \quad \Omega = \frac{rx_0}{K} + \frac{\alpha x_0 y_0(2 + bx_0)}{(ax_0^2 + bx_0 + 1)^2}.$$

Then $F(1) = \Omega s\delta^2 > 0$ and $F(-1) = 4 + 2\Delta\delta + \Omega s\delta^2$. From Lemma 2.2 we have

Proposition 2.2. *Let* (x_0, y_0) *be the positive fixed point of (1.2).*

(i) It is a sink if one of the following conditions holds
  - *(i.1)* $-2\sqrt{\Omega s} \le \Delta < 0$ *and* $0 < \delta < -\frac{\Delta}{\Omega s}$*;*
  - *(i.2)* $\Delta < -2\sqrt{\Omega s}$ *and* $0 < \delta < \frac{-\Delta - \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}$*.*

(ii) It is a source if one of the following conditions holds
  - *(ii.1)* $-2\sqrt{\Omega s} \le \Delta < 0$ *and* $\delta > -\frac{\Delta}{\Omega s}$*;*
  - *(ii.2)* $\Delta < -2\sqrt{\Omega s}$ *and* $\delta > \frac{-\Delta - \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}$*;*
  - *(ii.2)* $\Delta \ge 0$*.*

(iii) It is a saddle if the following conditions hold

$$\Delta < -2\sqrt{\Omega s} \text{ and } \frac{-\Delta - \sqrt{\Delta^2 - 4\Omega s}}{\Omega s} < \delta < \frac{-\Delta + \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}.$$

(iv) It is non-hyperbolic if one of the following conditions holds
  - *(iv.1)* $\Delta < -2\sqrt{\Omega s}$ *and* $\delta = \frac{-\Delta \pm \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}$ *and* $\delta \neq -\frac{2}{\Delta}, -\frac{4}{\Delta}$*;*
  - *(iv.2)* $-2\sqrt{\Omega s} < \Delta < 0$ *and* $\delta = -\frac{\Delta}{\Omega s}$*.*

From Lemma 2.2, we can see that if $(iv.1)$ of (2.2) holds one of the eigenvalues of the positive fixed point (x_0, y_0) is -1 and the other is neither 1 nor -1. Also, we can see that the eigenvalues of the fixed point (x_0, y_0) are complex conjugate numbers if the condition $(iv.2)$ of (2.2) holds. We rewrite the condition $(iv.1)$ of Proposition 2.2 as the following sets

$$F_{B_1} = \left\{(r, K, \alpha, a, b, s, h, \delta) : \delta = \frac{-\Delta - \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}, \Delta < -2\sqrt{\Omega s}\right\},$$

or

$$F_{B_2} = \left\{ (r, K, \alpha, a, b, s, h, \delta) : \delta = \frac{-\Delta + \sqrt{\Delta^2 - 4\Omega s}}{\Omega s}, \Delta < -2\sqrt{\Omega s} \right\}.$$

When the condition $(iv.2)$ of Proposition 2.2 holds, we deduce that the eigenvalues of the fixed point (x_0, y_0) are a pair of conjugate complex numbers with module one. The conditions in $(iv.2)$ of Proposition 2.2 can be written as the following set

$$H_B = \left\{ (r, K, \alpha, a, b, s, h, \delta) : \delta = -\frac{\Delta}{\Omega s}, -2\sqrt{\Omega s} < \Delta < 0 \right\}.$$

In the following section, we will study the flip bifurcation of the positive fixed point (x_0, y_0) when parameters of the system vary in the small neighborhood of F_{B_1} (or F_{B_2}), and the Neimark-Sacker bifurcation if parameters vary in the small neighborhood of H_B.

3. Flip bifurcation and Neimark-Sacker bifurcation

In the sequel, based on the argument given in the previous section and choosing the step size δ as the bifurcation parameter, we consider the Flip and the Neimark-Sacker bifurcation of the positive fixed point (x_0, y_0).

First, we study the Flip bifurcation of the positive fixed point (x_0, y_0) when the parameters of the system vary in a small neighborhood of F_{B_1}. The case of F_{B_2} can be handled similarly.

Take parameters $(r, K, \alpha, a, b, s, h, \delta_0)$ arbitrarily from F_{B_1}. Using $u = x - x_0$, $v = y - y_0$ and $\delta_* = \delta - \delta_0$, we transform the fixed point (x_0, y_0) to the origin and consider the parameter δ_* as the new bifurcation parameter. After Taylor expansion, system (1.2) is equivalent to the following system

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + b_1 u\delta_* + b_2 v\delta_* + e_1 u^3 \\ +e_2 u^2 v + b_3 u^2 \delta_* + b_4 uv\delta_* + O((|u| + |v| + |\delta_*|)^4) \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + a_{25}v^2 + c_1 u\delta_* + c_2 v\delta_* \\ +d_1 u^3 + d_2 u^2 v + d_3 uv^2 + c_3 u^2 \delta_* + c_4 uv\delta_* + c_5 v^2 \delta_* \\ +O((|u| + |v| + |\delta_*|)^4) \end{pmatrix}, \tag{3.1}$$

where

$$a_{11} = 1 + r\delta_0 - 2\,\frac{r\delta_0\, x_0}{K} - \frac{\alpha\, x_0 y_0 \delta_0\, (2 + bx_0)}{(1 + bx_0 + ax_0{}^2)^2}, \quad a_{12} = -\frac{\alpha\, x_0{}^2 \delta_0}{1 + bx_0 + ax_0{}^2},$$

$$a_{13} = -\frac{r\delta_0}{K} + \frac{\alpha\, y_0 \delta_0 \left(-1 + 3\, ax_0{}^2 + bx_0{}^3 a\right)}{(1 + bx_0 + ax_0{}^2)^3}, \quad a_{14} = -\frac{\alpha\, x_0 \delta_0\, (bx_0 + 2)}{(1 + bx_0 + ax_0{}^2)^2},$$

$$b_1 = r - 2\,\frac{rx_0}{K} - \frac{\alpha\, x_0 y_0\, (2 + bx_0)}{(1 + bx_0 + ax_0{}^2)^2}, \quad b_2 = -\frac{\alpha\, x_0{}^2}{1 + bx_0 + ax_0{}^2},$$

$$e_1 = -\frac{\alpha\, y_0 \delta_0 \left(-1 + ax_0{}^2\right)\left(abx_0{}^2 + 4\, ax_0 + b\right)}{(1 + bx_0 + ax_0{}^2)^4}, \quad e_2 = \frac{\alpha\, \delta_0 \left(-1 + 3\, ax_0{}^2 + bx_0{}^3 a\right)}{(1 + bx_0 + ax_0{}^2)^3},$$

$$b_3 = -\frac{r}{K} + \frac{\alpha\, y_0\left(-1 + 3\,a{x_0}^2 + b{x_0}^3 a\right)}{\left(1 + bx_0 + a{x_0}^2\right)^3}, \quad b_4 = -\frac{\alpha\, x_0\,(2 + bx_0)}{\left(1 + bx_0 + a{x_0}^2\right)^2},$$

$$a_{21} = \frac{s\delta_0 y_0}{x_0}, \quad a_{22} = 1 - s\delta_0, \quad a_{23} = -\frac{s\delta_0 y_0}{x_0^2}, \quad a_{24} = \frac{2s\delta_0}{x_0}, \quad a_{25} = -\frac{\delta_0 hs}{x_0},$$

$$d_1 = \frac{\delta_0 s y_0}{x_0^3}, \quad d_2 = -\frac{2\delta_0 s}{x_0^2}, \quad d_3 = \frac{\delta_0 hs}{x_0^2}, \quad c_1 = \frac{sy_0}{x_0},$$

$$c_2 = -s, \quad c_3 = -\frac{sy_0}{x_0^2}, \quad c_4 = \frac{2s}{x_0}, \quad c_5 = -\frac{hs}{x_0}.$$

We construct an invertible matrix

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}.$$

Under the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix},$$

and again using x and y instead of $\tilde{x}$ and $\tilde{y}$ respectively, the map (3.1) becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y,\delta_*) \\ g(x,y,\delta_*) \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} f(x,y,\delta_*) =& \frac{(a_{13}\,(\lambda_2 - a_{11}) - a_{23}a_{12})}{a_{12}\,(\lambda_2+1)}u^2 + \frac{(a_{14}\,(\lambda_2 - a_{11}) - a_{24}a_{12})}{a_{12}\,(\lambda_2+1)}uv - \frac{a_{25}}{\lambda_2+1}v^2 \\ &+ \frac{(b_1\,(\lambda_2 - a_{11}) - c_1 a_{12})}{a_{12}\,(\lambda_2+1)}u\delta_* + \frac{(b_2\,(\lambda_2 - a_{11}) - c_2 a_{12})}{a_{12}\,(\lambda_2+1)}v\delta_* \\ &+ \frac{(e_1\,(\lambda_2 - a_{11}) - d_1 a_{12})}{a_{12}\,(\lambda_2+1)}u^3 + \frac{(e_2\,(\lambda_2 - a_{11}) - d_2 a_{12})}{a_{12}\,(\lambda_2+1)}u^2 v - \frac{d_3}{\lambda_2+1}uv^2 \\ &+ \frac{(b_3\,(\lambda_2 - a_{11}) - c_3 a_{12})}{a_{12}\,(\lambda_2+1)}u^2\delta_* + \frac{(b_4\,(\lambda_2 - a_{11}) - c_4 a_{12})}{a_{12}\,(\lambda_2+1)}uv\delta_* \\ &- \frac{c_5}{\lambda_2+1}v^2\delta_* + O((|u| + |v| + |\delta_*|)^4), \\ g(x,y,\delta_*) =& \frac{(a_{13}\,(1 + a_{11}) + a_{23}a_{12})}{a_{12}\,(\lambda_2+1)}u^2 + \frac{(a_{14}\,(1 + a_{11}) + a_{24}a_{12})}{a_{12}\,(\lambda_2+1)}uv + \frac{a_{25}}{\lambda_2+1}v^2 \\ &+ \frac{(b_1\,(1 + a_{11}) + c_1 a_{12})}{a_{12}\,(\lambda_2+1)}u\delta_* + \frac{(b_2\,(1 + a_{11}) + c_2 a_{12})}{a_{12}\,(\lambda_2+1)}v\delta_* \\ &+ \frac{(e_1\,(1 + a_{11}) + d_1 a_{12})}{a_{12}\,(\lambda_2+1)}u^3 + \frac{(e_2\,(1 + a_{11}) + d_2 a_{12})}{a_{12}\,(\lambda_2+1)}u^2 v \\ &+ \frac{d_3}{\lambda_2+1}uv^2 + \frac{(b_3\,(1 + a_{11}) + c_3 a_{12})}{a_{12}\,(\lambda_2+1)}u^2\delta_* \\ &+ \frac{(b_4\,(1 + a_{11}) + c_4 a_{12})\,uv\delta}{a_{12}\,(\lambda_2+1)} + \frac{c_5 v^2\delta}{\lambda_2+1} + O((|u| + |v| + |\delta_*|)^4), \end{aligned}$$

and

$$u = a_{12}(x+y), \quad v = -(1+a_{11})x + (\lambda_2 - a_{11})y.$$

According to the center manifold theorem [9], the dynamic of the map (3.7) around the fixed point $(0,0)$ for parameter values near $\delta_* = 0$ can be analyzed using the behavior of a one parameter family of maps on the center manifold, which can be written as:

$$W^c(0) = \{(x, y, \delta_*) \in \mathbb{R}^3 \ : \ y = h_2(x, \delta_*), \ h_2(0,0) = 0, Dh_2(0,0) = 0\}.$$

Assume that $h_2(x, \delta_*)$ has the following form

$$h_2(x, \delta_*) = a_1 x^2 + a_2 x \delta_* + a_3 \delta_*^2 + O((|x| + |\delta_*|)^3). \tag{3.3}$$

Then, the center manifold (3.3) must satisfy

$$N(h(x, \delta_*)) = h(-x + f(x, h(x, \delta_*)), \delta_*) - \lambda_2 h(x, \delta_*) - g(x, h(x, \delta_*), \delta_*) = 0. \tag{3.4}$$

Substituting (3.7) and (3.3) into (3.4), we obtain

$$\begin{aligned}
a_1 &= -\frac{1}{{\lambda_2}^2 - 1}\left\{a_{25} + 2\,a_{25}a_{11} + a_{25}{a_{11}}^2 + a_{12}a_{13}\,(1 + a_{11}) + a_{23}{a_{12}}^2\right.\\
&\quad \left. - a_{14} - 2\,a_{14}a_{11} - a_{14}{a_{11}}^2 - a_{24}a_{12} - a_{24}a_{12}a_{11}\right\},\\
a_2 &= -\frac{a_{12}b_1\,(1 + a_{11}) + c_1{a_{12}}^2 - b_2 - 2\,b_2a_{11} - b_2{a_{11}}^2 - c_2a_{12} - c_2a_{12}a_{11}}{a_{12}\,(\lambda_2 + 1)^2},\\
a_3 &= 0.
\end{aligned}$$

Thus, the map restricted to the center manifold is given by

$$F_1 : x \mapsto -x + h_1 x^2 + h_2 x \delta_* + h_3 x^2 \delta_* + h_4 x \delta_*^2 + h_5 x^3 + O((|x| + |\delta_*|)^4), \tag{3.5}$$

where

$$\begin{aligned}
h_1 &= \frac{1}{\lambda_2 + 1}\left\{a_{12}a_{13}\,(\lambda_2 - a_{11}) - a_{23}{a_{12}}^2 - a_{25} - 2\,a_{25}a_{11} - a_{25}{a_{11}}^2\right.\\
&\quad \left. - a_{14}\lambda_2 - a_{14}\lambda_2 a_{11} + a_{14}a_{11} + a_{14}{a_{11}}^2 + a_{24}a_{12} + a_{24}a_{12}a_{11}\right\},\\
h_2 &= \frac{1}{a_{12}\,(\lambda_2 + 1)}\left\{a_{12}b_1\,(\lambda_2 - a_{11}) - c_1{a_{12}}^2 - b_2\lambda_2 - b_2\lambda_2 a_{11} + b_2 a_{11}\right.\\
&\quad \left. + b_2{a_{11}}^2 + c_2a_{12} + c_2a_{12}a_{11}\right\},\\
h_3 &= \frac{1}{a_{12}\,(\lambda_2 + 1)}\left\{a_2a_{12}a_{14}{\lambda_2}^2 - a_{12}b_4\lambda_2 a_{11} + {a_{12}}^2 b_3\,(\lambda_2 - a_{11}) + c_4{a_{12}}^2 a_{11}\right.\\
&\quad + a_{12}b_4a_{11} - a_{12}b_4\lambda_2 + 2\,{a_{12}}^2 a_2a_{13}\,(\lambda_2 - a_{11}) + a_2{a_{12}}^2 a_{24} + a_2a_{12}a_{14}a_{11}\\
&\quad - a_2a_{12}a_{14}\lambda_2 - a_2{a_{12}}^2 a_{24}\lambda_2 + a_1c_2a_{12}a_{11} - a_1{a_{12}}^2 c_1 - a_1c_2a_{12}\lambda_2 - {a_{12}}^3 c_3\\
&\quad + c_4{a_{12}}^2 - c_5a_{12} - 2\,{a_{12}}^3 a_2a_{23} + a_1a_{12}b_1\,(\lambda_2 - a_{11}) + a_1b_2{a_{11}}^2 + a_1b_2{\lambda_2}^2\\
&\quad - 2\,a_{25}a_2a_{12}a_{11} + 2\,a_{25}a_2a_{12}\lambda_2 + 2\,a_2{a_{12}}^2 a_{24}a_{11} + 2\,a_2a_{12}a_{14}{a_{11}}^2\\
&\quad - 2\,a_{25}a_2a_{12}{a_{11}}^2 + 2\,a_{25}a_2a_{12}a_{11}\lambda_2 - 3\,a_2a_{12}a_{14}\lambda_2 a_{11} - 2\,c_5a_{12}a_{11}\\
&\quad \left. - 2\,a_1b_2\lambda_2 a_{11} - c_5a_{12}{a_{11}}^2 + a_{12}b_4{a_{11}}^2\right\},\\
h_4 &= \frac{a_2}{a_{12}\,(\lambda_2 + 1)}\left\{a_{12}b_1\,(\lambda_2 - a_{11}) - c_1{a_{12}}^2 + b_2{\lambda_2}^2 - 2\,b_2\lambda_2 a_{11}\right.
\end{aligned}$$

$$+b_2{a_{11}}^2 - c_2a_{12}\lambda_2 + c_2a_{12}a_{11}\} ,$$

$$h_5 = \frac{1}{\lambda_2+1}\{-a_{12}e_2\lambda_2 - 2\,a_{25}a_1a_{11} - a_1a_{14}\lambda_2 + a_1a_{24}a_{12} - 2\,a_{25}a_1{a_{11}}^2 + a_{12}e_2a_{11}$$
$$-2\,{a_{12}}^2a_1a_{23} - d_3a_{12}{a_{11}}^2 + 2\,a_{25}a_1\lambda_2 - {a_{12}}^3d_1 + {a_{12}}^2e_1(\lambda_2 - a_{11}) + a_{12}e_2{a_{11}}^2$$
$$+a_1a_{14}a_{11} + a_1a_{14}{\lambda_2}^2 - 2\,d_3a_{12}a_{11} + 2\,a_1a_{14}{a_{11}}^2 + 2\,a_{12}a_1a_{13}(\lambda_2 - a_{11})$$
$$+d_2{a_{12}}^2a_{11} - 3\,a_1a_{14}\lambda_2a_{11} - a_1a_{24}a_{12}\lambda_2 + 2\,a_1a_{24}a_{12}a_{11} + 2\,a_{25}a_1a_{11}\lambda_2$$
$$+d_2{a_{12}}^2 - d_3a_{12} - a_{12}e_2\lambda_2a_{11}\} .$$

If the map (3.5) undergoes a flip bifurcation, then it must satisfy the following conditions

$$\alpha_1 = \left[\frac{\partial F_1}{\partial \delta_*}.\frac{\partial^2 F_1}{\partial u^2} + 2\frac{\partial^2 F_1}{\partial u \partial \delta_*}\right]|_{(0,0)} \neq 0,$$

and

$$\alpha_2 = \left[\frac{1}{2}\left(\frac{\partial^2 F_1}{\partial u^2}\right)^2 + \frac{1}{3}\frac{\partial^3 F_1}{\partial u^3}\right]|_{(0,0)} \neq 0.$$

It is noted that

$$\alpha_1 = h_2, \quad \alpha_2 = h_5 + h_1^2.$$

We summarize the above analysis into the following theorem.

Theorem 3.1. *The map (1.2) undergoes a flip bifurcation at the positive fixed point (x_0, y_0) if the following conditions are satisfied*

$$\alpha_1 \neq 0 \quad \text{and} \quad \alpha_2 \neq 0.$$

Moreover, if $\alpha_2 > 0$ (< 0), the 2-period points that bifurcate from this point are stable (resp. unstable).

Next, we give the condition of existence of Neimark-Sacker bifurcation using the Neimark-Sacker bifurcation theorem [9], where δ is chosen as a bifurcation parameter. We recall that the bifurcation corresponding to the presence of $\lambda_{1,2} = e^{\pm i\theta_0}$, $(0 < \theta_0 < \pi)$ is called a Neimark-Sacker (or torus) bifurcation [9]. Taking parameters $(r, K, \alpha, a, b, s, h, \delta_2)$ arbitrarily from H_B, we consider the map (1.2) at the positive fixed point (x_0, y_0). Since $\delta_2 = -\frac{\Delta}{\Omega s}$, choosing γ as a bifurcation parameter, we consider a perturbation of (1.2) as follows

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + (\delta_2+\gamma)x\left[r(1-\frac{x}{K}) - \frac{\alpha xy}{ax^2+bx+1}\right] \\ y + (\delta_2+\gamma)sy\left(1-\frac{hy}{x}\right) \end{pmatrix}. \tag{3.6}$$

Let $u = x - x_0,\ v = y - y_0$. Then we transform the fixed point (x_0, y_0) of (3.6) to the origin. We get

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} a_{11}u + a_{12}v + a_{13}u^2 + a_{14}uv + e_1u^3 + e_2u^2v \\ +O((|u|+|v|)^4) \\ a_{21}u + a_{22}v + a_{23}u^2 + a_{24}uv + a_{25}v^2 + d_1u^3 \\ +d_2u^2v + d_3uv^2 + O((|u|+|v|)^4) \end{pmatrix}, \tag{3.7}$$

where the coefficients are those that are given in (3.1) by substituting δ by $\delta_2+\gamma$.

The eigenvalues of the characteristic equation (2.4) are

$$\lambda_{1,2}=\frac{-p(\gamma)\pm\sqrt{p^2(\gamma)-4q(\gamma)}}{2},$$

where

$$p(\gamma)=-2-\Delta(\delta_2+\gamma),\qquad q(\gamma)=1+\Delta(\delta_2+\gamma)+\Omega s(\delta_2+\gamma).$$

Hence, since parameters belong to H_B we have

$$\lambda_{1,2}=1+\frac{\Delta(\delta_2+\gamma)}{2}\pm i\frac{\delta_2+\gamma}{2}\sqrt{4\Omega s-\Delta}, \tag{3.8}$$

and we have

$$|\lambda_{1,2}|=\sqrt{q(\gamma)},\qquad l=\frac{d|\lambda|}{d\gamma}|_{\gamma=0}=-\frac{\Delta}{2}\neq 0.$$

In addition, $p(0)\neq 0,1$ leads to

$$-\Delta\delta_2\neq 2,3, \tag{3.9}$$

then we obtain $\lambda_{1,2}^n\neq 1,\ n=1,2,3,4$.

Next we study the normal form of (3.6) when $\gamma=0$. Let $\mu=1+\frac{\Delta\delta_2}{2}$, $\omega=\frac{\delta_2}{2}\sqrt{4\Omega s-\Delta^2}$ $T=\begin{pmatrix} a_{12} & 0\\ \mu-a_{11} & -\omega\end{pmatrix}$ and use the translation

$$\begin{pmatrix} u\\ v\end{pmatrix}=T\begin{pmatrix} X\\ Y\end{pmatrix}.$$

Again, for the sake of simplicity, we use x and y instead of X and Y, respectively. Under the translation, the map (3.7) becomes

$$\begin{pmatrix} x\\ y\end{pmatrix}\mapsto\begin{pmatrix} \mu & -\omega\\ \omega & \mu\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix} \tilde{f}(x,y)\\ \tilde{g}(x,y)\end{pmatrix}, \tag{3.10}$$

where

$$\begin{aligned}
\tilde{f}(x,y)&=\frac{1}{a_{12}}\{a_{13}u^2+a_{14}uv+e_1u^3+e_2u^2v\}+O((|u|+|v|)^4),\\
\tilde{g}(x,y)&=\left(\frac{(\mu-a_{12})\,a_{13}}{a_{12}\omega}-\frac{a_{23}}{\omega}\right)u^2+\left(\frac{(\mu-a_{12})\,a_{14}}{a_{12}\omega}-\frac{a_{24}}{\omega}\right)uv\\
&\quad-\frac{a_{25}}{\omega}v^2+\left(\frac{(\mu-a_{12})\,e_1}{a_{12}\omega}-\frac{d_1}{\omega}\right)u^3+\left(\frac{(\mu-a_{12})\,e_2}{a_{12}\omega}-\frac{d_2}{\omega}\right)u^2v\\
&\quad-\frac{d_3uv^2}{\omega}+O((|u|+|v|)^4),
\end{aligned}$$

and $u=a_{12}x,\ \ v=(\mu-a_{11})x-\omega y$.

Next, we compute the stability coefficient of the Neimark-Sacker bifurcation of the map (3.10) using the method given in [5]. The stability coefficient is given as follows

$$\bar{a} = -Re\left(\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}\xi_{20}\xi_{11}\right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + Re(\bar{\lambda}\xi_{21}), \tag{3.11}$$

where

$$\begin{aligned}
\xi_{20} &= \frac{1}{8}[(\tilde{f}_{xx} - \tilde{f}_{yy} + 2\tilde{g}_{xy}) + i(\tilde{g}_{xx} - \tilde{g}_{yy} - 2\tilde{f}_{xy})],\\
\xi_{11} &= \frac{1}{4}[(\tilde{f}_{xx} + \tilde{f}_{yy}) + i(\tilde{g}_{xx} + \tilde{g}_{yy})],\\
\xi_{02} &= \frac{1}{8}[(\tilde{f}_{xx} - \tilde{f}_{yy} - 2\tilde{g}_{xy}) + i(\tilde{g}_{xx} - \tilde{g}_{yy} + 2\tilde{f}_{xy})],\\
\xi_{21} &= \frac{1}{16}[(\tilde{f}_{xxx} + \tilde{f}_{xyy} + \tilde{g}_{xxy} + \tilde{g}_{yyy}) + i(\tilde{g}_{xxx} + \tilde{g}_{xyy} - \tilde{f}_{xxy} - \tilde{f}_{yyy})].
\end{aligned}$$

After some calculation we get

$$\begin{aligned}
&\tilde{f}_{xx} = (2\,\mu - 2\,a_{11})\,a_{14} + 2\,a_{13}a_{12}, \quad \tilde{f}_{xy} = -a_{14}\omega, \quad \tilde{f}_{yy} = 0,\\
&\tilde{f}_{xxx} = 6\,a_{12}\,(e_1a_{12} + e_2(\mu - a_{11}))\,, \quad \tilde{f}_{xxy} = -2\,a_{12}e_2\omega,\ \tilde{f}_{xyy} = \tilde{f}_{yyy} = 0,\\
&\tilde{g}_{xx} = 2\left(\frac{(\mu - a_{12})\,a_{13}}{a_{12}\omega} - \frac{a_{23}}{\omega}\right)a_{12}{}^2 + 2\left(\frac{(\mu - a_{12})\,a_{14}}{a_{12}\omega} - \frac{a_{24}}{\omega}\right)a_{12}\,(\mu - a_{11})\\
&\qquad - 2\,\frac{a_{25}\,(\mu - a_{11})^2}{\omega},\\
&\tilde{g}_{xy} = -a_{14}\mu + a_{14}a_{12} + a_{24}a_{12} + 2\,a_{25}\mu - 2\,a_{25}a_{11}, \quad \tilde{g}_{yy} = -2\,a_{25}\omega,\\
&\tilde{g}_{xxx} = 6\left(\frac{(\mu - a_{12})\,e_1}{a_{12}\omega} - \frac{d_1}{\omega}\right)a_{12}{}^3 + 6\left(\frac{(\mu - a_{12})\,e_2}{a_{12}\omega} - \frac{d_2}{\omega}\right)a_{12}{}^2\,(\mu - a_{11})\\
&\qquad - 6\,\frac{d_3a_{12}\,(\mu - a_{11})^2}{\omega},\\
&\tilde{g}_{xxy} = 2\,a_{12}\,(-e_2\mu + a_{12}e_2 + d_2a_{12} + 2\,d_3\mu - 2\,d_3a_{11})\,,\\
&\tilde{g}_{xyy} = -2\,d_3a_{12}\omega, \quad \tilde{g}_{yyy} = 0.
\end{aligned}$$

From above calculations and the theorem in [5, 9, 13], we get the following result.

Theorem 3.2. *If the condition (3.9) holds and $\bar{a} \neq 0$, then the map (3.6) undergoes Neimark-Sacker bifurcation at the positive fixed point (x_0, y_0) when the parameter γ varies in the small neighborhood of the origin. Moreover, if $\bar{a} < 0$ (resp., $\bar{a} > 0$), then an attracting (resp., repelling) invariant closed curve bifurcates from the fixed point for $\gamma > 0$ (resp., $\gamma < 0$).*

4. Numerical Simulations

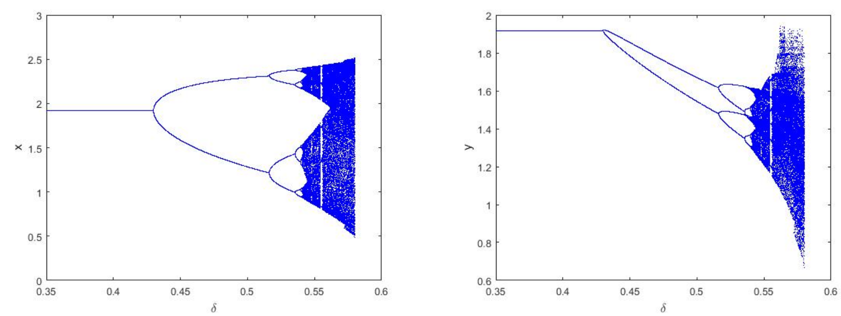
In this section, we will present bifurcation diagrams, maximum Lyapunov exponent, and phase portraits of system (1.2) to illustrate the above theoretical analysis and find new interesting complex dynamics using numerical tools. The bifurcation parameters are considered in the following two cases.

Case (**i**) Varying δ in the range $0.35 \le \delta \le 0.6$, and fixing $K = 2,\ r = 5,\ s = 1,\ h = 1,\ \alpha = 1,\ a = 3,\ b = 3$.

Case (**ii**) Varying δ in the range $1 \le \delta \le 1.35$, and fixing $K = \frac{5}{2},\ r = 3,\ s = 0.5,\ h = 1,\ \alpha = 9,\ a = 2,\ b = 2$.

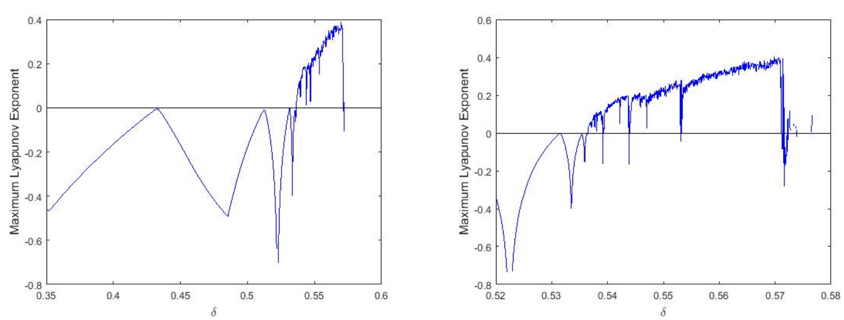
For case (**i**). $K = 2,\ r = 5,\ s = 1,\ h = 1,\ \alpha = 1,\ a = 3,\ b = 3$; based on Lemma 2.2, we know that the system (1.2) has a unique positive fixed point $(1.917299545, 1.917299545)$. The flip bifurcation emerges from this positive fixed point at $\delta = 0.433$ with $\alpha_1 = -2.899433016$ and $\alpha_2 = 0.02983774197$ and $(r, K, \alpha, a, b, s, h, \delta) \in F_{B_1}$. This confirms the results of Theorem 3.1.

According to the bifurcation diagrams shown in Figs. 1(a) and 1(b), the positive fixed point is stable for $\delta < 0.433$, then it loses its stability at the flip bifurcation parameter $\delta = 0.433$. We observe that there is a cascade of period doubling for $0.433 < \delta < 0.6$.



(a) in (δ, x) plane (b) in (δ, y) plane

Figure 1. Bifurcation diagram of the system (1.2) for $K = 2,\ r = 5,\ s = 1,\ h = 1,\ \alpha = 1,\ a = 3,\ b = 3$ with initial value $(1, 1)$.

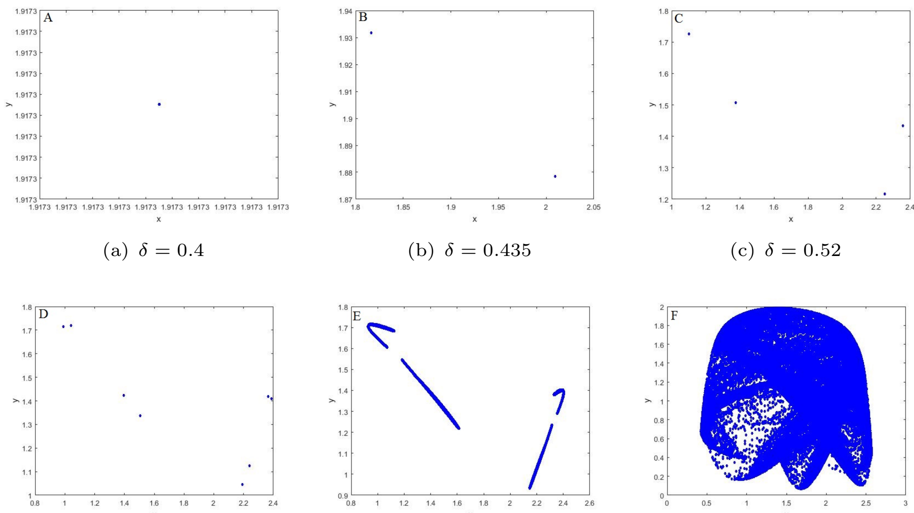


(a) $\delta \in [0.35, 0.6]$ (b) local magnification for $\delta \in [0.52, 0.56]$

Figure 2. Maximum Lyapunov exponent corresponding to bifurcation diagrams in Fig.1.

The maximum Lyapunov exponent corresponding to bifurcation diagram Figs. 1(a) and 1(b) is computed in Fig. 2(a). The local magnification of Fig. 2(a) for $\delta \in [0.52, 0.56]$ is presented in Fig. 2(b). The maximum Lyapunov exponent corresponding to $\delta = 0.57$ is $h_1 = 0.3637726139 > 0$, which confirms the existence of the chaotic sets. In general the positive Lyapunov exponent is considered to be one of the characteristics implying the existence of chaos [1,3,4].

The phase portrait of system (1.2) corresponding to bifurcation diagram Fig. 1 is shown in Fig. 3. We observe that there are period-1, period-2, period-4, period-8 orbits, quasi-periodic and when $\delta = 0.57$ we can see an attracting chaotic sets.

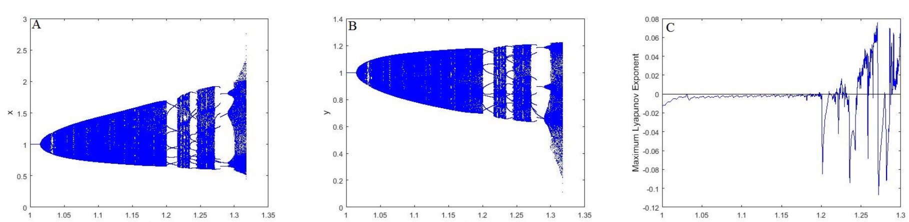


(a) $\delta = 0.4$ (b) $\delta = 0.435$ (c) $\delta = 0.52$

(d) $\delta = 0.533$ (e) $\delta = 0.54$ (f) $\delta = 0.57$

Figure 3. Phase portrait for various values of δ corresponding to Fig. 1.

For case (ii). $K = \frac{5}{2},\ r = 3,\ s = 0.5,\ h = 1,\ \alpha = 9,\ a = 2,\ b = 2$; by Lemma 2.2 we know that the system (1.2) has a unique positive fixed point at $(1, 1)$. The Neimark-Sacker bifurcation appears from the fixed point $(1, 1)$ at $\delta = 1.0151$, its eigenvalues are $\lambda_{1,2} = 0.31995 \pm 0.947327291647401i$ and $|\lambda_{1,2}| = 1,\ l = 0.680872 > 0,\ \bar{a} = -5.483464001$ and $(r, K, \alpha, a, b, s, h, \delta) \in H_B$. This confirms the result of Theorem 3.2.



(a) in (δ, x) plane (b) in (δ, y) plane (c) Maximum Lyapunov exponent corresponding to bifurcation diagrams (a) and (b)

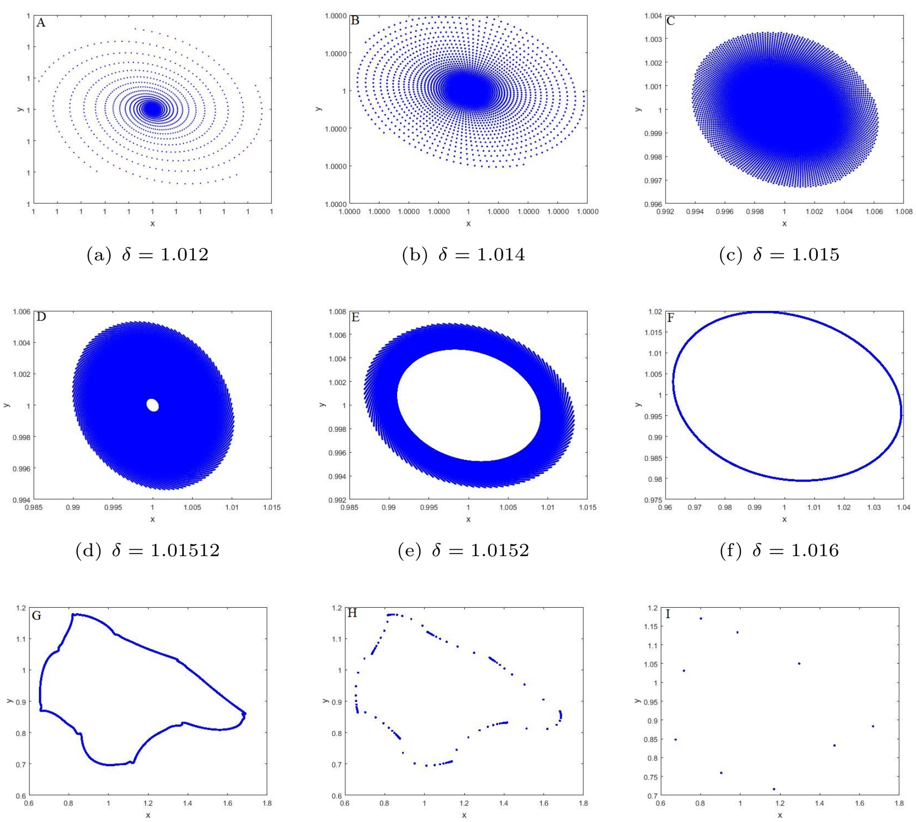
Figure 4. Bifurcation diagram of the system (1.2) for $K = \frac{5}{2},\ r = 3,\ s = 0.5,\ h = 1,\ \alpha = 9,\ a = 2,\ b = 2$ with initial value $(0.7, 0.8)$.

Figs. 4(a) and 4(b) show that the positive fixed point $(1, 1)$ is stable for $\delta < 1.0151$, then it looses its stability at $\delta = 1.0151$. For $\delta > 1.0151$ an invariant attracting cycle appears.

The maximum Lyapunov exponents corresponding to Figs. 4(a) and 4(b) are calculated and plotted in Fig. 4(c), that shows the existence of chaotic region and periodic orbits (non-chaotic region) as the parameter δ varying.

From Fig. 4(c) we can easily see that that the maximum Lyapunov exponents are negative for the parameter $\delta \in (1, 1.07)$, that is to say, the non-chaotic region is smaller than the chaotic region $(1.07, 1.3)$.

The phase portraits corresponding to Fig. 4 are plotted in Fig. 5, which demonstrates the process of how an invariant periodic orbit bifurcates from the stable fixed point. When δ exceeds 1.0152 a closed curve enclosing the fixed point is born. When δ increases at certain values, for instance, at $\delta = 1.201$, the periodic orbit disappears and a period-9 orbit emerges, and some cascades of period doubling bifurcations lead to chaos.



(a) $\delta = 1.012$ (b) $\delta = 1.014$ (c) $\delta = 1.015$

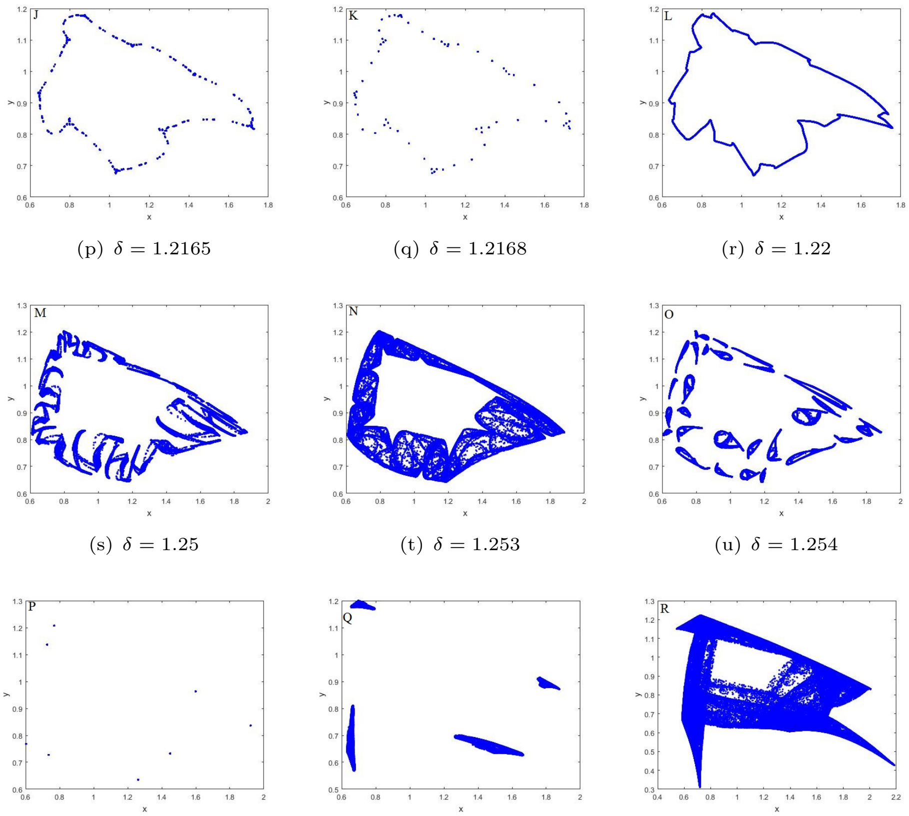
(d) $\delta = 1.01512$ (e) $\delta = 1.0152$ (f) $\delta = 1.016$

(g) $\delta = 1.198$ (h) $\delta = 1.2004$ (i) $\delta = 1.201$

Figure 5. Phase portrait for various values of δ corresponding to Fig. 4.

5. Controlling and suppressing chaos

It is known that chaotic maps are characterized by an exponential separation of nearby orbits in forward iterations (positive Lyapunov exponent). This feature of



(p) $\delta = 1.2165$ (q) $\delta = 1.2168$ (r) $\delta = 1.22$

(s) $\delta = 1.25$ (t) $\delta = 1.253$ (u) $\delta = 1.254$

(v) $\delta = 1.274$ (w) $\delta = 1.298$ (x) $\delta = 1.31$

Figure 5. Phase portrait for various values of δ corresponding to Fig. 4(continued).

chaos has been traditionally seen as a troublesome property, especially in practical settings, because even the tiniest perturbation might modify the system's behavior in an unpredictable way and lead the system to a catastrophic situation. Chaotic behavior is therefore undesirable in many practical settings, and one is interested in controlling the system to obtain regular behavior. This can be done by taking advantage of the infinite number of unstable periodic orbits coexisting with the chaotic attractor. The idea of controlling chaos consists of stabilizing some of these unstable orbits, thus leading to regular and predictable behavior [2, 6].

Feedback control is an algorithm that has been recognized as one of the methods to be useful for stabilizing unstable periodic orbits [12]. In this section, we shall apply the state feedback control method to stabilize chaotic orbits near an unstable fixed point of (1.2).

Consider the positive fixed point of the map (1.2) at (x_0, y_0). Using the notation introduced in [12], for values of δ close to δ_2 in a small neighborhood of (x_0, y_0), the map (1.2) can be approximated by the linear map given by

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11}u_n + a_{12}v_n \\ a_{21}u_n + a_{22}v_n \end{pmatrix},$$

where $u_n = x_n - x_0,\ v_n = y_n - y_0$ and a_{11}, a_{12}, a_{21} and a_{22} are given in (3.1). Consider the following controlled form of the above system

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11}u_n + a_{12}v_n + p_n \\ a_{21}u_n + a_{22}v_n \end{pmatrix}, \tag{5.1}$$

with the following feedback control law as the control force

$$p_n = -k_1 u_n - k_2 v_n,$$

where k_1 and k_2 are the feedback gain. The Jacobian matrix of the controlled system (5.1) at $(0, 0)$ is given by

$$J = \begin{pmatrix} a_{11} - k_1 & a_{12} - k_2 \\ a_{21} & a_{22} \end{pmatrix}.$$

The characteristic equation of J is

$$\lambda^2 - (a_{11} + a_{22} - k_1)\lambda + a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2) = 0.$$

Suppose that the eigenvalues (regulator poles) are given by λ_1 and λ_2, then

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} - k_1 \ \text{ and } \ \lambda_1\lambda_2 = a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2). \tag{5.2}$$

The lines of marginal stability are determined by solving the equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These conditions guarantee that the eigenvalues λ_1 and λ_2 have modulus less than unity. Suppose that $\lambda_1\lambda_2 = 1$. Then

$$l_1 \ : \ k_1 a_{22} - k_2 a_{21} = a_{11}a_{22} - a_{12}a_{21} - 1.$$

Now, first assume that $\lambda_1 = 1$ then from (5.2) we have

$$l_2 \ : \ k_1(1 - a_{22}) - k_2 a_{21} = a_{11} + a_{22} - 1 - a_{11}a_{22} + a_{12}a_{21}.$$

Next, assume that $\lambda_1 = -1$, then by (5.2) we have

$$l_3 \ : \ k_1(1 + a_{22}) - k_2 a_{21} = a_{11} + a_{22} + 1 + a_{11}a_{22} - a_{12}a_{21}.$$

The stable eigenvalues (regulator poles) lie within a triangular region as depicted in Fig. 6. Select $k_1 = -1$ and $k_2 = -2$. This point lies inside the triangular region as depicted in Fig. 6. The perturbed system (1.2) becomes

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n + \delta x_n \left[r(1 - \frac{x_n}{K}) - \frac{\alpha x_n y_n}{a x_n^2 + b x_n + 1} \right] - k_1(x_n - x_0) - k_2(y_n - y_0) \\ y_n + \delta s y_n \left(1 - \frac{h y_n}{x_n} \right) \end{pmatrix}. \tag{5.3}$$

We have applied a numerical simulations to see how the state feedback control method controls the unstable positive fixed point $(1, 1)$. Parameter values are fixed as $r = 3,\ h = 1,\ s = 0.5,\ K = 2.5,\ \alpha = 9,\ a = b = 2,\ \delta = 1.253$. The initial value is $(0.8, 0.8)$ and the feedback gain $k_1 = -1$ and $k_2 = -2$. By Fig. 6 we see that a chaotic trajectory is stabilized at the fixed point $(1, 1)$.

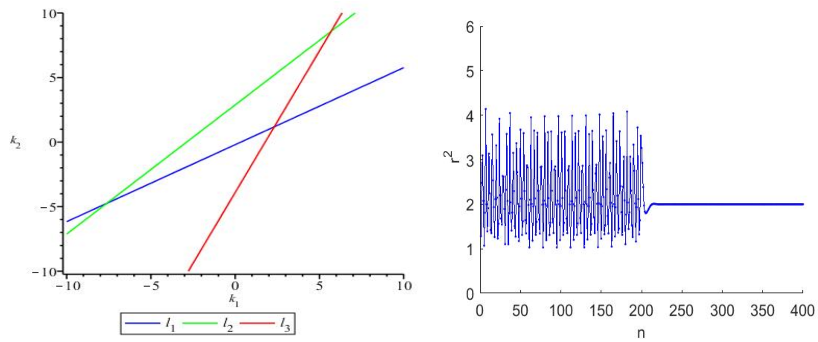


Figure 6. Left: The bounded region for the eigenvalues of the perturbed system (5.3) for $h = 1$, $s = 0.5$, $r = 3$, $\alpha = 9$, $K = 2.5$, $a = b = 2$ and $\delta = 1.253$. Right: Time series data for the map (5.3) with and without control, $r^2 = x^2 + y^2$. The control is activated after the 198th iterate.

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