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SOME QUANTUM ESTIMATES OF HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

Wenjun Liu† and Hefeng Zhuang

Abstract In this study, based on a new quantum integral identity, we establish some quantum estimates of Hermite-Hadamard type inequalities for convex functions. These results generalize and improve some known results given in literatures.

Keywords Convex functions, Hermite-Hadamard inequality, Hölder integral inequality, Quantum estimate.

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1. Introduction

Convex functions play an important role in mathematical inequalities [11–13, 16, 20, 29]. One of the most famous inequalities for convex functions is Hermite-Hadamard inequality, which is stated as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, where $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This famous result can be considered as a necessary and sufficient condition for a function to be convex.

Hermite-Hadamard inequality has drawn many researchers' interest, a variety of refinements and generalizations have been found (see for example, [1, 2, 4, 5, 7–9, 14, 15, 17, 18, 21–23, 28, 33]).

In [20], Özdemir established the following lemma.

Lemma 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on $I^0, a, b \in I$ with $a < b$ and f'' be integrable on $[a, b]$. Then the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\\ =&\frac{(b-a)^2}{2}\int_0^1 s(1-s)f''(sa+(1-s)b)ds. \end{aligned} \tag{1.1}$$

†the corresponding author. Email address: wjliu@nuist.edu.cn (W. J. Liu)
College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

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In [21], Özdemir etc gave some estimates for the result of Lemma 1.1 via m-convexity.

Theorem 1.1. *Let $f : I^0 \to \mathbb{R}$, where $I^0 \subset [0,\infty)$ be a twice differentiable function on I^0, $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a,b]$. If $|f''|^r$ is m-convex on $[a,b]$ for some fixed $r > 1$ and $m \in (0,1]$ then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^r+m\left|f''\left(\frac{b}{m}\right)\right|^r}{2}\right)^{\frac{1}{r}},\end{aligned} \tag{1.2}$$

where $p = \frac{r}{r-1}$.

Theorem 1.2. *Let $f : I^0 \to \mathbb{R}$, where $I^0 \subset [0,\infty)$ be a twice differentiable function on I^0, $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a,b]$. If $|f''|^r$ is m-convex on $[a,b]$ for some fixed $r > 1$ and $m \in (0,1]$ then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{(b-a)^2}{2}\left(|f''(a)|^r+m(r+1)\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}.\end{aligned} \tag{1.3}$$

Theorem 1.3. *Let $f : I^0 \to \mathbb{R}$, where $I^0 \subset [0,\infty)$ be a twice differentiable function on I^0, $a, b \in I$ with $a < b$ and suppose that $f'' \in L[a,b]$. If $|f''|^r$ is m-convex on $[a,b]$ for some fixed $r \geq 1$ and $m \in (0,1]$ then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{(b-a)^2}{4}\left(2|f''(a)|^r+m(r+1)\left|f''\left(\frac{b}{m}\right)\right|^r\right)^{\frac{1}{r}}.\end{aligned} \tag{1.4}$$

In recent years, the topic of quantum calculus has attracted the attention of several scholars [10,27]. Quantum calculus appeared as a connection between mathematics and physics. It has large applications in many mathematical areas such as number theory, special functions, quantum mechanics and mathematical inequalities [19, 24–26]. At present, q-analogues of many identities and inequalities have been established.

In [19], Noor etc established the following lemma and developed some quantum estimates of it.

Lemma 1.2. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx-\frac{qf(a)+f(b)}{1+q}\\ =&\frac{q(b-a)}{1+q}\int_0^1(1-(1+q)t)\,{}_aD_qf((1-t)a+tb)\,{}_0d_qt.\end{aligned}$$

Theorem 1.4. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q f\right|^r, r \geq 1$ is a convex function, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx - \frac{qf(a)+f(b)}{1+q}\right| \\ \leq &\frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\left(\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(a)\right|^r\right. \\ &\left.+\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

Theorem 1.5. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I^0 (the interior of I) with ${}_aD_q$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q f\right|^r$ is a convex function where $p, r > 1$, $\frac{1}{p}+\frac{1}{r}=1$, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx - \frac{qf(a)+f(b)}{1+q}\right| \\ \leq &\frac{q(b-a)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{\frac{1}{p}}\left(\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(a)\right|^r\right. \\ &\left.+\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

The main purpose of this paper is to establish a new quantum integral identity similar to the one given in Lemma 1.1 and develop some quantum estimates of Hermite-Hadamard type inequalities for convex functions. Our results in special cases recapture Lemma 1.1 and Theorems 1.1-1.3.

2. Preliminaries

In this section, we first recall some previously known concepts on q-calculus which will be used in this paper.

Let $J = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant.

Definition 2.1 ([26]). Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then q-derivative on J of function f at x is defined as

$${}_aD_qf\left(x\right) = \frac{f\left(x\right) - f\left(qx+(1-q)\,a\right)}{\left(1-q\right)\left(x-a\right)}, x \neq a, \quad {}_aD_qf(a) = \lim_{x\to a} {}_aD_qf(x). \tag{2.1}$$

We say that f is q-differentiable on J provided ${}_aD_qf\left(x\right)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_qf = D_qf$, where D_q is the well-known q-derivative of the function $f\left(x\right)$ defined by

$$D_qf\left(x\right) = \frac{f\left(x\right) - f\left(qx\right)}{\left(1-q\right)x}. \tag{2.2}$$

Definition 2.2 ([26]). Let $f : J \to \mathbb{R}$ be a continuous function. We define the second-order q-derivative on interval J, which is denoted as ${}_aD_q^2 f$, provided ${}_aD_q f$ is q-differentiable on J with ${}_aD_q^2 f = {}_aD_q({}_aD_q f) : J \to \mathbb{R}$. Similarly, we define higher order q-derivative on J, ${}_aD_q^n : J \to \mathbb{R}$.

Definition 2.3 ([26]). Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Then q-integral on J is defined by

$$\int_a^x f(t)\, {}_ad_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.3}$$

for $x \in J$. Note that if $a = 0$, then we have the classical q-integral, which is defined by

$$\int_0^x f(t)\, {}_0d_q t = (1-q)\, x \sum_{n=0}^{\infty} q^n f(q^n x) \tag{2.4}$$

for $x \in [0, +\infty)$.

Theorem 2.1 ([26]). *Assume $f, g : J \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

$$\int_a^x [f(t) + g(t)]\, {}_ad_q t = \int_a^x f(t)_a d_q t + \int_a^x g(t)_a d_q t;$$
$$\int_a^x (\alpha f)(t)_a d_q t = \alpha \int_a^x f(t)_a d_q t.$$

In addition, we introduce the q-analogues of a and $(x-a)^n$ and the definition of q-Beta function.

Definition 2.4 ([10]). For any real number a,

$$[a] = \frac{1-q^a}{1-q} \tag{2.5}$$

is called the q-analogue of a. In particular, for $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{1-q^n}{1-q} = q^{n-1} + \ldots + q + 1.$$

Definition 2.5 ([10]). If n is an integer, the q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1, & \text{if} \quad n = 0, \\ (x-a)(x-qa)\ldots(x-q^{n-1}a), & \text{if} \quad n \geq 1. \end{cases} \tag{2.6}$$

Definition 2.6 ([10]). For any $t, s > 0$,

$$\beta_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\, {}_0d_q x \tag{2.7}$$

is called the q-Beta function. Note that

$$\beta_q(t,1) = \int_0^1 x^{t-1}\, {}_0d_q x = \frac{1}{[t]}, \tag{2.8}$$

where $[t]$ is the q-analogue of t.

3. Some simple calculations

In this section, we present some simple calculations that will be used in this paper.

Lemma 3.1. *Let $f(x) = 1$ in Definition 2.3, then we have*

$$\int_0^1 {}_0d_qx = (1-q)\sum_{n=0}^{\infty} q^n = 1.$$

Lemma 3.2. *Let $f(x) = x$ in Definition 2.3 for $x \in [0,1]$, then we have*

$$\int_0^1 x_0d_qx = (1-q)\sum_{n=0}^{\infty} q^{2n} = \frac{1}{1+q}.$$

Lemma 3.3. *Let $f(x) = x^2$ in Definition 2.3 for $x \in [0,1]$, then we have*

$$\int_0^1 x^2{}_0d_qx = (1-q)\sum_{n=0}^{\infty} q^{3n} = \frac{1}{1+q+q^2}.$$

Lemma 3.4. *Let $f(x) = 1-x$ in Definition 2.3 for $x \in [0,1]$, then we have*

$$\int_0^1 (1-x)_0d_qx = \int_0^1 {}_0d_qx - \int_0^1 x_0d_qx = 1 - \frac{1}{1+q} = \frac{q}{1+q}.$$

Lemma 3.5. *Let $f(x) = 1-qx$ in Definition 2.3 for $x \in [0,1]$ where $0<q<1$ be a constant, then we have*

$$\int_0^1 (1-qx)_0d_qx = \int_0^1 {}_0d_qx - q\int_0^1 x_0d_qx = 1 - q\frac{1}{1+q} = \frac{1}{1+q}.$$

Lemma 3.6. *Let $f(x) = x(1-x)$ in Definition 2.3 for $x \in [0,1]$, then we have*

$$\int_0^1 x(1-x)_0d_qx = \int_0^1 x_0d_qx - \int_0^1 x^2{}_0d_qx = \frac{q^2}{(1+q)(1+q+q^2)}.$$

Lemma 3.7. *Let $f(x) = x(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0<q<1$ be a constant, then we have*

$$\int_0^1 x(1-qx)_0d_qx = \int_0^1 x_0d_qx - q\int_0^1 x^2{}_0d_qx = \frac{1}{(1+q)(1+q+q^2)}.$$

Lemma 3.8. *Let $f(x) = (1-x)(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0<q<1$ be a constant, then we have*

$$\int_0^1 (1-x)(1-qx)_0d_qx = \int_0^1 (1-x)_0d_qx - q\int_0^1 x(1-x)_0d_qx = \frac{q}{1+q+q^2}.$$

Lemma 3.9. *Let $f(x) = x^2(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0<q<1$ be a constant, then we have*

$$\int_0^1 x^2(1-qx)_0d_qx = \int_0^1 (x^2-qx^3)_0d_qx = \frac{1}{(1+q+q^2)(1+q+q^2+q^3)}.$$

Lemma 3.10. *Let $f(x) = x(1-x)(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0 < q < 1$ be a constant, then we have*

$$\int_0^1 x(1-x)(1-qx)_0 d_q x = \int_0^1 x_0 d_q x - (1+q)\int_0^1 x^2{}_0 d_q x + q\int_0^1 x^3{}_0 d_q x$$
$$= \frac{q^2}{(1+q+q^2)(1+q+q^2+q^3)}.$$

Lemma 3.11. *Let $f(x) = x^r(1-x)$ in Definition 2.3 for $x \in [0,1]$ where $r \geq 1$ be a constant, then we have*

$$\int_0^1 x^r(1-x)_0 d_q x = \frac{q^{r+1}}{[r+1]\,[r+2]},$$

where $[r+1]$, $[r+2]$ are the q-analogues of $r+1$ and $r+2$.

Proof. By using (2.5) and (2.8), we have

$$\int_0^1 x^r(1-x)_0 d_q x = \int_0^1 x^r{}_0 d_q x - \int_0^1 x^{r+1}{}_0 d_q x = \frac{1-q}{1-q^{r+1}} - \frac{1-q}{1-q^{r+2}}$$
$$= (1-q)\frac{q^{r+1}(1-q)}{(1-q^{r+1})(1-q^{r+2})} = \frac{q^{r+1}}{[r+1]\,[r+2]}.$$

The proof is completed. □

Lemma 3.12. *Let $f(x) = x^{r+1}(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0 < q < 1$ and $r \geq 1$ be constants, then*

$$\int_0^1 x^{r+1}(1-qx)_0 d_q x = \frac{1}{[r+2]\,[r+3]},$$

where $[r+2]$, $[r+3]$ are the q-analogues of $r+2$ and $r+3$.

Proof. By using (2.5) and (2.8), we have

$$\int_0^1 x^{r+1}(1-qx)_0 d_q x = \int_0^1 x^{r+1}{}_0 d_q x - q\int_0^1 x^{r+2}{}_0 d_q x = \frac{1}{[r+2]} - \frac{q}{[r+3]}$$
$$= \frac{(1-q)(1-q)}{(1-q^{r+2})(1-q^{r+3})} = \frac{1}{[r+2]\,[r+3]}.$$

The proof is completed. □

Lemma 3.13. *Let $f(x) = x^r(1-x)(1-qx)$ in Definition 2.3 for $x \in [0,1]$ where $0 < q < 1$ and $r \geq 1$ be constants, then we have*

$$\int_0^1 x^r(1-x)(1-qx)_0 d_q x = \frac{(1+q)q^{r+1}}{[r+1]\,[r+2]\,[r+3]},$$

where $[r+1]$, $[r+2]$, $[r+3]$ are the q-analogues of $r+1$, $r+2$ and $r+3$.

Proof. By using (2.5) and (2.8), we have

$$\begin{aligned}\int_0^1 x^r(1-x)(1-qx)_0 d_q x =& \frac{1}{[r+1]} - \frac{1+q}{[r+2]} + \frac{q}{[r+3]} \\ =& \Big(\frac{(1-q^{r+2})(1-q^{r+3})}{(1-q)(1-q)} - \frac{(1+q)(1-q^{r+1})(1-q^{r+3})}{(1-q)(1-q)} \\ &+ \frac{q(1-q^{r+1})(1-q^{r+2})}{(1-q)(1-q)} \Big) \times \left(\frac{1}{[r+1]\,[r+2]\,[r+3]} \right) \\ =& \frac{(1+q)q^{r+1}}{[r+1]\,[r+2]\,[r+3]}.\end{aligned}$$

The proof is completed. □

4. A new quantum integral identity

In this section, we establish a new q-integral identity.

Lemma 4.1. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0<q<1$. Then the following identity holds:*

$$\begin{aligned}&\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_a d_q x \\ =& \frac{q^2(b-a)^2}{1+q}\int_0^1 t(1-qt)\,{}_a D_q^2 f((1-t)a+tb)\,{}_0 d_q t. \qquad (4.1)\end{aligned}$$

Proof. *From Definition 2.1 and 2.2, we have*

$$\begin{aligned}&{}_aD_q^2 f((1-t)a+tb) \\ =& {}_aD_q({}_aD_q f((1-t)a+tb)) \\ =& \frac{{}_aD_q f((1-t)a+tb) - {}_aD_q f((1-qt)a+qtb)}{(1-q)(b-a)t} \\ =& \Bigg[\frac{f((1-t)a+tb)-f((1-qt)a+qtb)}{(1-q)(b-a)t} \\ &- \frac{f((1-qt)a+qtb)-f((1-q^2t)a+q^2tb)}{(1-q)q(b-a)t} \Bigg] \div [(1-q)(b-a)t] \\ =& \frac{qf((1-t)a+tb)-(1+q)f((1-qt)a+qtb)+f((1-q^2t)a+q^2tb)}{(1-q)^2(b-a)^2qt^2}.\end{aligned}$$

Applying this calculation and Definition 2.3, we have

$$\begin{aligned}&\int_0^1 t(1-qt)\,{}_aD_q^2 f((1-t)a+tb)\,{}_0d_q t \\ =& \int_0^1 t(1-qt)\Bigg(\frac{qf((1-t)a+tb)-(1+q)f((1-qt)a+qtb)}{(1-q)^2(b-a)^2qt^2} \\ &+ \frac{f((1-q^2t)a+q^2tb)}{(1-q)^2(b-a)^2qt^2} \Bigg)\,{}_0d_q t\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \frac{q(1-q)\sum\limits_{n=0}^{\infty} f((1-q^n)a+q^n b)}{(1-q)^2(b-a)^2 q} - \frac{(1+q)(1-q)\sum\limits_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1} b)}{(1-q)^2(b-a)^2 q} \right.\\
&\quad \left. + \frac{(1-q)\sum\limits_{n=0}^{\infty} f((1-q^{n+2})a+q^{n+2} b)}{(1-q)^2(b-a)^2 q} \right\}\\
&\quad - q\left\{ \frac{q(1-q)(b-a)\sum\limits_{n=0}^{\infty} q^n f((1-q^n)a+q^n b)}{(1-q)^2(b-a)^3 q} \right.\\
&\quad - \frac{(1+q)(1-q)(b-a)\sum\limits_{n=0}^{\infty} q^{n+1} f((1-q^{n+1})a+q^{n+1} b)}{(1-q)^2(b-a)^3 q^2}\\
&\quad \left. + \frac{(1-q)(b-a)\sum\limits_{n=0}^{\infty} q^{n+2} f((1-q^{n+2})a+q^{n+2} b)}{(1-q)^2(b-a)^3 q^3} \right\}\\
&= \left\{ \frac{q\left[\sum\limits_{n=0}^{\infty} f((1-q^n)a+q^n b) - \sum\limits_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1} b)\right]}{(1-q)(b-a)^2 q} \right\}\\
&\quad - \left\{ \frac{\sum\limits_{n=0}^{\infty} f((1-q^{n+1})a+q^{n+1} b) - \sum\limits_{n=0}^{\infty} f((1-q^{n+2})a+q^{n+2} b)}{(1-q)(b-a)^2 q} \right\}\\
&\quad - q\left\{ \frac{\int_a^b f(x)_a d_q x}{(1-q)^2(b-a)^3} - \frac{(1+q)\int_a^b f(x)_a d_q x - (1+q)(1-q)(b-a)f(b)}{(1-q)^2(b-a)^3 q^2} \right.\\
&\quad \left. + \frac{\int_a^b f(x)_a d_q x - (1-q)(b-a)f(b) - (1-q)(b-a)q f((1-q)a+qb)}{(1-q)^2(b-a)^3 q^3} \right\}\\
&= \left\{ \frac{q(f(b)-f(a))}{(1-q)(b-a)^2 q} - \frac{f((1-q)a+qb)-f(a)}{(1-q)(b-a)^2 q} \right\}\\
&\quad - \left\{ \frac{1+q}{(b-a)^3 q^2}\int_a^b f(x)_a d_q x + \frac{q^2+q-1}{(1-q)(b-a)^2 q^2} f(b) - \frac{q f((1-q)a+qb)}{(1-q)(b-a)^2 q^2} \right\}\\
&= \frac{q f(a)+f(b)}{(b-a)^2 q^2} - \frac{1+q}{(b-a)^3 q^2}\int_a^b f(x)_a d_q x.
\end{aligned}
$$

Multiplying both sides by $\frac{q^2(b-a)^2}{1+q}$, *we complete the proof.* □

Remark 4.1. If $q \to 1$ and substitute $sa+(1-s)b$ for $(1-t)a+tb$, then (4.1) reduces to identity (1.1) in Lemma 1.1.

5. Hermite-Hadamard inequalities for convex functions

In this section, we will give some estimates for the left-hand side of (4.1) through convex functions.

Theorem 5.1. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a, b]$ for $r \geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\ \leq &\frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}}\left(m_1\left|{}_aD_q^2 f(a)\right|^r + m_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}},\end{aligned} \tag{5.1}$$

where

$$m_1 = (1-q)\sum_{n=0}^{\infty}(q^{2n} - q^{3n})(1-q^{n+1})^r, \quad m_2 = (1-q)\sum_{n=0}^{\infty} q^{3n}(1-q^{n+1})^r.$$

Proof. Using Lemma 4.1 and Hölder inequality, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\ \leq &\frac{q^2(b-a)^2}{1+q}\int_0^1 t(1-qt)\left|{}_aD_q^2 f((1-t)a+tb)\right|\,{}_0d_qt \\ \leq &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t\,{}_0d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1 t(1-qt)^r\left|{}_aD_q^2 f((1-t)a+tb)\right|^r\,{}_0d_qt\right)^{\frac{1}{r}}.\end{aligned}$$

Applying Lemma 3.2 and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\ \leq &\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 t(1-t)(1-qt)^r\,{}_0d_qt\right. \\ &\left. + \left|{}_aD_q^2 f(b)\right|^r\int_0^1 t^2(1-qt)^r\,{}_0d_qt\right)^{\frac{1}{r}} \\ = &\frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}}\left(m_1\left|{}_aD_q^2 f(a)\right|^r + m_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}m_1 &= \int_0^1 t(1-t)(1-qt)^r\,{}_0d_qt = (1-q)\sum_{n=0}^{\infty}(q^{2n}-q^{3n})(1-q^{n+1})^r, \\ m_2 &= \int_0^1 t^2(1-qt)^r\,{}_0d_qt = (1-q)\sum_{n=0}^{\infty}q^{3n}(1-q^{n+1})^r.\end{aligned}$$

The proof is completed. □

Remark 5.1. If $q \to 1$, then we have

$$\int_0^1 t(1-t)^{r+1}dt = \frac{1}{(r+2)(r+3)}, \quad \int_0^1 t^2(1-t)^r dt = \frac{2}{(r+1)(r+2)(r+3)},$$

and (5.1) reduces to the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{2^{2-\frac{1}{r}}}\left(\frac{(r+1)\left|f''(a)\right|^r + 2\left|f''(b)\right|^r}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}.$$

Corollary 5.1. *If r is a positive integer, then*

$$(1-qt)^r \le (1-qt)_q^r, \quad (1-t)(1-qt)^r \le (1-qt)_q^{r+1}.$$

Theorem 5.1 reduces to

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \le &\frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}}\left(\beta_q(2,r+2)\left|{}_aD_q^2 f(a)\right|^r + \beta_q(3,r+1)\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

Theorem 5.2. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ where $p, r > 1$, $\frac{1}{p}+\frac{1}{r}=1$, then*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \le &\frac{q^2(b-a)^2}{1+q}(k_1)^{\frac{1}{p}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r + (1+q)\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}, \qquad (5.2)\end{aligned}$$

where

$$k_1 = (1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^p.$$

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \le &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t(1-qt)^p {}_0d_q t\right)^{\frac{1}{p}}\left(\int_0^1 t\left|{}_aD_q^2 f((1-t)a+tb)\right|^r {}_0d_q t\right)^{\frac{1}{r}} \\ \le &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t(1-qt)^p {}_0d_q t\right)^{\frac{1}{p}} \\ &\times\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 t(1-t)_0 d_q t + \left|{}_aD_q^2 f(b)\right|^r\int_0^1 t^2{}_0 d_q t\right)^{\frac{1}{r}}.\end{aligned}$$

Applying Lemma 3.3 and 3.6, we have

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\leq &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t(1-qt)^p{}_0 d_q t\right)^{\frac{1}{p}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r+(1+q)\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}} \\
= &\frac{q^2(b-a)^2}{1+q}(k_1)^{\frac{1}{p}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r+(1+q)\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}.
\end{aligned}$$

It is easy to check that

$$k_1=\int_0^1 t(1-qt)^p{}_0 d_q t=(1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^p.$$

The proof is completed. □

Remark 5.2. If $q\to 1$, then we have

$$\int_0^1 t(1-t)^p dt=\frac{1}{(p+1)(p+2)},$$

and (5.2) reduces to the following inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq &\frac{(b-a)^2}{2}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^r+2\left|f''(b)\right|^r}{6}\right)^{\frac{1}{r}}.
\end{aligned}$$

Corollary 5.2. *If p is a positive integer, $p>1$, then*

$$(1-qt)^p\leq(1-qt)_q^p.$$

Theorem 5.2 reduces to

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\leq &\frac{q^2(b-a)^2}{1+q}\left(\beta_q(2,p+1)\right)^{\frac{1}{p}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r+(1+q)\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}.
\end{aligned}$$

Theorem 5.3. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0<q<1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ for $r\geq 1$, then*

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\leq &\frac{q^2(b-a)^2}{1+q}\left(h_1\left|{}_aD_q^2 f(a)\right|^r+h_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}, \qquad (5.3)
\end{aligned}$$

where

$$\begin{aligned} h_1 =&(1-q)\sum_{n=0}^{\infty}(q^n)^{r+1}(1-q^n)(1-q^{n+1})^r, \\ h_2 =&(1-q)\sum_{n=0}^{\infty}(q^n)^{r+2}(1-q^{n+1})^r. \end{aligned}$$

Proof. Using Lemma 4.1 and Hölder inequality, we have

$$\begin{aligned} &\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \le&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 {}_0d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^r(1-qt)^r\left|{}_aD_q^2 f((1-t)a+tb)\right|^r {}_0d_q t\right)^{\frac{1}{r}}. \end{aligned}$$

Applying Lemma 3.1 and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned} &\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \le&\frac{q^2(b-a)^2}{1+q}\Bigg(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 t^r(1-qt)^r(1-t)_0 d_q t \\ &+\left|{}_aD_q^2 f(b)\right|^r\int_0^1 t^{r+1}(1-qt)^r{}_0d_q t\Bigg)^{\frac{1}{r}} \\ =&\frac{q^2(b-a)^2}{1+q}\left(h_1\left|{}_aD_q^2 f(a)\right|^r+h_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} h_1 &= \int_0^1 t^r(1-qt)^r(1-t)_0 d_q t = (1-q)\sum_{n=0}^{\infty}(q^n)^{r+1}(1-q^n)(1-q^{n+1})^r, \\ h_2 &= \int_0^1 t^{r+1}(1-qt)^r{}_0d_q t = (1-q)\sum_{n=0}^{\infty}(q^n)^{r+2}(1-q^{n+1})^r. \end{aligned}$$

The proof is completed. □

Remark 5.3. If $q\to 1$, then we have

$$\begin{aligned} \int_0^1 t^r(1-t)^{r+1}dt =&\beta(r+1,r+2), \\ \int_0^1 t^{r+1}(1-t)^r dt =&\beta(r+2,r+1), \end{aligned}$$

and (5.4) reduces to the following inequality:

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le&\frac{(b-a)^2}{2}\left(\beta(r+1,r+2)\left|f''(a)\right|^r+\beta(r+2,r+1)\left|f''(b)\right|^r\right)^{\frac{1}{r}}. \end{aligned}$$

Corollary 5.3. *If r is a positive integer, then*

$$(1-qt)^r \le (1-qt)_q^r, \quad (1-t)(1-qt)^r \le (1-qt)_q^{r+1}.$$

Theorem 5.3 reduces to

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \le \frac{q^2(b-a)^2}{1+q}\left(\beta_q(r+1,r+2)\left|{}_aD_q^2 f(a)\right|^r + \beta_q(r+2,r+1)\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.$$

Theorem 5.4. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ where $p,r > 1$, $\frac{1}{p}+\frac{1}{r} = 1$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \le \frac{q^2(b-a)^2}{1+q}(u_1)^{\frac{1}{p}}\left(\frac{q\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}, \tag{5.4}$$

where

$$u_1 = (1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p.$$

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\le &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t^p(1-qt)^p {}_0d_q t\right)^{\frac{1}{p}}\left(\int_0^1 \left|{}_aD_q^2 f((1-t)a+tb)\right|^r {}_0d_q t\right)^{\frac{1}{r}} \\
\le &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t^p(1-qt)^p {}_0d_q t\right)^{\frac{1}{p}} \\
&\times\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 (1-t)_0 d_q t + \left|{}_aD_q^2 f(b)\right|^r\int_0^1 t_0 d_q t\right)^{\frac{1}{r}}.
\end{aligned}$$

Applying Lemma 3.2 and 3.4, we have

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \le \frac{q^2(b-a)^2}{1+q}(u_1)^{\frac{1}{p}}\left(\frac{q\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}.$$

It is easy to check that

$$u_1 = \int_0^1 t^p(1-qt)^p {}_0d_q t = (1-q)\sum_{n=0}^{\infty}(q^n)^{p+1}(1-q^{n+1})^p.$$

The proof is completed. □

Remark 5.4. If $q \to 1$, then

$$\int_0^1 t^p(1-t)^p dt = \beta(p+1, p+1).$$

Using the properties of Beta function, that is, $\beta(x,x) = 2^{1-2x}\beta(\frac{1}{2}, x)$ and $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(xy)}$, we obtain

$$\beta(p+1, p+1) = 2^{1-2(p+1)}\beta\left(\frac{1}{2}, p+1\right) = 2^{-2p-1}\frac{\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)},$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(t)$ is Gamma function:

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \quad t > 0.$$

Thus, inequality (5.4) reduces to the $m = 1$ case of inequality (1.2), due to the fact that

$$\begin{aligned}
&\frac{(b-a)^2}{2}\left(\beta(p+1,p+1)\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^r + |f''(b)|^r}{2}\right)^{\frac{1}{r}} \\
\leq &\frac{(b-a)^2}{8}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^r + |f''(b)|^r}{2}\right)^{\frac{1}{r}}.
\end{aligned}$$

Corollary 5.4. *If p is a positive integer, $p > 1$, then*

$$(1-qt)^p \leq (1-qt)_q^p.$$

Theorem 5.4 reduces to

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\leq &\frac{q^2(b-a)^2}{1+q}\left(\beta_q(p+1,p+1)\right)^{\frac{1}{p}}\left(\frac{q\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{1+q}\right)^{\frac{1}{r}}.
\end{aligned}$$

Theorem 5.5. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then*

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\
\leq &\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}}\left(z_1\left|{}_aD_q^2 f(a)\right|^r + z_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}, \qquad (5.5)
\end{aligned}$$

where

$$\begin{aligned}
z_1 =&(1-q)\sum_{n=0}^\infty (q^n - q^{2n})(1-q^{n+1})^r, \\
z_2 =&(1-q)\sum_{n=0}^\infty q^{2n}(1-q^{n+1})^r,
\end{aligned}$$

and $[p+1]$ is the q-analogue of $p+1$.

Proof. Using Lemma 4.1 and Hölder inequality, we have

$$\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|$$
$$\le\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t^p{}_0d_qt\right)^{\frac{1}{p}}\left(\int_0^1 (1-qt)^r\left|{}_aD_q^2 f((1-t)a+tb)\right|^r{}_0d_qt\right)^{\frac{1}{r}}.$$

Applying (2.8) in Definition 2.6 and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|$$
$$\le\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}}\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 (1-qt)^r(1-t)_0d_qt\right.$$
$$\left.+\left|{}_aD_q^2 f(b)\right|^r\int_0^1 t(1-qt)^r{}_0d_qt\right)^{\frac{1}{r}}$$
$$=\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}}\left(z_1\left|{}_aD_q^2 f(a)\right|^r+z_2\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.$$

It is easy to check that

$$z_1=\int_0^1 (1-qt)^r(1-t)_0d_qt=(1-q)\sum_{n=0}^{\infty}(q^n-q^{2n})(1-q^{n+1})^r,$$
$$z_2=\int_0^1 t(1-qt)^r{}_0d_qt=(1-q)\sum_{n=0}^{\infty}q^{2n}(1-q^{n+1})^r.$$

The proof is completed. □

Remark 5.5. If $q\to 1$, then

$$\int_0^1 (1-t)^{r+1}dt=\frac{1}{r+2},\quad \int_0^1 t(1-t)^r dt=\frac{1}{(r+1)(r+2)},$$

and (5.5) reduces to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{(b-a)^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{(r+1)\left|f''(a)\right|^r+\left|f''(b)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}}.$$

Corollary 5.5. *If r is a positive integer, $r>1$, then*

$$(1-t)^{r+1}\le(1-qt)_q^{r+1},\quad (1-t)^r\le(1-qt)_q^r.$$

Theorem 5.5 reduces to

$$\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|$$
$$\le\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}}\left(\beta_q(1,r+2)\left|{}_aD_q^2 f(a)\right|^r+\beta_q(2,r+1)\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.$$

Theorem 5.6. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \le \frac{q^2(b-a)^2}{1+q}(\lambda_1)^{\frac{1}{p}}\left(\left(\frac{q^{r+1}}{[r+1]\,[r+2]}\right)\left|{}_aD_q^2 f(a)\right|^r + \frac{1}{[r+2]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}, \quad (5.6)$$

where

$$\lambda_1 = (1-q)\sum_{n=0}^{\infty} q^n(1-q^{n+1})^p,$$

and $[r+1]$, $[r+2]$ are the q-analogues of $r+1$ and $r+2$.

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\
\le&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 (1-qt)^p\right)^{\frac{1}{p}}\left(\int_0^1 t^r\left|{}_aD_q^2 f((1-t)a+tb)\right|^r {}_0d_qt\right)^{\frac{1}{r}} \\
\le&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 (1-qt)^p {}_0d_qt\right)^{\frac{1}{p}} \\
&\times\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 t^r(1-t)_0d_qt + \left|{}_aD_q^2 f(b)\right|^r\int_0^1 t^{r+1}{}_0d_qt\right)^{\frac{1}{r}}.
\end{aligned}$$

Applying Lemma 3.11 and (2.8) in Definition 2.6, we have

$$\begin{aligned}
&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)\,{}_ad_qx\right| \\
\le&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 (1-qt)^p {}_0d_qt\right)^{\frac{1}{p}} \\
&\times\left(\left(\frac{q^{r+1}}{[r+1]\,[r+2]}\right)\left|{}_aD_q^2 f(a)\right|^r + \frac{1}{[r+2]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}} \\
=&\frac{q^2(b-a)^2}{1+q}(\lambda_1)^{\frac{1}{p}}\left(\left(\frac{q^{r+1}}{[r+1]\,[r+2]}\right)\left|{}_aD_q^2 f(a)\right|^r + \frac{1}{[r+2]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.
\end{aligned}$$

It is easy to check that

$$\lambda_1 = \int_0^1 (1-qt)^p {}_0d_qt = (1-q)\sum_{n=0}^{\infty} q^n(1-q^{n+1})^p.$$

The proof is completed. □

Remark 5.6. If $q \to 1$, then

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1}.$$

Since

$$\lim_{p\to\infty} \left(\frac{1}{1+p}\right)^{\frac{1}{p}} = 1 \quad \text{and} \quad \lim_{p\to 1^+} \left(\frac{1}{1+p}\right)^{\frac{1}{p}} = \frac{1}{2},$$

we have

$$\frac{1}{2} < \left(\frac{1}{1+p}\right)^{\frac{1}{p}} < 1, \quad r \in (1,\infty),$$

hence for $r \in (1,\infty)$,

$$\left(\frac{1}{1+p}\right)^{\frac{1}{p}} < 1 \quad \text{and} \quad \frac{1}{(r+1)(r+2)} < 1.$$

Then inequality (5.6) reduces to the $m=1$ case of inequality (1.3).

Corollary 5.6. *If p is a positive integer, $p > 1$, then*

$$(1-qt)^p \le (1-qt)_q^p.$$

Theorem 5.6 reduces to

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right|$$
$$\le \frac{q^2(b-a)^2}{1+q}\left(\beta_q(1,p+1)\right)^{\frac{1}{p}}\left(\left(\frac{q^{r+1}}{[r+1]\,[r+2]}\right)\left|{}_aD_q^2 f(a)\right|^r + \frac{1}{[r+2]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.$$

Theorem 5.7. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|$ is convex on $[a,b]$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \le \frac{q^2(b-a)^2\left(q^2\left|{}_aD_q^2 f(a)\right| + \left|{}_aD_q^2 f(b)\right|\right)}{(1+q)(1+q+q^2)(1+q+q^2+q^3)}. \tag{5.7}$$

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|$, we have

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right|$$
$$\le \frac{q^2(b-a)^2}{1+q}\left(\left|{}_aD_q^2 f(a)\right|\int_0^1 t(1-t)(1-qt)_0 d_q t + \left|{}_aD_q^2 f(b)\right|\int_0^1 t^2(1-qt)_0 d_q t\right).$$

Applying Lemmas 3.9 and 3.10, we get (5.7). The proof is completed. □

Remark 5.7. If $q \to 1$, then (5.7) reduces to the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2(f''(a)+f''(b))}{24}.$$

Theorem 5.8. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ for $r \geq 1$, then*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \leq \frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}}. \tag{5.8}$$

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \leq &\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t(1-qt)_0 d_q t\right)^{1-\frac{1}{r}} \\ &\times\left(\left|{}_aD_q^2 f(a)\right|^r \int_0^1 t(1-t)(1-qt)_0 d_q t + \left|{}_aD_q^2 f(b)\right|^r \int_0^1 t^2(1-qt)_0 d_q t\right)^{\frac{1}{r}}.\end{aligned}$$

Applying Lemmas 3.7, 3.9 and 3.10, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \leq &\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{r}} \\ = &\frac{q^2(b-a)^2}{(1+q)^{2-\frac{1}{r}}}\left(\frac{q^2\left|{}_aD_q^2 f(a)\right|^r + \left|{}_aD_q^2 f(b)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}}.\end{aligned}$$

The proof is completed. □

Remark 5.8. If $q \to 1$, then (5.8) reduces to the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2^{2+\frac{1}{r}}}\left(|f''(a)|^r + |f''(b)|^r\right)^{\frac{1}{r}}.$$

Theorem 5.9. *Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0 < q < 1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ for $r \geq 1$, then*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x\right| \\ \leq &\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{(1+q)q^{r+1}}{[r+1]\,[r+2]\,[r+3]}\left|{}_aD_q^2 f(a)\right|^r \right. \\ &\left. + \frac{1}{[r+2]\,[r+3]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}},\end{aligned} \tag{5.9}$$

where $[r+1]$, $[r+2]$, $[r+3]$ are the q-analogues of $r+1$, $r+2$ and $r+3$.

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 (1-qt)_0 d_q t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^r(1-qt)\left|{}_aD_q^2 f((1-t)a+tb)\right|^r{}_0d_q t\right)^{\frac{1}{r}}\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 (1-qt)_0 d_q t\right)^{1-\frac{1}{r}}\\ &\times\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1 t^r(1-t)(1-qt)_0 d_q t+\left|{}_aD_q^2 f(b)\right|^r\int_0^1 t^{r+1}(1-qt)_0 d_q t\right)^{\frac{1}{r}}.\end{aligned}$$

Applying Lemmas 3.5, 3.12 and 3.13, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{(1+q)q^{r+1}}{[r+1]\,[r+2]\,[r+3]}\left|{}_aD_q^2 f(a)\right|^r\right.\\ &\left.+\frac{1}{[r+2]\,[r+3]}\left|{}_aD_q^2 f(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

The proof is completed. □

Remark 5.9. If $q\to 1$, then (5.9) reduces to

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{(b-a)^2}{4}\left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}\left(2\left|f''(a)\right|^r+(r+1)\left|f''(b)\right|^r\right)^{\frac{1}{r}}.\end{aligned}\tag{5.10}$$

Since $\left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}\leq 1, r\in[1,\infty)$, then inequality (5.10) reduces to the $m=1$ case of inequality (1.4).

Theorem 5.10. *Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a twice q-differentiable function on I^0 with ${}_aD_q^2 f$ be continuous and integrable on I where $0<q<1$. If $\left|{}_aD_q^2 f\right|^r$ is convex on $[a,b]$ where $p,r>1$, $\frac{1}{p}+\frac{1}{r}=1$, then*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\beta_q(p+1,2)\right)^{\frac{1}{p}}\left(\frac{(q+q^2)\left|{}_aD_q^2 f(a)\right|^r+\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}.\end{aligned}\tag{5.11}$$

Proof. Using Lemma 4.1, Hölder inequality and the convexity of $\left|{}_aD_q^2 f\right|^r$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\int_0^1 t^p(1-qt)_0 d_q t\right)^{\frac{1}{p}}\\ &\times\left(\left|{}_aD_q^2 f(a)\right|^r\int_0^1(1-t)(1-qt)_0 d_q t+\left|{}_aD_q^2 f(b)\right|^r\int_0^1 t(1-qt)_0 d_q t\right)^{\frac{1}{r}}.\end{aligned}$$

Applying Lemmas 3.7, 3.8 and the fact that $(1-qt)=(1-qt)_q^1$, we have

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)_a d_q x\right|\\ \leq&\frac{q^2(b-a)^2}{1+q}\left(\beta_q(p+1,2)\right)^{\frac{1}{p}}\left(\frac{(q+q^2)\left|{}_aD_q^2 f(a)\right|^r+\left|{}_aD_q^2 f(b)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}.\end{aligned}$$

The proof is completed. □

Remark 5.10. If $q\to 1$, then

$$\beta(p+1,2)=\int_0^1 t^p(1-t)dt=\frac{1}{(p+1)(p+2)},$$

and (5.11) reduces to

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{(b-a)^2}{2}\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\left|f''(a)\right|^r+\left|f''(b)\right|^r}{6}\right)^{\frac{1}{r}}.\end{aligned}$$

6. Conclusions

Quantum calculus has large applications in many areas such as number theory, special functions, quantum mechanics and mathematical inequalities. In this paper, we first establish a new quantum integral identity and then develop some quantum estimates of Hermite-Hadamard type inequalities for convex functions. These results in some special cases recapture the known results. We hope that our results will motivate other researchers who are exclusively working in the field of integral inequalities for convex function and may be helpful for further study in various applications areas (see [3, 6, 30–32]).

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