Journal of Applied Analysis and Computation
Volume 7, Number 2, May 2017, 659–669
Website:http://jaac-online.com/
DOI:10.11948/2017041

KIRCHHOFF TYPE PROBLEMS INVOLVING *P*-BIHARMONIC OPERATORS AND CRITICAL EXPONENTS*

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Abstract In this paper, we study the existence of solutions for a class of Kirchhoff type problems involving p-biharmonic operators and critical exponents. The proof is essentially based on the mountain pass theorem due to Ambrosetti and Rabinowitz [2] and the Concentration Compactness Principle due to Lions [18, 19].

Keywords Kirchhoff type problems, p-biharmonic operators, critical exponents, mountain pass theorem.

MSC(2010) 47A75, 35B38, 35P30, 34L05, 34L30.

1. Introduction and Preliminaries

In this paper, we are interested in the existence of solutions for the following Kirchhoff type problem

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p\,dx\right)\Delta\Big(|\Delta u|^{p-2}\Delta u\Big) = \lambda f(x,u) + |u|^{p^{**}-2}u, & x\in\Omega,\\ u = \frac{\partial u}{\partial \nu} = 0, \quad x\in\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^2 boundary, $N \geq 2$, Δ is the Laplace operator and $\frac{\partial u}{\partial \nu}$ is the outer normal derivative, λ is a positive parameter, $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, p^{**} is the critical exponent, i.e.

$$p^{**} = \begin{cases} \frac{Np}{N-2p}, & \text{if } 2p < N,\\ +\infty, & \text{if } 2p \geq N, \end{cases} \tag{1.2}$$

where $p \in (2, +\infty)$.

Throughout this paper we assume that $M : [0, +\infty) \to \mathbb{R}$ is a continuous and increasing function that satisfies:

(M_0) There exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in [0, +\infty)$.

Since problem (1.1) contains integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [10, 17]. Kirchhoff type problems for

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*The work is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED).

p-Laplace operators have been studied by many mathematicians, see [4, 5, 9, 13, 14, 21, 22, 24]. In a recent paper [11], Colasuonno and Pucci have introduced p-polyharmonic operators. Using variational methods, they studied the multiplicity of solutions for a class of $p(x)$-polyharmonic elliptic Kirchhoff equations. In [20], V.F. Lubyshev studied the existence of solutions for an even-order nonlinear problem with convex-concave nonlinearity. In [6], the author extended the previous results in [11] to nonlocal higher-order problems. Althought the Kirchhoff function $M(t)$ in [6, 11] was assumed to be degenerate at zero, the nonlinearity is not critical. Finally, we refer the readers to two related interesting papers [7, 25]. In [7], G. Autuori et al. considered a class of Kirchhoff type problems involving a fractional elliptic operator and a critical nonlinearity while in [25], L. Zhao et al. studied the existence of solutions for a higher order Kirchhoff type problem with exponential critical growth.

In this paper, we study the existence of solutions for Kirchhoff type problems involving p-biharmonic operators and critical exponents. We are motivated by the results introduced in [1, 6, 11] and some papers on p-biharmonic operators with critical exponents [3, 8, 12, 15, 16, 23]. We also get a priori estimates of the obtained solution. We believe that this is the first contribution to the study of the existence of solutions for Kirchhoff type problems involving p-biharmonic operators with critical exponents. Due to the presence of the critical exponents, the problem considered here is lack of compactness. To overcome this difficulty, we use the Concentration Compactness Principle due to Lions [18, 19]. The existence of a nontrivial solution is then obtained by the mountain pass theorem due to Ambrosetti and Rabinowitz in [2].

In order to state the main result, we assume that $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(F_1) $|f(x,t)| \le C(1+|t|^{q-1})$ for all $(x,t) \in \Omega\times\mathbb{R}$, where $p < q < p^{**}$ and p^{**} is defined by (1.2);

(F_2) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-1}} = 0$ uniformly for $x \in \Omega$;

(F_3) There exists $\theta \in (p, p^{**})$ such that

$$0 < \theta F(x,t) \le f(x,t)t, \quad \forall x \in \Omega, \quad \forall t \in \mathbb{R}\backslash\{0\},$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

Let $W_0^{2,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p\,dx\right)^{\frac{1}{p}}, \quad u \in W_0^{2,p}(\Omega).$$

We then have that $W_0^{2,p}(\Omega)$ is continuously and compactly embedded into the Lebesgue space $L^r(\Omega)$ endowed the norm $|u|_r = \left(\int_\Omega |u|^r\,dx\right)^{\frac{1}{r}}$, $2 < r < p^{**}$. Denote by S_r the best constant for this embedding, that is, $S_r|u|_r \le \|u\|$ for all $u \in W_0^{2,p}(\Omega)$.

Definition 1.1. We say that $u \in W_0^{2,p}(\Omega)$ is a weak solution of problem (1.1) if

$$M\left(\|u\|^p\right)\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v\,dx - \lambda\int_\Omega f(x,u)v\,dx - \int_\Omega |u|^{p^{**}-2}uv\,dx = 0$$

for all $v \in W_0^{2,p}(\Omega)$.

The main result of this paper can be stated as follows.

Theorem 1.1. *Assume that* (M_0), (F_1)-(F_3) *are satisfied. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$, *problem (1.1) has a nontrivial solution. Moreover, if* u_λ *is a solution of problem (1.1) then* $\lim_{\lambda\to+\infty} \|u_\lambda\| = 0$.

2. Proof of the main result

Here we are assuming, without loss of generality, that the Kirchhoff function $M(t)$ is unbounded. Contrary case, the truncation on $M(t)$ is not necessary. Since we are intending to work with $N \geq 2$, we shall make a truncation on M as follows. From (M_0), given $a \in \mathbb{R}$ such that $m_0 < a < \frac{\theta}{p} m_0$, there exists $t_0 > 0$ such that $M(t_0) = a$. We set

$$M_a(t) := \begin{cases} M(t), & 0 \leq t \leq t_0, \\ a, & t \geq t_0. \end{cases} \tag{2.1}$$

From (M_0) we get

$$M_a(t) \leq a, \quad \forall t \geq 0. \tag{2.2}$$

As we shall see, the proof of Theorem 1.1 is based on a careful study of the solutions of the following auxiliary problem

$$\begin{cases} -M_a\left(\|u\|^p\right) \Delta\Big(|\Delta u|^{p-2}\Delta u\Big) = \lambda f(x,u) + |u|^{p^{**}-2}u, & x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where f, N, p, λ are as in Section 1. We shall prove the following auxiliary result.

Theorem 2.1. *Assume that* (M_0), (F_1)-(F_3) *are satisfied. Then, there exists* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *and all* $a \in (m_0, \frac{\theta}{p} m_0)$, *problem (2.3) has a nontrivial solution.*

We recall that $u \in W_0^{2,p}(\Omega)$ is a weak solution of problem (2.3) if

$$M_a\left(\|u\|^p\right) \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v \, dx - \lambda \int_\Omega f(x,u)v \, dx - \int_\Omega |u|^{p^{**}-2}uv \, dx = 0,$$

for all $v \in W_0^{2,p}(\Omega)$. Hence, we shall look for nontrivial solutions of (2.3) by finding critical points of the C^1- functional $I_{a,\lambda} : W_0^{2,p}(\Omega) \to \mathbb{R}$ given by the formula

$$I_{a,\lambda}(u) = \frac{1}{p}\widehat{M}_a\left(\|u\|^p\right) - \lambda \int_\Omega F(x,u)\, dx - \frac{1}{p^{**}} \int_\Omega |u|^{p^{**}} dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(x,t) = \int_0^t f(x,s)\, ds$. Note that

$$I'_{a,\lambda}(u)(v) = M_a\left(\|u\|^p\right) \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v \, dx - \lambda \int_\Omega f(x,u)v \, dx - \int_\Omega |u|^{p^{**}-2}uv \, dx$$

for all $u, v \in W_0^{2,p}(\Omega)$.

We say that a sequence $\{u_n\} \subset W_0^{2,p}(\Omega)$ is a Palais-Smale sequence for the functional $I_{a,\lambda}$ at level $c \in \mathbb{R}$ if

$$I_{a,\lambda}(u_n) \to c \text{ and } I'_{a,\lambda}(u_n) \to 0 \text{ in } (W_0^{2,p}(\Omega))^*,$$

where $(W_0^{2,p}(\Omega))^*$ is the dual space of $W_0^{2,p}(\Omega)$. If every Palais-Smale sequence of $I_{a,\lambda}$ has a strong convergent subsequence, then one says that $I_{a,\lambda}$ satisfies the Palais-Smale condition ((PS) condition for short).

Lemma 2.1. *For all $\lambda > 0$, there exist positive constants ρ and r such that $I_{a,\lambda}(u) \geq r > 0$ for all $u \in W_0^{2,p}(\Omega)$ with $\|u\| = \rho$.*

Proof. From (F_1) and (F_2), for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|f(x,t)| \leq \epsilon |t|^{p-1} + C_\epsilon |t|^{q-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

which implies that

$$|F(x,t)| \leq \frac{\epsilon}{p}|t|^p + \frac{C_\epsilon}{q}|t|^q, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.4}$$

From (2.4) and (M_0), for all $u \in W_0^{2,p}(\Omega)$, we get

$$\begin{aligned} I_{a,\lambda}(u) &= \frac{1}{p}\widehat{M}_a\left(\|u\|^p\right) - \lambda \int_\Omega F(x,u)\,dx - \frac{1}{p^{**}}\int_\Omega |u|^{p^{**}}\,dx \\ &\geq \frac{m_0}{p}\|u\|^p - \lambda \int_\Omega \left(\frac{\epsilon}{p}|u|^p + \frac{C_\epsilon}{q}|u|^q\right)dx - \frac{1}{p^{**}}\int_\Omega |u|^{p^{**}}\,dx \\ &\geq \frac{m_0}{p}\|u\|^p - \lambda\epsilon S_p^{-p}\|u\|^p - \lambda C_\epsilon S_q^{-q}\|u\|^q - \frac{1}{p^{**}S^{\frac{p^{**}}{p}}}\|u\|^{p^{**}}, \end{aligned}$$

where S is the best constant of the embedding $W_0^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)$, that is,

$$S = \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^p\,dx}{\left(\int_\Omega |u|^{p^{**}}\,dx\right)^{\frac{p}{p^{**}}}}. \tag{2.5}$$

For $\lambda > 0$, let $\epsilon = \frac{m_0 S_p^p}{2p\lambda}$, we get

$$\begin{aligned} I_{a,\lambda}(u) &\geq \frac{m_0}{2p}\|u\|^p - \lambda C_\epsilon S_q^{-q}\|u\|^q - \frac{1}{p^{**}S^{\frac{p^{**}}{p}}}\|u\|^{p^{**}} \\ &= \|u\|^p\left(\frac{m_0}{2p} - \lambda C_\epsilon S_q^{-q}\|u\|^{q-p} - \frac{1}{p^{**}S^{\frac{p^{**}}{p}}}\|u\|^{p^{**}-p}\right). \end{aligned}$$

Since $p < q < p^{**}$, there exist positive constants ρ and r such that $I_{a,\lambda}(u) \geq r > 0$ for all $u \in W_0^{2,p}(\Omega)$ with $\|u\| = \rho$. □

Lemma 2.2. *For all $\lambda > 0$, there exists $e \in W_0^{2,p}(\Omega)$ with $I_{a,\lambda}(e) < 0$ and $\|e\| > \rho$.*

Proof. From (F_3), there are positive constans C_1 and C_2 such that

$$F(x,t) \geq C_1 t^\theta - C_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{2.6}$$

Fix $u_0 \in C_0^\infty(\Omega)\setminus\{0\}$ with $\|u_0\| = 1$. Using (2.2) and (2.6), for all $t > 0$ large enough, we have

$$\begin{aligned} I_{a,\lambda}(tu_0) &= \frac{1}{p}\widehat{M}_a\left(\|tu_0\|^p\right) - \lambda\int_\Omega F(x,tu_0)\,dx - \frac{1}{p^{**}}\int_\Omega |tu_0|^{p^{**}}\,dx \\ &\leq \frac{a}{p}t^p - \lambda C_1 t^\theta \int_\Omega |u_0|^\theta\,dx + \lambda C_2 - \frac{t^{p^{**}}}{p}\int_\Omega |u_0|^{p^{**}}\,dx. \end{aligned}$$

Since $\theta \in (p, p^{**})$, the result follows by considering $e = t_* u_0$ for some $t_* > 0$ large enough. $\square$

Using a version of the Mountain pass theorem due to Ambrosetti and Rabinowitz [2], without (PS) condition, there exists a sequence $\{u_n\} \subset W_0^{2,p}(\Omega)$ such that

$$I_{a,\lambda}(u_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty,$$

where

$$c_{a,\lambda} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{a,\lambda}(\gamma(t)),$$

and

$$\Gamma = \left\{\gamma \in C([0,1], W_0^{2,p}(\Omega)) : \ \gamma(0) = 0, \ I_{a,\lambda}(\gamma(1)) < 0\right\}.$$

Lemma 2.3. *It holds that*

$$\lim_{\lambda\to+\infty} c_{a,\lambda} = 0.$$

Proof. Since the functional $I_{a,\lambda}$ has the mountain pass geometry, it follows that there exists $t_\lambda > 0$ verifying $I_{a,\lambda}(t_\lambda u_0) = \max_{t\geq 0} I_{a,\lambda}(tu_0)$, where u_0 is the function given by Lemma 2.2. Hence, $\frac{d}{dt} I_{a,\lambda}(t_\lambda u_0)(t_\lambda u_0) = 0$ or

$$0 = M_a\left(\|t_\lambda u_0\|^p\right) \int_\Omega |\Delta t_\lambda u_0|^p \, dx - \lambda \int_\Omega f(x, t_\lambda u_0) t_\lambda u_0 \, dx - t_\lambda^{p^{**}} \int_\Omega |u_0|^{p^{**}} \, dx.$$

Hence,

$$t_\lambda^p M_a(t_\lambda^p) = \lambda \int_\Omega f(x, t_\lambda u_0) t_\lambda u_0 \, dx + t_\lambda^{p^{**}} \int_\Omega |u_0|^{p^{**}} \, dx. \tag{2.7}$$

From (2.2), (2.7) and (F_3), we get

$$a \geq t_\lambda^{p^{**}-p} \int_\Omega |u_0|^{p^{**}} \, dx,$$

which implies that $\{t_\lambda\}$ is bounded. Thus, there exist a sequence $\lambda_n \to +\infty$ and $\bar{t} \geq 0$ such that $t_{\lambda_n} \to \bar{t}$ as $n \to \infty$. Consequently, there is $C_3 > 0$ such that

$$t_{\lambda_n}^p M_a(t_{\lambda_n}^p) \leq C_3, \quad \forall n \in \mathbb{N},$$

and $\forall n \in \mathbb{N}$,

$$\lambda_n \int_\Omega f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx + t_{\lambda_n}^{p^{**}} \int_\Omega |u_0|^{p^{**}} \, dx \leq C_3. \tag{2.8}$$

If $\bar{t} > 0$, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx = \int_\Omega f(x, \bar{t} u_0) \bar{t} u_0 \, dx$$

and thus (2.8) leads to

$$\lim_{n\to\infty} \left(\lambda_n \int_\Omega f(x, t_{\lambda_n} u_0) t_{\lambda_n} u_0 \, dx + t_{\lambda_n}^{p^{**}} \int_\Omega |u_0|^{p^{**}} \, dx\right) = +\infty,$$

which is an absurd. Thus, we conclude that $\bar{t} = 0$. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0,1]$, which belongs to Γ, to get the following estimate

$$0 < c_{a,\lambda} \leq \max_{t\in[0,1]} I_{a,\lambda}(\gamma_*(t)) = I_{a,\lambda}(t_\lambda u_0) \leq \widehat{M}_a(t_\lambda^p).$$

In this way,

$$\lim_{\lambda \to +\infty} \widehat{M}_a(t_\lambda^p) = 0,$$

which leads to $\lim_{\lambda \to +\infty} c_{a,\lambda} = 0$. □

Lemma 2.4. *Let $\{u_n\} \subset W_0^{2,p}(\Omega)$ be a sequence such that*

$$I_{a,\lambda}(u_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty. \tag{2.9}$$

Then $\{u_n\}$ is bounded.

Proof. Assuming by contradiction that $\{u_n\}$ is not bounded in $W_0^{2,p}(\Omega)$, up to a subsequence, we may assume that $\|u_n\| \to +\infty$ as $n \to \infty$. It follows from (2.1), (M_0) and (F_3) that for n large enough

$$\begin{aligned}
&1 + c_{a,\lambda} + \|u_n\| \\
\geq& I_{a,\lambda}(u_n) - \frac{1}{\theta} I'_{a,\lambda}(u_n)(u_n) \\
=& \frac{1}{p}\widehat{M}_a\left(\|u_n\|^p\right) - \lambda \int_\Omega F(x,u_n)\,dx - \frac{1}{p^{**}}\int_\Omega |u_n|^{p^{**}}\,dx \\
&- \frac{1}{\theta} M_a\left(\|u_n\|^p\right)\int_\Omega |\nabla u_n|^p\,dx + \frac{\lambda}{\theta}\int_\Omega f(x,u_n)u_n\,dx + \frac{1}{\theta}\int_\Omega |u_n|^{p^{**}}\,dx \\
\geq& \left(\frac{m_0}{p} - \frac{a}{\theta}\right)\|u_n\|^p - \frac{\lambda}{\theta}\int_\Omega \left(f(x,u_n)u_n - \theta F(x,u_n)\right)\,dx + \left(\frac{1}{\theta} - \frac{1}{p^{**}}\right)\int_\Omega |u_n|^{p^{**}}\,dx \\
\geq& \left(\frac{m_0}{p} - \frac{a}{\theta}\right)\|u_n\|^p - C_4,
\end{aligned}$$

where C_4 is a positive constant. Since $a < \frac{m_0}{p}\theta$ and $\theta < p^{**}$, the sequence $\{u_n\}$ is bounded. □

Proof of Theorem 2.1. From Lemma 2.3, we have

$$\lim_{\lambda \to +\infty} c_{a,\lambda} = 0. \tag{2.10}$$

Therefore, there exists $\lambda_0 > 0$ such that

$$c_{a,\lambda} < \left(\frac{1}{\theta} - \frac{1}{p^{**}}\right) S^{\frac{N}{2p}}, \tag{2.11}$$

for all $\lambda \geq \lambda_0$, where S is given by (2.5). Now, fix $\lambda \geq \lambda_0$ and let us show that problem (2.1) admits a nontrivial solution. From Lemmas 2.3 and 2.4, there exists a bounded sequence $\{u_n\} \subset W_0^{2,p}(\Omega)$ verifying

$$I_{a,\lambda}(u_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(u_n) \to 0 \text{ as } n \to \infty. \tag{2.12}$$

Hence, up to subsequences, we may assume that $\{u_n\}$ converges weakly to $u \in W_0^{2,p}(\Omega)$, $\{u_n\}$ converges strongly to u in $L^q(\Omega)$, $2 < q < p^{**}$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$ as $n \to \infty$.

Using the Concentration Compactness Principle due to Lions [18,19], if $|\Delta u_n|^p \rightharpoonup \mu$, $|u_n|^{p^{**}} \rightharpoonup \nu$ weakly-* in the sense of measures, where μ and ν are bounded non-negative measures on $\mathbb{R}^N$, then there exist an at most countable set J, sequences

$(x_j)_{j\in J}\subset\overline{\Omega}$ and $(\nu_j)_{j\in J}$, $(\mu_j)_{j\in J}$ nonnegative numbers such that

$$\begin{aligned}
&\nu=|u|^{p^{**}}+\sum_{j\in J}\nu_j\delta_{x_j},\quad \nu_j>0,\\
&\mu\geq|\Delta u|^{p}+\sum_{j\in J}\mu_j\delta_{x_j},\quad \mu_j>0,\\
&\nu_j^{\frac{p}{p^{**}}}\leq\frac{\mu_j}{S},
\end{aligned}\tag{2.13}$$

for all $j\in J$, where δ_{x_j} is the Dirac mass at $x_j\in\overline{\Omega}$.

Now, we claim that $J=\emptyset$. Arguing by contradiction, assume that $J\neq\emptyset$ and fix $j\in J$. For $\epsilon>0$, consider $\phi_{j,\epsilon}\in C^\infty(\mathbb{R}^N)$ such that $\phi_{j,\epsilon}\equiv 1$ in $B_\epsilon(x_j)$, $\phi_{j,\epsilon}\equiv 0$ on $\Omega\backslash B_{2\epsilon}(x_j)$ and $|\nabla\phi_{j,\epsilon}|_\infty\leq\frac{2C}{\epsilon}$, and $|\Delta\phi_{j,\epsilon}|\leq\frac{2C}{\epsilon^2}$, where $x_j\in\overline{\Omega}$ belongs to the support of ν. Since $\{\phi_{j,\epsilon}u_n\}$ is bounded in the space $W_0^{2,p}(\Omega)$, it then follows from (2.1) that $I'_{a,\lambda}(u_n)(\phi_{j,\epsilon}u_n)\to 0$ as $n\to\infty$, that is,

$$\begin{aligned}
&2M_a\left(\|u_n\|^p\right)\int_\Omega|\Delta u_n|^{p-2}\Delta u_n(\nabla u_n\nabla\phi_{j,\epsilon})dx\\
&+M_a\left(\|u_n\|^p\right)\int_\Omega u_n|\Delta u_n|^{p-2}\Delta u_n\Delta\phi_{j,\epsilon}\,dx\\
=&-M_a\left(\|u_n\|^p\right)\int_\Omega|\nabla u_n|^p\phi_{j,\epsilon}\,dx\\
&+\lambda\int_\Omega f(x,u_n)\phi_{j,\epsilon}u_n\,dx+\int_\Omega|u_n|^{p^{**}}\phi_{j,\epsilon}\,dx+o_n(1).
\end{aligned}\tag{2.14}$$

Using the Hölder inequality and the boundedness of the sequence $\{u_n\}$, we have

$$\begin{aligned}
&\left|\int_\Omega|\Delta u_n|^{p-2}\Delta u_n(\nabla u_n\nabla\phi_{j,\epsilon})\,dx\right|\\
=&\left|\int_{B_{2\epsilon}(x_j)\cap\Omega}|\Delta u_n|^{p-2}\Delta u_n(\nabla u_n\nabla\phi_{j,\epsilon})\,dx\right|\\
\leq&\int_{B_{2\epsilon}(x_j)\cap\Omega}|\Delta u_n|^{p-1}|\nabla u_n||\nabla\phi_{j,\epsilon}|\,dx\\
\leq&\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla u_n|^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla u_n|^p|\nabla\phi_{j,\epsilon}|^p\,dx\right)^{\frac{1}{p}}\\
\leq&C_3\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla u_n|^p|\nabla\phi_{j,\epsilon}|^p\,dx\right)^{\frac{1}{p}}\\
\leq&C_3\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla u_n|^{p^{**}}\right)^{\frac{1}{p^{**}}}\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla\phi_{j,\epsilon}|^{2N}\right)^{\frac{2}{N}}\\
\leq&\overline{C}_3\left(\int_{B_{2\epsilon}(x_j)\cap\Omega}|\nabla u_n|^{p^{**}}\right)^{\frac{1}{p^{**}}}\to 0\text{ as }n\to\infty\text{ and }\epsilon\to 0.
\end{aligned}\tag{2.15}$$

Since $\{u_n\}$ is bounded, we may assume that $\|u_n\|\to t_1\geq 0$ as $n\to\infty$. Observing that $M(t)$ is continuous, we then have $M(\|u_n\|^p)\to M(t_1^p)\geq m_0>0$ as $n\to\infty$.

Hence,

$$M_a(\|u_n\|^p)\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\nabla u_n \nabla \phi_{j,\epsilon})\,dx \to 0 \text{ as } n\to\infty \text{ and } \epsilon\to 0. \tag{2.16}$$

Similarly, we have

$$\lim_{n\to\infty} M_a\left(\|u_n\|^p\right)\int_\Omega u_n|\Delta u_n|^{p-2}\Delta u_n \Delta\phi_{j,\epsilon}\,dx = 0. \tag{2.17}$$

On the other hand, by (F_2) and the boundedness the sequence $\{u_n\}$ in $W_0^{2,p}(\Omega)$ we also have

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)\phi_{j,\epsilon}u_n\,dx = 0. \tag{2.18}$$

From (2.14)-(2.18), letting $n\to\infty$, we deduce that

$$\int_\Omega d\nu \geq M_a(t_1^p)\int_\Omega \phi_{j,\epsilon}d\mu + o_\epsilon(1).$$

Letting $\epsilon\to 0$ and using the standard theory of Radon measures, we conclude that $\nu_j \geq M_a(t_1^p)\mu_j \geq m_0\mu_j$. Using (2.13) we have

$$\nu_j \geq S^{\frac{N}{2p}}, \tag{2.19}$$

where S is given by (2.5).

Now, we shall prove that (2.19) cannot occur, and therefore the set $J=\emptyset$. Indeed, arguing by contradiction, let us suppose that $\nu_j \geq S^{\frac{N}{2p}}$ for some $j\in J$. Since $\{u_n\}$ is a $(PS)_{c_{a,\lambda}}$ sequence for the functional $I_{a,\lambda}$, from the conditions (F_3) and (M_0), and $m_0 < a < \frac{\theta}{p}m_0$ we have

$$\begin{aligned} c_{a,\lambda} &= I_{a,\lambda}(u_n) - \frac{1}{\theta}I'_{a,\lambda}(u_n)(u_n) + o_n(1)\\ &\geq \frac{1}{p}\widehat{M}_a(\|u_n\|^p) - \frac{1}{\theta}M_a(\|u_n\|^p)\|u_n\|^p + \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)\int_\Omega |u_n|^{p^{**}}\,dx + o_n(1)\\ &\geq \left(\frac{m_0}{p}-\frac{a}{\theta}\right)\|u_n\|^p + \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)\int_\Omega |u_n|^{p^{**}}\,dx + o_n(1)\\ &\geq \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)\int_\Omega |u_n|^{p^{**}}\,dx + o_n(1)\\ &\geq \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)\int_\Omega |u_n|^{p^{**}}\phi_{j,\epsilon}\,dx + o_n(1). \end{aligned} \tag{2.20}$$

Letting $n\to\infty$ in (2.20), we get

$$c_{a,\lambda} \geq \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)\int_\Omega \phi_{j,\epsilon}d\nu$$

and then

$$c_{a,\lambda} \geq \left(\frac{1}{\theta}-\frac{1}{p^{**}}\right)S^{\frac{N}{2p}}, \tag{2.21}$$

which contradicts (2.11). Thus, $J=\emptyset$ and it follows that

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p^{**}}\,dx = \int_\Omega |u|^{p^{**}}\,dx. \tag{2.22}$$

We also have $u_n(x) \to u(x)$ a.e. $x \in \Omega$ as $n \to \infty$, so by the condition (F_1) and the Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty} \int_\Omega (f(x,u_n)u_n - f(x,u)u)\,dx = 0. \tag{2.23}$$

From (2.22)-(2.23) we deduce since $I'_{a,\lambda}(u_n)(u_n) \to 0$ as $n \to \infty$ that

$$\lim_{n\to\infty} M_a\left(\|u_n\|^p\right)\|u_n\|^p = \lambda \int_\Omega f(x,u)u\,dx + \int_\Omega |u|^{p^{**}}\,dx. \tag{2.24}$$

On the other hand, by (2.12), and the boundedness of $\|u_n - u\|$ we have $I'_{a,\lambda}(u_n)(u_n - u) \to 0$ as $n \to \infty$, that is,

$$\begin{aligned} &M_a\left(\|u_n\|^p\right)\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\Delta u_n - \Delta u)\,dx - \lambda\int_\Omega f(x,u_n)(u_n - u)\,dx \\ &- \int_\Omega |u_n|^{p^{**}-2}u_n(u_n - u)\,dx \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.25}$$

Using Hölder's inequality we have

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n - u)\,dx\right| &\le \int_\Omega |f(x,u_n)||u_n - u|\,dx \\ &\le C\int_\Omega (1 + |u_n|^{q-1})|u_n - u|\,dx \\ &\le C\left(|\Omega|^{\frac{q-1}{q}} + |u_n|^{q-1}_{L^q(\Omega)}\right)|u_n - u|_{L^q(\Omega)} \\ &\to 0 \text{ as } n \to \infty \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \left|\int_\Omega |u_n|^{p^{**}-2}u_n(u_n - u)\,dx\right| &\le \int_\Omega |u_n|^{p^{**}-1}|u_n - u|\,dx \\ &\le |u_n|^{p^{**}-1}_{L^{p^{**}}(\Omega)}|u_n - u|_{L^{p^{**}}(\Omega)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.27}$$

From (2.25)-(2.27) we get

$$M_a\left(\|u_n\|^p\right)\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\Delta u_n - \Delta u)\,dx \to 0 \text{ as } n \to \infty. \tag{2.28}$$

Using the condition (M_0), it follows that

$$\int_\Omega |\Delta u_n|^{p-2}\Delta u_n(\Delta u_n - \Delta u)\,dx \to 0 \text{ as } n \to \infty. \tag{2.29}$$

By standard arguments, we can show that $\{u_n\}$ converges stronlgy to u in $W_0^{2,p}(\Omega)$. This completes the proof of Theorem 2.1. □

Now, we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Let λ_0 be as in Theorem 2.1 and, for $\lambda \ge \lambda_0$, let $u_\lambda \in W_0^{2,p}(\Omega)$ be the nontrival solution of problem (2.3) found in Theorem 2.1. We claim that there exists $\lambda^* \geq \lambda_0$ such that $\|u_\lambda\|^p \leq t_0$ for all $\lambda \geq \lambda^*$. If this is the case, it follows from the definition of $M_a(t)$ that $M_a(\|u_\lambda\|^p) = M(\|u_\lambda\|^p)$. Thus, u_λ is a nontrivial weak solution of problem (1.1).

We argue by contradiction that, there is a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $\lambda_n \to +\infty$ as $n \to \infty$ and $\|u_{\lambda_n}\|^p \geq t_0$. Then we have

$$\begin{aligned} c_{a,\lambda_n} \geq \frac{1}{p}\widehat{M}_a(\|u_{\lambda_n}\|^p) - \frac{1}{\theta}M_a(\|u_{\lambda_n}\|^p)\|u_{\lambda_n}\|^p &\geq \left(\frac{m_0}{p} - \frac{a}{\theta}\right)\|u_{\lambda_n}\|^p \\ &\geq \left(\frac{m_0}{p} - \frac{a}{\theta}\right)t_0, \end{aligned} \tag{2.30}$$

which is a contradiction since $\lim_{n\to\infty} c_{a,\lambda_n} = 0$ and $a \in (m_0, \frac{\theta}{p}m_0)$.

Finally, we shall prove that $\lim_{\lambda\to+\infty}\|u_\lambda\| = 0$. Indeed, by (M_0) and the fact that $\|u_\lambda\|^p \leq t_0$, it follows that $M(\|u_\lambda\|^p) \leq M(t_0) = a$. Hence, using (M_0) and (F_4) we have

$$\begin{aligned} c_{a,\lambda} \geq \frac{1}{p}\widehat{M}(\|u_\lambda\|^p) - \frac{1}{\theta}M(\|u_\lambda\|^p)\|u_\lambda\|^p &\geq \frac{m_0}{p}\|u_\lambda\|^p - \frac{a}{\theta}\|u_\lambda\|^p \\ &= \left(\frac{m_0}{p} - \frac{a}{\theta}\right)\|u_\lambda\|^p. \end{aligned} \tag{2.31}$$

Using Lemma 2.4 again we have that $\lim_{\lambda\to+\infty} c_{a,\lambda} = 0$. Therefore, it follows since $a \in (m_0, \frac{\theta}{p}m_0)$ that $\lim_{\lambda\to+\infty}\|u_\lambda\| = 0$. The proof of Theorem 1.1 is now completed. □

Acknowledgements

The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. This research is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) (Grant N.101.02.2017.04).

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