Journal of Applied Analysis and Computation
Volume 7, Number 1, February 2017, 92–101
Website:http://jaac-online.com/
DOI:10.11948/2017007

PROXIMAL AND SYNDETICAL PROPERTIES IN NONAUTONOMOUS DISCRETE SYSTEMS*

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Abstract This paper is mainly concerned with a class of nonautonomous discrete systems $(X, f_{1,\infty})$. New definitions of proximity relations and sensitivity in nonautonomous discrete systems are given. Some relations among $P(f_{1,\infty})$, $L(f_{1,\infty})$, $R(f_{1,\infty})$, $S(f_{1,\infty})$ and $P(f_{1,\infty})(x)$ are derived. And some chaotic properties of $f_{1,\infty}$ are proved.

Keywords Nonautonomous discrete system, proximity, syndeticity, sensitivity.

MSC(2010) 37B55, 37D45, 54H20.

1. Introduction

The proximal relation in a group of transformations was first studied by Ellis and Gottschalk in [6]. The syndetically proximal relation was then introduced by Clay in [3]. In [12], Shoenfeld introduced the notion of regular homomorphisms which are defined by extending regular minimal sets to homomorphisms with a minimal range. And Yu in [17] introduced the concepts regular relation and syndetically regular relation. For more resent results on proximal and syndetically proximal properties, we refer to [7, 10, 13–16] and others.

Most of papers studied complexity in autonomous discrete systems (X, f). However, if various perturbations of a system are described as different functions, then there are a sequence of maps to map the points in the system. This means that many systems in engineering applications are nonautonomous systems. The aim of this paper is to investigate some proximity relations and chaotic properties in nonautonomous systems. Nonautonomous discrete systems were precisely introduced in [8], in connection with nonautonomous difference equations (see [5, 11] and some references therein).

Let X be a compact metric space and consider a sequence of continuous maps $f_n : X \to X, n \in \mathbb{N}$ (where $\mathbb{N}$ is the set of nonzero natural numbers), denoted by $f_{1,\infty} = (f_1, f_2, \cdots)$. This sequence defines a nonautonomous discrete system $(X, f_{1,\infty})$. Denote $f_i^k = f_{i+k-1} \circ \cdots \circ f_{i+1} \circ f_i$ and the orbit of any point $x \in X$

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*The authors were supported by National Natural Science Foundation of China (11501391, 61573010), the Opening Project of Artificial Intelligence Key Laboratory of Sichuan Province (2015RZJ01), and the Opening Project of Bridge Non-destruction Detecting and Engineering Computing Key Laboratory of Sichuan Province (2014QZJ02, 2015QYJ01).

is given by the sequence $(f_1^n(x)) = Orb(x, f_{1,\infty}), n \in \mathbb{N}$. In this paper, it is always assumed that all the maps f_n, $n \in \mathbb{N}$, are surjective. It should be noted that this condition is needed by most papers dealing with this kind of systems (for example, [1,4,9]).

Write

$$N(U,V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \phi\} \text{ and } N(x,U) = \{n \in \mathbb{N} : f_1^n(x) \in U\}.$$

A dynamical system $(X, f_{1,\infty})$ is *transitive* if for each pair of nonempty open sets U, V of X, $N(U,V)$ is nonempty. A point $x \in X$ is *transitive* if the orbit $Orb(x, f_{1,\infty})$ is dense in X. A set S is a *minimal set* if every $x \in S$ is transitive. The set

$$\omega(x, f_{1,\infty}) = \{y : \textit{ there exists an increasing sequence } \{n_i\}_{i=1}^{\infty} \textit{ such that } y = \lim_{i\to\infty} f_1^{n_i}(x)\}$$

is said to be the *ω-limit set* of x. A point $x \in X$ is a *minimal point* if the ω-limit set of x is a minimal set.

2. Proximal relation and syndetically proximal relation

A set $F \subset \mathbb{N}$ is called (i) *syndetic* if there exists a positive integer a such that $\{i, i+1, \ldots, i+a\} \cap F \neq \phi$ for any $i \in \mathbb{N}$; (ii) *thick* if $\mathbb{N} - F$ is not syndetic, i.e., F contains arbitrarily long runs of positive integers; (iii) *thickly syndetic* if $\{n \in \mathbb{N} : n + j \in F \text{ for } 0 \leq j \leq k\}$ is syndetic for each $k \in \mathbb{N}$.

A pair of points $(x, y) \in X \times X$ is *proximal* if $\liminf_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) = 0$ and *distal* if it is not proximal. A pair of points $(x, y) \in X \times X$ is *syndetically proximal* if for every $\varepsilon > 0$ the set

$$A_{xy}^{\varepsilon} = \{j \in \mathbb{N} : \rho(f_1^j(x), f_1^j(y)) < \varepsilon\}$$

is syndetic.

The *proximal relation* and the *syndetically proximal relation* on $X \times X$ are defined respectively by

$$P(f_{1,\infty}) = \{(x,y) \in X \times X : (x,y) \textit{ is proximal}\};$$
$$L(f_{1,\infty}) = \{(x,y) \in X \times X : (x,y) \textit{ is syndetically proximal}\}.$$

And the set

$$P(f_{1,\infty})(x) = \{y \in X : (x,y) \textit{ is proximal}\}$$

is called the *proximal cell* of x.

$f_{1,\infty}$ is said *proximal* if $P(f_{1,\infty}) = X \times X$ and *syndetically proximal* if $L(f_{1,\infty}) = X \times X$.

A continuous, equivariant map is called a *homomorphism*. A one-one homomorphism of X onto X is called an *automorphism* of X. Denote the set of automorphisms of X by $H(X)$ and the following concepts are given.

Two points $x, y \in X$ are said to be *regular* if there exists $h \in H(X)$ such that $(h(x), y) \in P(f_{1,\infty})$. The set of regular pairs in X is called the *regular relation* and is denoted by $R(f_{1,\infty})$. Two points $x, y \in X$ are said to be *syndetically regular* if there exists $h \in H(X)$ such that $(h(x), y) \in L(f_{1,\infty})$. The set of syndetically regular pairs in X is called the *syndetically regular relation* and is denoted by $S(f_{1,\infty})$.

The following shows some relations among $P(f_{1,\infty})$, $L(f_{1,\infty})$, $R(f_{1,\infty})$, $S(f_{1,\infty})$ and $P(f_{1,\infty})(x)$ in nonautonomous systems.

Theorem 2.1. *Given a nonautonomous discrete system $(X, f_{1,\infty})$, the following statements are true:*

(i) $\triangle_X \subset L(f_{1,\infty})$ *(where $\triangle_X = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$);*

(ii) $L(f_{1,\infty}) \subset P(f_{1,\infty}) \subset R(f_{1,\infty})$;

(iii) $L(f_{1,\infty}) \subset S(f_{1,\infty}) \subset R(f_{1,\infty})$;

(iv) If $P(f_{1,\infty}) = L(f_{1,\infty})$, *then* $R(f_{1,\infty}) = S(f_{1,\infty})$;

(v) If $P(f_{1,\infty})$ *is closed, then* $P(f_{1,\infty}) = L(f_{1,\infty})$.

Proof. *(i)* Because $\rho(f_1^j(x), f_1^j(x)) = 0 < \varepsilon$ for every $(x,x) \in X \times X$, one has $\triangle_X \subset L(f_{1,\infty})$.

(ii) Since the identity map is an automorphism of X, $P(f_{1,\infty}) \subset R(f_{1,\infty})$ is obvious. The following shows that $L(f_{1,\infty}) \subset P(f_{1,\infty})$.

Assume $(x,y) \in L(f_{1,\infty})$. Then, $\forall \varepsilon > 0$, there exists $a \in \mathbb{N}$ such that, $\forall i \in \mathbb{N}$, one has

$$\{j \in \mathbb{N} : \rho(f_1^j(x), f_1^j(y)) < \varepsilon\} \cap \{i, i+1, \ldots, i+a\} \neq \phi.$$

Then, for $\varepsilon = \frac{1}{2}$, $\exists a_1 \in \mathbb{N}$, put $i_1 = 1$, $\exists j_1 \in \{1, 2, \ldots, a_1 + 1\}$ such that

$$\rho(f_1^{j_1}(x), f_1^{j_1}(y)) < \frac{1}{2}.$$

For $\varepsilon = \frac{1}{2^2}$, $\exists a_2 \in \mathbb{N}$, put $i_2 = a_1 + 2$, $\exists j_2 \in \{a_1 + 2, a_1 + 3, \ldots, a_1 + 2 + a_2\}$ such that

$$\rho(f_1^{j_2}(x), f_1^{j_2}(y)) < \frac{1}{2^2}.$$

For $\varepsilon = \frac{1}{2^3}$, $\exists a_3 \in \mathbb{N}$, put $i_3 = a_1 + a_2 + 3$, $\exists j_3 \in \{a_1 + a_2 + 3, a_1 + a_2 + 4, \ldots, a_1 + a_2 + 3 + a_3\}$ such that

$$\rho(f_1^{j_3}(x), f_1^{j_3}(y)) < \frac{1}{2^3}.$$

Continuing this process, one can obtain an increasing sequence $\{j_k\}_{k=1}^{\infty}$ such that

$$\rho(f_1^{j_k}(x), f_1^{j_k}(y)) < \frac{1}{2^k} (k = 1, 2, \ldots).$$

Then

$$\liminf_{n \to \infty} \rho(f_1^n(x), f_1^n(y)) = 0.$$

So, $(x,y) \in P(f_{1,\infty})$. Hence, $L(f_{1,\infty}) \subset P(f_{1,\infty})$.

(iii) First, let h be the identity map. Then $h \in H(x)$. So, $L(f_{1,\infty}) \subset S(f_{1,\infty})$ holds.

And by (2), $L(f_{1,\infty}) \subset P(f_{1,\infty})$. Assume $(x, y) \in S(f_{1,\infty})$. Then, $\exists h \in H(x)$ such that $(h(x), y) \in L(f_{1,\infty})$. So, $(h(x), y) \in P(f_{1,\infty})$. One can obtain that $(x, y) \in R(f_{1,\infty})$.

Hence, $S(f_{1,\infty}) \subset R(f_{1,\infty})$ is hold.

(iv) By the definitions of $R(f_{1,\infty})$ and $S(f_{1,\infty})$, the conclusion is true.

(v) For any $(x, y) \in P(f_{1,\infty})$, i.e.,

$$\liminf_{n\to\infty} \rho(f_1^n(x), f_1^n(y)) = 0,$$

there exists an increasing sequence $\{n_s\}_{s=1}^{+\infty} \subset \mathbb{N}$ such that

$$\lim_{s\to\infty} \rho(f_{n_s} \circ \cdots \circ f_{n_1}(x), f_{n_s} \circ \cdots \circ f_{n_1}(y)) = 0.$$

For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\rho(f_{n_k} \circ \cdots \circ f_{n_1}(x), f_{n_k} \circ \cdots \circ f_{n_1}(y)) < \varepsilon$$

for every $n_k \in \{n_s\}_{s=1}^{+\infty} : n_k > N_\varepsilon$.

Put $a = \max\{N_\varepsilon + 1, \sup_{s\in\mathbb{N}} |n_{s+1} - n_s|\}$. Then, for every $i \in \mathbb{N}$, one has

$$\begin{aligned} &A_{xy}^\varepsilon \cap \{i, i+1, \ldots, i+a\} \\ =&\{j \in \mathbb{N} : \rho(f_1^j(x), f_1^j(y)) < \varepsilon\} \cap \{i, i+1, \ldots, i+a\} \\ \neq&\phi. \end{aligned}$$

So, $(x, y) \in L(f_{1,\infty})$.

This completes the proof. □

Theorem 2.2. *Suppose $(X, f_{1,\infty})$ is minimal and proximal. Then*

$$L(f_{1,\infty}) = S(f_{1,\infty}) \subset P(f_{1,\infty}) = R(f_{1,\infty}).$$

To prove this theorem, three lemmas are established first.

Lemma 2.1. *If $(X, f_{1,\infty})$ is a minimal system, then X is closed and invariant for every f_i $(i = 1, 2, \ldots)$.*

Proof. First, $\forall x \in X$, $\overline{Orb(x, f_{1,\infty})} = X$, namely X is closed.

For every f_i $(i = 1, 2, \ldots)$, put $x_0 \in X$ and

$$f_i(X) = f_i(\overline{Orb(x_0, f_{1,\infty})}) \subset \overline{f_i(Orb(x_0, f_{1,\infty}))} \subset X.$$

Then, X is invariant for every f_i $(i = 1, 2, \ldots)$.

This completes the proof. □

Lemma 2.2. *If $(X, f_{1,\infty})$ is proximal and $(Y, f_{1,\infty})$ is a minimal system, then there is a unique minimal set in $X \times Y$ (where X and Y are two compact metric spaces).*

Proof. Suppose M_1, M_2 are two minimal sets in $X \times Y$, and Π_X and Π_y are projections in X and Y respectively.

Since $(Y, f_{1,\infty})$ is a minimal system, one has $\Pi_y(M_i) = Y$ $(i = 1, 2)$. Let y is an arbitrary point in Y. Then, there exist $x_1, x_2 \in X$ such that $(x_i, y) \in M_i$ $(i = 1, 2)$.

And, because $(X, f_{1,\infty})$ is proximal, one has

$$\liminf_{j\to\infty} \rho(f_1^j(x_1), f_1^j(x_2)) = 0.$$

One can find a sequence $\{f_{i_k}\}_{k=1}^{\infty} \in \{f_i\}_{i=1}^{\infty}$ such that, for some z in X,

$$\lim_{k\to\infty} f_{i_k} \circ \cdots \circ f_{i_1}(x_1) = \lim_{k\to\infty} f_{i_k} \circ \cdots \circ f_{i_1}(x_2) = z.$$

And, there exists $y_1 \in Y$ such that

$$\lim_{k\to\infty} f_{i_k} \circ \cdots \circ f_{i_1}(y) = y_1.$$

By Lemma 2.1, M_1, M_2 are closed and invariant, hence $(z, y_1) \in M_1 \cap M_2$, which means that $M_1 = M_2$.

This completes the proof. □

Lemma 2.3. *Let X and Y be two compact metric spaces. If $(X, f_{1,\infty})$ is proximal and $(Y, f_{1,\infty})$ is a minimal system, then there is a unique homomorphism from $(Y, f_{1,\infty})$ to $(X, f_{1,\infty})$. Specifically, the homomorphism is the identity.*

Proof. Suppose $\varphi_1 : (Y, f_{1,\infty}) \to (X, f_{1,\infty})$ and $\varphi_2 : (Y, f_{1,\infty}) \to (X, f_{1,\infty})$ are two arbitrary homomorphism maps. Then $W_i = \{(\varphi_i(y), y) : y \in Y\}$ $(i = 1, 2)$ are two minimal sets in $X \times Y$. By Lemma 2.2, $W_1 = W_2$, hence $\varphi_1 = \varphi_2$.

Notice that the identity is a homomorphism from Y to X, hence φ_1 is the identity.

This completes the proof. □

Now, the proof of Theorem 2.2 is given.

Proof. (1) $L(f_{1,\infty}) \subset S(f_{1,\infty})$ is obtained in Theorem 2.1 *(iii)*. $S(f_{1,\infty}) \subset L(f_{1,\infty})$ is shown next.

For every $(x, y) \in S(f_{1,\infty})$, there exists $h \in H(x)$ such that $(h(x), y) \in L(f_{1,\infty})$. However, $(X, f_{1,\infty})$ is minimal and proximal. By Lemma 2.3, h is the identity. So, $(x, y) \in L(f_{1,\infty})$.

(2) Since $P(f_{1,\infty}) = X \times X$, one has $S(f_{1,\infty}) \subset P(f_{1,\infty})$.

(3) For every pair $(x, y) \in X \times X$ and arbitrary $h \in H(X)$, one has $(h(x), y) \in X \times X$. Since $P(f_{1,\infty}) = X \times X$, one has $(h(x), y) \in P(f_{1,\infty})$. This implies that $(x, y) \in R(f_{1,\infty})$. So, $R(f_{1,\infty}) = X \times X = P(f_{1,\infty})$.

This completes the proof. □

Corollary 2.1. *Suppose $H(X) = \{1_X\}$, where $\{1_X\}$ is the identity homomorphism of X. Then*

$$L(f_{1,\infty}) = S(f_{1,\infty}) \subset P(f_{1,\infty}) = R(f_{1,\infty}).$$

Theorem 2.3. *For a minimal dynamical system $(X, f_{1,\infty})$, the following are equivalent.*

(i) For every $x \in X$, the proximal cell $P(f_{1,\infty})(x)$ is dense in X;

(ii) For some $x \in X$, the proximal cell $P(f_{1,\infty})(x)$ is dense in X;

(iii) $P(f_{1,\infty})$ is dense in $X \times X$.

Proof. (i) $\Rightarrow$ (ii) It is clearly.

(ii) $\Rightarrow$ (iii) Minimal dynamical systems are transitive. The following shows that, for any transitive system, the fact that $P(f_{1,\infty})(x)$ is dense for some $x \in \mathrm{Trans}(f_{1,\infty})$ implies that $P(f_{1,\infty})$ is dense in $X \times X$.

Let $(z,w) \in X \times X$ and $\varepsilon > 0$. Since x is a transitive point, there exists a positive integer n_ε such that $\rho(f_1^{n_\varepsilon}(x), z) < \varepsilon$. Since $f_i (\forall i = 1, 2, \ldots)$ are surjective, there exists $u \in X$ such that $f_1^{n_\varepsilon}(u) = w$. Let $\delta > 0$ be an ε modulus of continuity for $f_1^{n_\varepsilon}$ at u, and because $P(f_{1,\infty})(x)$ is dense in X, one can choose $y \in P(f_{1,\infty})(x)$ such that $\rho(u, y) < \delta$. Then

$$\rho(f_1^{n_\varepsilon}(u), f_1^{n_\varepsilon}(y)) = \rho(f(w), f_1^{n_\varepsilon}(y)) < \varepsilon.$$

So, $(f_1^{n_\varepsilon}(x), f_1^{n_\varepsilon}(y)) \in B((z,w), \varepsilon)$.

Since $(x,y) \in P(f_{1,\infty})$, i.e., $\liminf_{k\to\infty} \rho(f_1^k(x), f_1^k(y)) = 0$, one has $(f_1^{n_\varepsilon}(x), f_1^{n_\varepsilon}(y)) \in P(f_{1,\infty})$. So, $P(f_{1,\infty})$ is dense in $X \times X$.

(iii) $\Rightarrow$ (i) Since $\overline{P(f_{1,\infty})} = X$, one has that for any $(x,y) \in X \times X$ and any $\varepsilon > 0$, there exists $(z,w) \in B((x,y),\varepsilon)$ such that $\liminf_{n\to\infty} \rho(f_1^n(z), f_1^n(w)) = 0$.

For any fixed $x_0 \in X$, $\forall y \in X$, $\forall \varepsilon > 0$, there exists $(x_0, w) \in B((x_0, y), \varepsilon)$, i.e., $w \in B(y, \varepsilon)$, such that $\liminf_{n\to\infty} \rho(f_1^n(x_0), f_1^n(w)) = 0$.

Then, for every open set $U \in X$,

$$\{w \in X : \liminf_{n\to\infty} \rho(f_1^n(x_0), f_1^n(w)) = 0\} \cap U \neq \phi.$$

By the arbitrariness of (x,y), changing x_0, one can obtain that

$$\{w \in X : \liminf_{n\to\infty} \rho(f_1^n(x), f_1^n(w)) = 0\} \cap U \neq \phi$$

for every $x \in X$ and every open set U in X.

So, $P(f_{1,\infty})(x)$ is dense in X.

This completes the proof. □

3. Chaotic properties

Let $(X, f_{1,\infty})$ be a nonautonomous system. For $x, y \in X$ and $\delta > 0$, (x,y) is called a *Li-Yorke pair of modulus* δ if

$$\limsup_{k\to\infty} \rho(f_1^k(x), f_1^k(y)) > \delta \text{ and } \liminf_{k\to\infty} \rho(f_1^k(x), f_1^k(y)) = 0.$$

Specially, if $\delta = 0$, (x,y) is called a *Li-Yorke pair*. The set of Li-Yorke pairs of modulus δ and the set of Li-Yorke pairs are denoted by $LY_\rho(f_{1,\infty}, \delta)$ and $LY_\rho(f_{1,\infty})$, respectively.

The sequence $f_{1,\infty}$ is *densely chaotic* if $LY_\rho(f_{1,\infty})$ is dense in $X \times X$ and *densely* δ*-chaotic* if $LY_\rho(f_{1,\infty}, \delta)$ is dense in $X \times X$.

Denote

$$\xi(x, y, t, n) = |\{i : \rho(f_1^i(x), f_1^i(y)) < t, 0 < i < n\}|,$$

where $|A|$ is the cardinality of the set A. By means of ξ, the following two functions are defined:

$$F_{xy}(t, f_{1,\infty}) = \liminf_{n\to\infty} \frac{1}{n} \xi(x,y,t,n) \text{ and } F^*_{xy}(t, f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \xi(x,y,t,n).$$

They are called *lower* and *upper distribution* functions, respectively. It is easy to see that

$$F_{xy}(t, f_{1,\infty}) = F^*_{xy}(t, f_{1,\infty}) = 0 \text{ for } t < 0$$

and

$$F_{xy}(t, f_{1,\infty}) = F^*_{xy}(t, f_{1,\infty}) = 1 \text{ for } t > \text{ diam}X,$$

(where diam$X$ is the diameter of X). The sequence $f_{1,\infty}$ is said to be *distributional chaotic* if there is an uncountable subset $S \subset X$ such that for any $x, y \in S, x \neq y$, one has that

(i) $F^*_{xy}(t, f_{1,\infty}) = 1$ for all $t > 0$;
(ii) $F_{xy}(s, f_{1,\infty}) = 0$ for some $s > 0$.

If there exists $s > 0$ such that $f_{1,\infty}$ is *distributional chaotic*, $f_{1,\infty}$ is said to be *uniformly distributionally chaotic.*

The sequence $f_{1,\infty}$ has *sensitive dependence on initial conditions (briefly, is sensitive)* if there exists a $\delta > 0$ such that for all $x \in X$, all $\varepsilon > 0$, and some positive integer n, there is some y which is within a distance ε of x such that $\rho(f^n_1(x), f^n_1(y)) \geq \delta$.

Write

$$N(U, \varepsilon) = \{n \in \mathbb{N} : \text{diam}(f^n_1(U)) > \varepsilon\}.$$

It is easy to see that $(X, f_{1,\infty})$ is sensitive if and only if $N(U, \varepsilon) \neq \phi$ for some $\varepsilon > 0$ and every nonempty open set $U \subset X$.

The sequence $f_{1,\infty}$ is *thickly sensitive* if $N(U, \varepsilon)$ is thick for some $\varepsilon > 0$ and every nonempty open set $U \subset X$. The sequence $f_{1,\infty}$ is *thickly syndetically sensitive* if $N(U, \varepsilon)$ is thickly syndetic for some $\varepsilon > 0$ and every nonempty open set $U \subset X$.

By the above definitions, the following conclusions are obtained.

Theorem 3.1. *If $f_{1,\infty}$ is densely chaotic or densely δ-chaotic, then $P(f_{1,\infty})$ is dense in $X \times X$.*

Proof. It is obvious. □

Theorem 3.2. *If $(X, f_{1,\infty})$ is syndetically proximal, then the lower distribution $F_{xy}(t, f_{1,\infty}) > 0$ for any $t > 0$.*

Proof. Since $f_{1,\infty}$ is syndetically proximal, for any $(x, y) \in X \times X : x \neq y$ and any $t > 0$, there exists $a_t \in \mathbb{N}$ such that

$$\begin{aligned} & A^t_{xy} \cap \{i, i+1, \ldots, i+a_t\} \\ =& \{j \in \mathbb{N} : \rho(f^j_1(x), f^j_1(y)) < t\} \cap \{i, i+1, \ldots, i+a_t\} \\ \neq& \phi \end{aligned}$$

for every $i \in \mathbb{N}$.

Noticing that $a_t[\frac{n}{a_t}] \leq n \leq (a_t + 1)[\frac{n}{a_t}]$ for any $n \in \mathbb{N}$, one has $\frac{1}{n}[\frac{n}{a_t}] \geq \frac{1}{a_t+1}$.

$$\frac{1}{n}[\frac{n}{a_t}] \leq \frac{1}{n}\xi(x, y, t, n) = \frac{1}{n}|\{j \in \mathbb{N} : \rho(f^j_1(x), f^j_1(y)) < t, 0 \leq i < n\}|,$$

thus $\frac{1}{n}\xi(x,y,t,n) \geq \frac{1}{a_t+1}$. So $F_{xy}(t, f_{1,\infty}) > 0$ for any $t > 0$.

This completes the proof. □

According to Theorem 3.2, the following conclusion holds.

Theorem 3.3. *If $(X, f_{1,\infty})$ is syndetically proximal, then $f_{1,\infty}$ is not distributional chaotic nor uniform distributional chaotic.*

Now, some results of sensitivity are established.

Theorem 3.4. *If $(X, f_{1,\infty})$ is minimal and sensitive, then $f_{1,\infty}$ is syndetically sensitive.*

Proof. For sensitive constant $\varepsilon > 0$ and any open set $U \subset X$, there exist $x, y \in U$ and $n \in \mathbb{N}$ such that $\rho(f_1^n(x), f_1^n(y)) > \varepsilon$. So, there exists an open subset $U_1 \subset U$ such that $\rho(f_1^n(x), f_1^n(U_1)) > \varepsilon$.

Since $(X, f_{1,\infty})$ is minimal, there exists $k \in \mathbb{N}$ such that $f_{n+1}^k(f_1^n(x)) \in U_1 \in U$. So,

$$\rho(f_1^n(x), f_1^{k+n}(x)) \geq \rho(f_1^n(x), f_1^n(U_1)) > \varepsilon.$$

For any $i \in \mathbb{N}$, f_i is uniformly continuous. So, there exists $\delta \in (0, \frac{\varepsilon}{4})$ such that for $\forall x_1, x_2 \in X : \rho(x_1, x_2) < \delta$, $\rho(f_t^j(x), f_t^j(x)) < \frac{\varepsilon}{4}$ $(j = 1, 2, \ldots, k), \forall t \in \mathbb{N}$.

Also, $f_1^n(x)$ is a minimal point, hence $N(f_1^n(x), B(f_1^n(x), \delta))$ is syndetic. For any $m \in N(f_1^n(x), B(f_1^n(x), \delta))$, $f_1^{m+n}(x) = f_{n+1}^m(f_1^n(x)) \in B(f_1^n(x), \delta)$, one has

$$\rho(f_1^n(x), f_1^{m+n}(x)) < \delta \text{ and } \rho(f_1^{k+n}(x), f_1^{k+m+n}(x)) < \frac{\varepsilon}{4}.$$

So

$$\begin{aligned} &\rho(f_1^{m+n}(x), f_1^{k+m+n}(x)) \\ \geq &\rho(f_1^n(x), f_1^{k+n}(x)) - \rho(f_1^n(x), f_1^{m+n}(x)) - \rho(f_1^{k+n}(x), f_1^{k+m+n}(x)) \\ > &\frac{\varepsilon}{2}. \end{aligned}$$

This implies that

$$N(U, \frac{\varepsilon}{2}) = \{n \in \mathbb{N} : \operatorname{diam}(f_t^n(U)) > \frac{\varepsilon}{2}, \forall t \in \mathbb{N}\}$$

is syndetic.

This completes the proof. □

Theorem 3.5. *If $(X, f_{1,\infty})$ is transitive, and the minimal points are dense in X, then $f_{1,\infty}$ is thickly syndetically sensitive.*

Proof. Let V be a nonempty open set of X and S, T be minimal sets of $f_{1,\infty}$ with $\rho(S, T) = a$.

Since $f_i(\forall i \in \mathbb{N})$ is uniformly continuous, for any $k \in \mathbb{N}$, there exists $\delta > 0$ such that, for any $x_1, x_2 \in X : \rho(x_1, x_2) < \delta$,

$$\rho(f_t^i(x_1), f_t^i(x_2)) < \frac{a}{4}(i = 1, 2, \ldots, k; \forall t \in \mathbb{N}).$$

For any transitive point $x \in V$ and minimal set S, $\exists m \in \mathbb{N}$ such that $\rho(f_1^m(x), S) < \frac{\delta}{2}$. Thus there exists minimal point $x' \in V$ with $\rho(f_1^m(x'), S) < \frac{\delta}{2}$.

Since x' is a minimal point, there exists a syndetic set $\{n_j\}_{j=1}^{\infty}$ satisfying $\rho(f_1^{n_j}(x'), S) < \delta$. Thus,

$$\rho(f_1^{n_j+i}(x'), S) < \frac{a}{4}(i = 1, 2, \dots, k; j = 1, 2, \dots).$$

So $N(V, B(S, \frac{a}{4}))$ is thickly syndetic.

In the same way one can show that $N(V, B(T, \frac{a}{4}))$ is thickly syndetic. Then, $N(V, B(S, \frac{a}{4})) \cap N(V, B(T, \frac{a}{4}))$ is syndetic. Thus, $\forall m \in N(V, B(S, \frac{a}{4})) \cap N(V, B(T, \frac{a}{4}))$, one has

$$\{m, m+1, \dots, m+k\} \subset N(V, B(S, \frac{a}{4})) \cap N(V, B(T, \frac{a}{4})).$$

By the arbitrariness of k, $N(V, B(S, \frac{a}{4})) \cap N(V, B(T, \frac{a}{4}))$ is thickly syndetic. That is, $N(V, \frac{a}{2})$ is thickly syndetic.

We thus conclude that $f_{1,\infty}$ is thickly syndetically sensitive.

This completes the proof. □

Remark 3.1. Considering that $f_i (i \in \mathbb{N})$ converges uniformly to a map f, what is the relationship between the complexity of $f_{1,\infty}$ and the complexity of f? Canovas [2] points that, on compact metric spaces, the chaotic of $f_{1,\infty}$ will not always imply the chaotic of f. Li-Yorke chaotic is an example for negative. However, by [2], an ω-limit set of $f_{1,\infty}$ is also an ω-limit set of f. Moreover, Kolyada [8] proved that, on compact metric spaces, $h_{top}(f_{1,\infty}) \leq h_{top}(f)$ if $f_i (i \in \mathbb{N})$ converges uniformly to a map f (where $h_{top}(\cdot)$ is topological entropy). A natural question arises. How about proximal and syndetical properties in this case?

Remark 3.2. There are some other problems for further research. In autonomous dynamical systems, distributional (p, q)-chaos, DC1, DC2, and DC3 are defined. These definitions can be extended to nonautonomous dynamical systems. What are the properties of them?

Acknowledgements

Many thanks to the referees for the careful reading of the manuscript and valuable suggestions.

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