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EXACT TRAVELING WAVE SOLUTIONS AND BIFURCATIONS FOR THE DULLIN-GOTTWALD-HOLM EQUATION*

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Abstract Utilizing the methods of dynamical system theory, the Dullin-Gottwald-Holm equation is studied in this paper. The dynamical behaviors of the traveling wave solutions and their bifurcations are presented in different parameter regions. Furthermore, the exact explicit forms of all possible bounded solutions, such as solitary wave solutions, periodic wave solutions and breaking loop wave solutions are obtained.

Keywords Dullin-Gottwald-Holm equation, bifurcation, solitary wave solution, periodic wave solution, breaking loop wave solution.

MSC(2010) 34C37, 34C23, 74J30.

1. Introduction

As we know, the Dullin-Gottwald-Holm (DGH) equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 2u_x u_{xx}) + \gamma u_{xxx} = 0, \tag{1.1}$$

was derived in [6] describing the unidirectional propagation of surface waves in a shallow water regime. In fact, the DGH equation (1.1) is connected with two separate equations. When $\alpha^2 \to 0$, the DGH equation (1.1) becomes the Korteweg-de Vries (KdV) equation. When $\gamma \to 0$, the DGH equation (1.1) reduces to the Camassa-Holm (CH) equation.

Since the work of [6], various studies were devoted to DGH system. For instance, different kinds of wave solutions of CH-γ equation have been studied in [3, 5, 8, 9, 15]. Rehman et al. [22] employed the phase plane method to analyze the singular traveling wave equations of some generalized CH equations and showed the traveling wave nature of these pulse and front solutions. Ha and Liu [11] investigated the traveling wave solutions to a class of dispersive models in terms of the parameter θ. Dullin et al. [7] presented three types of equations for shallow water waves. Using Exp-function method, Xiao et al. [25] discussed exact solutions of the reduction of

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DGH equation. Naz et al. [20], Biswas and Kara [2] constructed the conservation laws for the DGH system and generalized DGH system, respectively. Meng et al. [19] presented new exact periodic wave solutions for the DGH equation. Liu and Yin [16,18], Yan and Yin [26] considered the local well-posedness for a generalized DGH equation by using Kato's theory and the Cauchy problem for the two-component DGH system respectively. Christov and Hakkaev [4], Ai and Gui [1] presented an algorithm for the inverse scattering problem associated to the DGH equation. Some phenomena, such as N peakons, wave-breaking and peakon-antipeakon interaction, have been researched in [10,17,27–29]. Shen et al [23], Sun [24] studied the problem for optimal control of the viscous DGH equation.

To current state of our knowledge, the possible bounded exact solutions of DGH equation in different parameter regions have not been presented entirely. Motivated by this, we attempt to investigate the dynamical behaviors of all traveling wave solutions of (1.1) and to find possible exact parametric representations of the bounded traveling wave solutions of (1.1) in this paper. To this end, let $u(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed. Then, system (1.1) becomes

$$-c\phi' + c_0\phi' + 3\phi\phi' - \alpha^2(-c\phi''' + \phi\phi''' + 2\phi'\phi'') + \gamma\phi''' = 0, \tag{1.2}$$

where " $'$ " reperents the derivative with respect to ξ. Integrating (1.2) once and setting the integration constant as 0, we obtain

$$\alpha^2(\phi - c - \frac{\gamma}{\alpha^2})\phi'' + \frac{1}{2}\alpha^2\phi'^2 - (c_0 - c)\phi - \frac{3}{2}\phi^2 = 0. \tag{1.3}$$

Denote

$$\beta = c + \frac{\gamma}{\alpha^2}, \quad s = c_0 - c,$$

for $\alpha^2 \neq 0$. Eq. (1.3) is expressed as

$$\alpha^2(\phi - \beta)\phi'' + \frac{1}{2}\alpha^2\phi'^2 - s\phi - \frac{3}{2}\phi^2 = 0,$$

which is equivalent to the system

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{\frac{3}{2}\phi^2 + s\phi - \frac{1}{2}\alpha^2 y^2}{\alpha^2(\phi - \beta)}. \end{aligned} \tag{1.4}$$

Obviously, this is a singular traveling wave system with the singular straight line $\phi = \beta$. The first integral of (1.4) is as follows:

$$H(\phi, y) = \frac{1}{2}\alpha^2(\phi - \beta)y^2 - \frac{1}{2}s\phi^2 - \frac{1}{2}\phi^3 = h. \tag{1.5}$$

According to the dynamical theory, the phase portraits of (1.4) determine all possible traveling wave solution of (1.1). Therefore, we first investigate the dynamical behaviors of (1.4). However, the straight line $\phi = \beta$ is the singular line for system (1.4). To overcome the difficult of discontinuous of the right side of system (1.4) at $\phi = \beta$, we use the method described in [12, 21]. That is, we introduce a transformation of the independent variable to obtain the regular system and discuss the dynamics of it. By using the known dynamical behaviors of regular system, we study the wave profiles determined by all bounded solutions of the system (1.4).

The paper is organized as follows: In section 2, the dynamics and bifurcations of the associated regular system of (1.4) are presented first under different parameter conditions. In section 3, corresponding to different orbits of the associated regular system of (1.4), the exact parametric representations for the possible bounded solution $\phi(\xi)$ are studied and analyzed. Finally, conclusions of this paper are drawn in section 4.

2. Bifurcations of the phase portraits of system

In this section, we discuss the bifurcations of the associated regular system of (1.4) in the parameter space (s, β).

Letting $d\xi = \alpha^2(\phi - \beta)d\zeta$, we obtain the associated regular system of (1.4) as follows:

$$\begin{aligned} \frac{d\phi}{d\zeta} &= \alpha^2(\phi - \beta)y, \\ \frac{dy}{d\zeta} &= \frac{3}{2}\phi^2 + s\phi - \frac{1}{2}\alpha^2 y^2. \end{aligned} \tag{2.1}$$

It is easily to know system (2.1) has two equilibria $O(0,0)$ and $P(-\frac{2s}{3}, 0)$. System (2.1) has two equilibria $Q_{1,2}(\beta, \pm\sqrt{Y})$ in the straight line $\phi = \beta$ when $Y = \frac{3\beta^2+2s\beta}{\alpha^2} > 0$. If and only if $\beta = -\frac{2s}{3}$, we get $P(-\frac{2s}{3}, 0) = Q_{1,2}(\beta, \pm\sqrt{Y})$.

Let $M(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium (ϕ_i, y_i). We have

$$\begin{aligned} &J(0,0) = \det(M(0,0)) = s\beta\alpha^2, \\ &J(\frac{-2s}{3}, 0) = \det(M(\frac{-2s}{3}, 0)) = -s\alpha^2(\frac{2s}{3} + \beta), \\ &J(\beta, \pm\sqrt{Y}) = \det(M(\beta, \pm\sqrt{Y})) = -\alpha^4 Y = -\alpha^2(3\beta^2 + 2s\beta), \\ &\text{trace}M(0,0) = \text{trace}M(\frac{-2s}{3}, 0) = \text{trace}M(\beta, \pm\sqrt{Y}) = 0. \end{aligned} \tag{2.2}$$

From the theory of planar dynamical systems, for an equilibrium of a integrable system, if $J < 0$, the equilibrium is a saddle; if $J > 0$, the equilibrium is a center; if $J = 0$, the equilibrium is a degenerated equilibrium.

Denote

$$\begin{aligned} h_0 &= H(0,0) = 0, \\ h_1 &= H(-\frac{2s}{3}, 0) = -\frac{2}{27}s^3, \\ h_2 &= H(\beta, \pm\sqrt{Y}) = -\frac{1}{2}\beta^2(s + \beta). \end{aligned} \tag{2.3}$$

Through simple analysis, we obtain four bifurcation curves in (s, β) parameter plane: $s = 0, \beta = 0, \beta = -s, \beta = -\frac{2}{3}s$. These curves partition (s, β) plane into eight regions $I_a - I_h$, see Fig. 1.

In different regions of the (s, β) parameter space, we obtain the bifurcations of phase portraits of (2.1), as shown in Fig. 2.

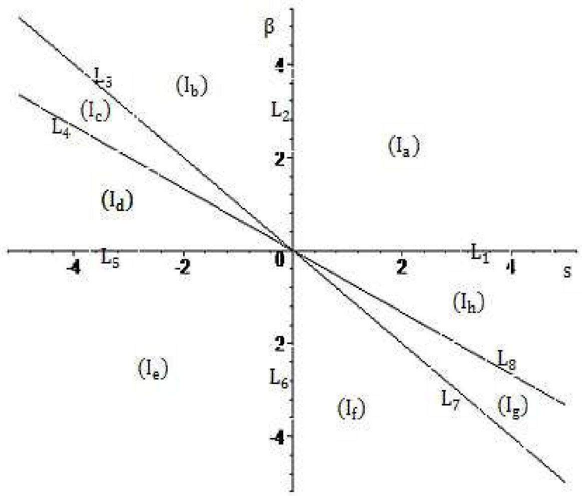


Figure 1. Bifurcation diagram for system (2.1).

3. Exact traveling wave solutions under different parameter regions

In this section, based on the phase portraits of solutions of system (2.1) in different parameter regions of the (s, β) plane in Fig. 2, we consider the corresponding traveling wave solutions $\phi(\xi)$ of (1.1) in different parameter regions of the (s, β) space and try to get the possible exact explicit parametric representations for all bounded functions $\phi(\xi)$ determined by (1.1).

Suppose that $\phi(\xi)$ is a continuous solution of the partial differential equation (1.1) for $\xi \in (-\infty, +\infty)$ and $\lim_{\xi\to+\infty} \phi(\xi) = a$, $\lim_{\xi\to-\infty} \phi(\xi) = b$. Then, $\phi(x, t)$ is called a solitary wave solution of (1.1) if $a = b$; $\phi(x, t)$ is called a kink or anti-kink solution of (1.1) if $a \neq b$. Acturally, according to the theory of dynamical systems (see [12]), a homoclinic orbit of traveling wave system (2.1) corresponds to a solitary wave solution of the partial differential equation (1.1), and a heteroclinic orbit of traveling wave system (2.1) corresponds to a kink (or anti-kink) wave solution of (1.1). Furthermore, a periodic orbit of the traveling wave system (2.1) corresponds to a periodically traveling wave solution of (1.1).

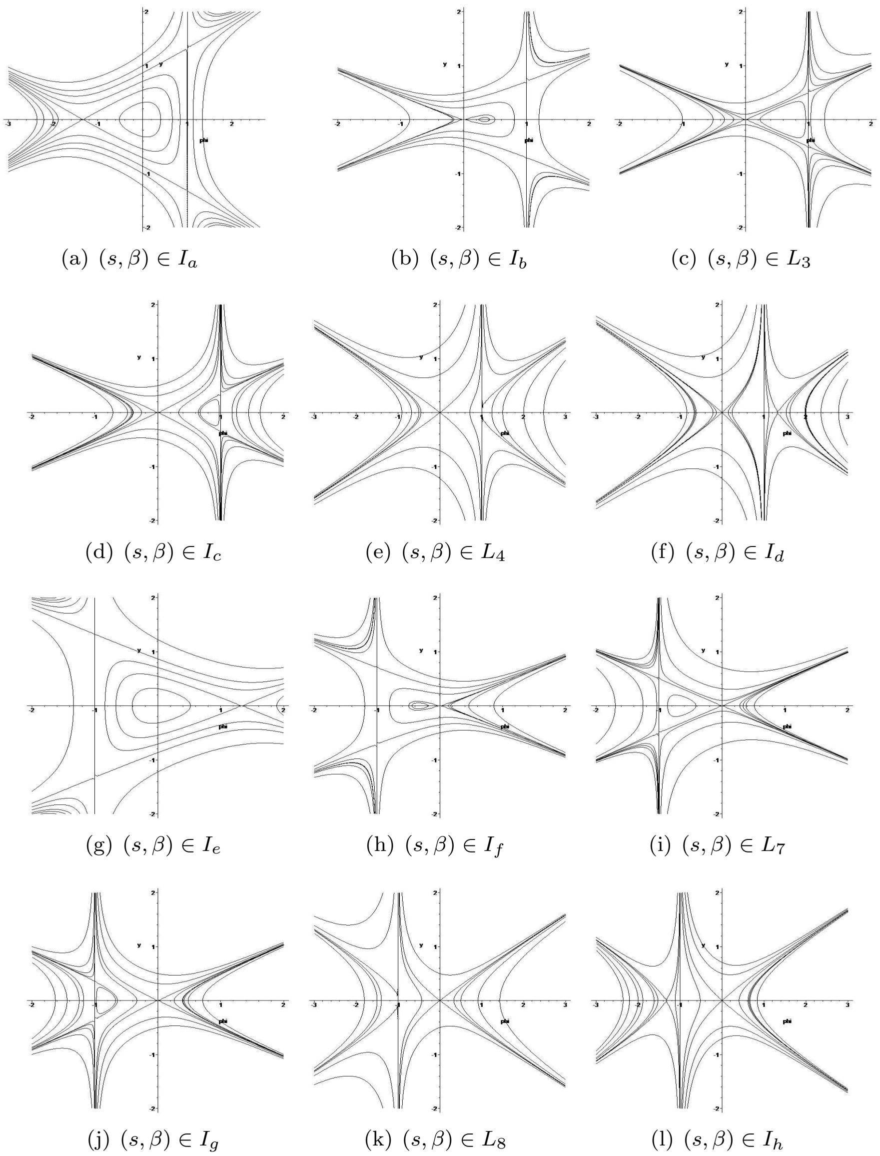
1. The case $s > 0, \beta > 0$, i.e., $(s, \beta) \in I_a$.

In this case, we have the phase portraits of (2.1) shown in Fig. 2(a). Corresponding to the homoclinic orbit defined by $H(\phi, y) = h_1 = -\frac{2}{27}s^3$, we have

$$y^2 = \frac{(\frac{s}{3} - \phi)(\phi + \frac{2s}{3})^2}{\alpha^2(\beta - \phi)}. \tag{3.1}$$

Together with the first equation of (1.4) and (3.1), we obtain the following exact solitary traveling wave solutions for (1.1) by introducing a new variable χ:

$$\phi(\chi) = \frac{2s(\beta + \frac{2s}{3})}{(\beta + \frac{5s}{3}) + (\beta - \frac{s}{3})\cosh(\sqrt{s(\beta + \frac{2s}{3})}\chi)} - \frac{2s}{3},$$



(a) $(s, \beta) \in I_a$ (b) $(s, \beta) \in I_b$ (c) $(s, \beta) \in L_3$

(d) $(s, \beta) \in I_c$ (e) $(s, \beta) \in L_4$ (f) $(s, \beta) \in I_d$

(g) $(s, \beta) \in I_e$ (h) $(s, \beta) \in I_f$ (i) $(s, \beta) \in L_7$

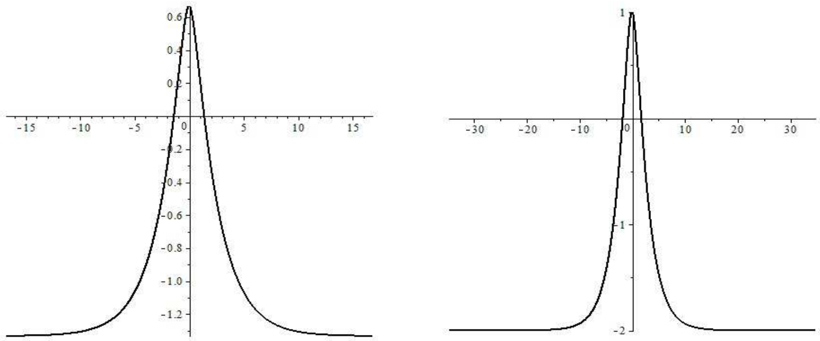
(j) $(s, \beta) \in I_g$ (k) $(s, \beta) \in L_8$ (l) $(s, \beta) \in I_h$

Figure 2. The bifurcation of phase portraits of system (2.1).

$$\xi(\chi) = |\alpha|[(\beta + \frac{2s}{3})\chi \mp \ln|\frac{6\sqrt{(\beta-\phi)(\frac{s}{3}-\phi)} + 6\phi - (3\beta + s)}{3\beta - s}|], \qquad (3.2)$$

for $\chi \in (-\infty, 0]$ and $\chi \in [0, +\infty)$, respectively. The profile of waves given by (3.2) is shown in Fig. 3.

Corresponding to the families of periodic orbits defined by $H(\phi, y) = h, h \in (h_1, 0)$, one can see the level curves in Fig. 4 defined by $H(\phi, y) = h$, for a fixed $h \in (h_1, 0)$. The level curves have two branches, passing through the points $(r_1, 0)$, $(r_2, 0)$ and $(r_3, 0)$, where $r_3 < r_2 < 0 < r_1 < \beta$.



(a) $s = 2, \beta = 1, \alpha = 2$ (b) $s = 3, \beta = 2, \alpha = 2$

Figure 3. The solitary wave solution given by (3.2) with $H(\phi, y) = h_1$, corresponding to the homoclinic orbit in Fig. 2(a).

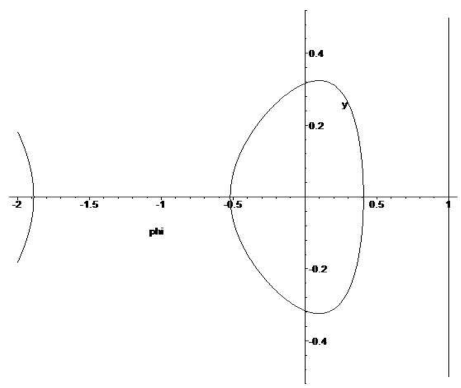


Figure 4. The level curve defined by $H(\phi, y) = h$ for $s > 0, \beta > 0$, $h \in (h_1, 0)$.

We rewrite $H(\phi, y) = h$ as

$$y^2 = \frac{-(2h + s\phi^2 + \phi^3)}{\alpha^2(\beta - \phi)} = \frac{(\phi - r_3)(\phi - r_2)(r_1 - \phi)}{\alpha^2(\beta - \phi)}. \tag{3.3}$$

From the first equation of (1.4) and (3.3), we can get the parametric representation of the periodic waves as follows:

$$\xi = |\alpha| \int_{r_2}^{\phi} \sqrt{\frac{\beta - \phi}{(r_1 - \phi)(\phi - r_2)(\phi - r_3)}} d\phi. \tag{3.4}$$

Introducing a new variable χ, we get the following periodic traveling wave solutions of (1.1):

$$\begin{aligned} \phi(\chi) &= r_3 + \frac{r_2 - r_3}{1 - \alpha_1^2 sn^2(\chi, k_1)}, \\ \xi(\chi) &= \frac{2|\alpha|}{\sqrt{(\beta - r_2)(r_1 - r_3)}}[(\beta - r_3)\chi - (r_2 - r_3)\Pi(\arcsin(sn(\chi, k_1)), \alpha_1^2, k_1)], \end{aligned} \tag{3.5}$$

where $k_1^2 = \frac{(r_1-r_2)(\beta-r_3)}{(\beta-r_2)(r_1-r_3)}$, $\alpha_1^2 = \frac{r_1-r_2}{r_1-r_3}$ and $\Pi(\cdot,\alpha_1^2,k_1)$ is the elliptic integral of the third kind.

2. The case $s<0, \beta>0, \beta>-s$, i.e., $(s,\beta)\in I_b$.

In Fig. 2(b), corresponding to the homoclinic orbit defined by $H(\phi,y)=h_0=0$, we have

$$y^2 = \frac{\phi^2(-\phi-s)}{\alpha^2(\beta-\phi)}. \tag{3.6}$$

Together with the first equation of (1.4) and (3.6), we obtain the following parameter representation of the solitary wave of (1.1):

$$\begin{aligned} \phi(\chi) &= \frac{-2\beta s}{(\beta-s)+(\beta+s)\cosh(\sqrt{-\beta s}\chi)}, \\ \xi(\chi) &= |\alpha|[\beta\chi \mp \ln|\frac{2\sqrt{(-\phi-s)(\beta-\phi)}+2\phi-(\beta-s)}{\beta+s}|], \end{aligned} \tag{3.7}$$

for $\chi\in(-\infty,0]$ and $\chi\in[0,+\infty)$, respectively.

Corresponding to the families of periodic orbits defined by $H(\phi,y)=h, h\in(0,h_1)$, similar to the case **1**, one can obtain the parametric representation of the periodic wave solutions as (3.5).

3. The case $\beta>0, \beta=-s$, i.e., $(s,\beta)\in L_3$.

From Fig. 2(c), two straight line orbits connecting to the equilibrium $(0,0)$ and $(\beta,\pm\sqrt{Y})$ defined by $H(\phi,y)=h_0=0$ have the expression

$$y^2 = \frac{\phi^2}{\alpha^2}. \tag{3.8}$$

Thus, we have

$$\phi = -se^{\frac{|\xi|}{|\alpha|}},\ 0\le|\xi|<+\infty. \tag{3.9}$$

For $h\in(0,h_1)$, corresponding to the family of periodic orbits defined by $H(\phi,y)=h$, we have the same periodic wave families as (3.5). It is indicated that system (1.1) has infinitely periodic traveling wave solutions. As $h\to 0$, these periodic traveling wave solutions will gradually lose their smoothness to become periodic cusp traveling wave solutions, finally, to converge to the cusp wave solution given by (3.9), see Fig. 5.

4. The case $\beta>0, -\frac{2s}{3}<\beta<-s$, i.e., $(s,\beta)\in I_c$.

In Fig. 2(d), corresponding to the arch orbit of system (2.1) passing through the straight line $\phi=\beta$, which is defined by $H(\phi,y)=h_2=-\frac{1}{2}\beta^2(s+\beta)$, we have

$$y^2 = \frac{\phi^3+s\phi^2-\beta^2(s+\beta)}{\alpha^2(\phi-\beta)} = \frac{\phi^2+(\beta+s)\phi+\beta(\beta+s)}{\alpha^2}, \tag{3.10}$$

which implies that

$$\begin{aligned} \phi(\xi) = &-\frac{\beta+s}{2}+\frac{1}{2}\sqrt{(s-3\beta)(s+\beta)}\cosh(\frac{\xi}{|\alpha|}), \\ &0\le|\xi|\le|\alpha|\text{arccosh}\frac{3\beta+s}{\sqrt{(\beta+s)(s-3\beta)}}. \end{aligned} \tag{3.11}$$

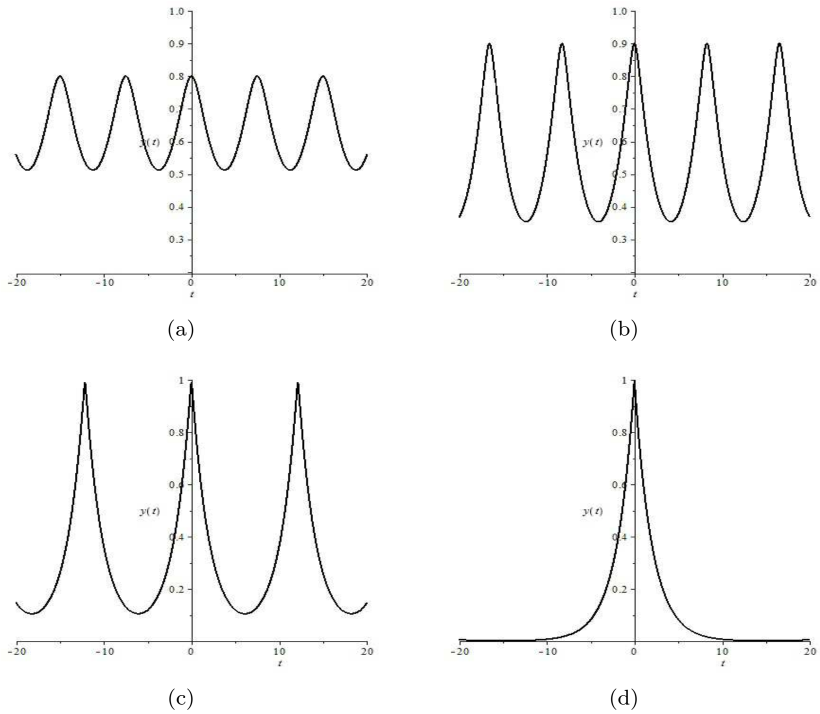


Figure 5. Solitary cusp wave and periodic wave solutions for $s = -1, \beta = 1, \alpha = 2$. (a) $\phi(0) = 0.8$; (b) $\phi(0) = 0.9$; (c) $\phi(0) = 0.99$; (d) $H(\phi, y) = 0$. The periodic wave solutions in (a), (b) and (c) correspond to the periodic orbits in Fig. 2(b) and the solitary cusp wave corresponds to the two straight line orbits connecting to the equilibrium $(0, 0)$.

Corresponding to the curves defined by $H(\phi, y) = h, h \in (0, h_2)$, we obtain the level curves defined by $H(\phi, y) = h$, for a fixed $h \in (0, h_2)$ in Fig. 6. The level curves have three open branches, passing through the points $(r_1, 0)$, $(r_2, 0)$ and $(r_3, 0)$, where $r_3 < 0 < r_2 < \beta < r_1$.

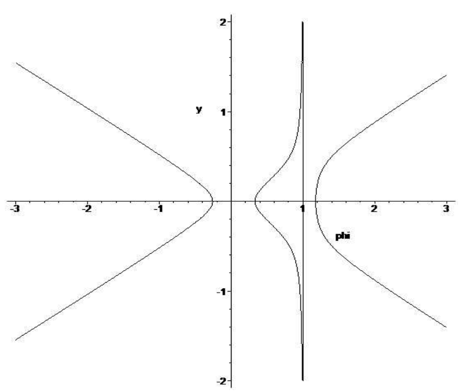


Figure 6. The level curve defined by $H(\phi, y) = h$ for $\beta > 0, -\frac{2}{3}s < \beta < -s$, $h \in (0, h_2)$.

To have a parametric representation of the open orbit passing through the point

$(r_2, 0)$, we begin with

$$y^2 = \frac{2h + s\phi^2 + \phi^3}{\alpha^2(\phi - \beta)} = \frac{(\phi - r_3)(\phi - r_2)(r_1 - \phi)}{\alpha^2(\beta - \phi)}. \tag{3.12}$$

From the first equation of (1.4) and (3.12), we get

$$\xi = |\alpha| \int_{r_2}^{\phi} \sqrt{\frac{\beta - \phi}{(\phi - r_3)(\phi - r_2)(r_1 - \phi)}} d\phi. \tag{3.13}$$

Similarly, introducing a variable χ, we obtain

$$\begin{aligned} \phi(\chi) &= r_3 + \frac{r_2 - r_3}{1 - \alpha_2^2 sn^2(\chi, k_2)}, \chi \in (-\chi_{20}, \chi_{20}), \\ \xi(\chi) &= \frac{2|\alpha|}{\sqrt{(\beta - r_3)(r_1 - r_2)}}[(r_1 - r_3)\chi - (r_2 - r_3)\Pi(\arcsin(sn(\chi, k_2)), \alpha_2^2, k_2)], \end{aligned} \tag{3.14}$$

where $k_2^2 = \frac{(r_1 - r_3)(\beta - r_2)}{(\beta - r_3)(r_1 - r_2)}$, $\alpha_2^2 = \frac{\beta - r_2}{\beta - r_3}$ and $\Pi(\cdot, \alpha_2^2, k_2)$ is the elliptic integral of the third kind. χ_{20} satisfies $r_3 + \frac{r_2 - r_3}{1 - \alpha_2^2 sn^2(\chi_{20}, k_2)} = \beta$.

It seems that (3.14) gives the loop solutions of system (1.1). In fact, the expression (3.14) corresponds to the bifurcation branch passing through the point $(r_2, 0)$ in Fig. 6. The time interval of existence of the traveling wave solution $\phi(\xi)$ with respect to ξ is finite. From the analysis in Li [13, 14], we know the solution (3.14) gives the breaking loop wave solutions of system (1.1). Fig. 7 indicates the breaking loop solutions of (1.1) with h is varied.

For $h \in (h_2, h_1)$, corresponding to the family of periodic orbits defined by $H(\phi, y) = h$, we have the same periodic wave families as (3.5).

5. The case $\beta > 0, \beta = -\frac{2s}{3}$, i.e., $(s, \beta) \in L_4$.

Due to $\beta = -\frac{2s}{3}$, we have $h_1 = h_2$. In Fig. 2(e), corresponding to the curves defined by $H(\phi, y) = h, h \in (0, h_1)$, we get a parametric representation of the open orbit passing through the point $(r_2, 0)$ as (3.14).

6. The case $\beta > 0, \beta < -\frac{2s}{3}$, i.e., $(s, \beta) \in I_d$.

In Fig. 2(f), corresponding to the curves defined by $H(\phi, y) = h, h \in (0, h_2)$, we also obtain a parametric representation of the open orbit passing through the point $(r_2, 0)$ as (3.14).

Corresponding to the curves defined by $H(\phi, y) = h, h \in (h_2, h_1)$, we can see the level curves defined by $H(\phi, y) = h$, for a fixed $h \in (h_2, h_1)$ in Fig. 8. The level curves have three open branches, passing through the points $(r_1, 0)$, $(r_2, 0)$ and $(r_3, 0)$, where $r_3 < 0 < \beta < r_2 < r_1$.

Similar to the case **4**, we get the parametric representation of the open orbit passing through the point $(r_2, 0)$ as follows:

$$\begin{aligned} \phi(\chi) &= r_1 + \frac{r_2 - r_1}{1 - \alpha_3^2 sn^2(\chi, k_3)}, \chi \in (-\chi_{30}, \chi_{30}), \\ \xi(\chi) &= -\frac{2|\alpha|}{\sqrt{(r_1 - \beta)(r_2 - r_3)}}[(r_1 - \beta)\chi - (r_1 - r_2)\Pi(\arcsin(sn(\chi, k_3)), \alpha_3^2, k_3)], \end{aligned} \tag{3.15}$$

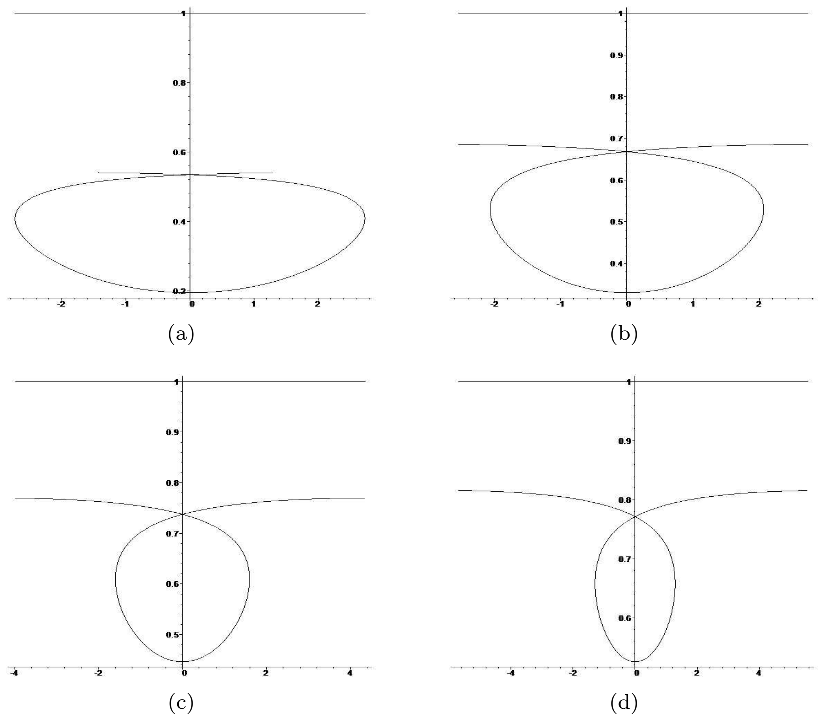


Figure 7. Breaking loop waves corresponding to an open branch of $H(\phi, y) = h, h \in (0, h_2)$ in Fig. 6 for $s = -\frac{5}{4}, \beta = 1, \alpha = 2$. (a) $h = 0.02$; (b) $h = 0.05$; (c) $h = 0.08$; (d) $h = 0.1$.

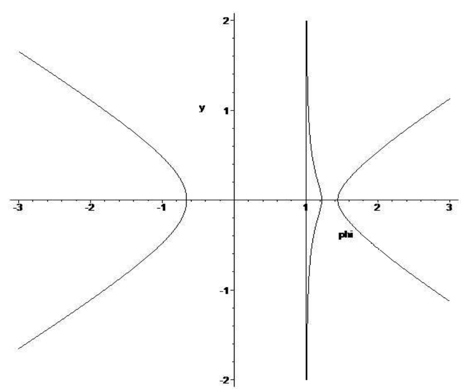
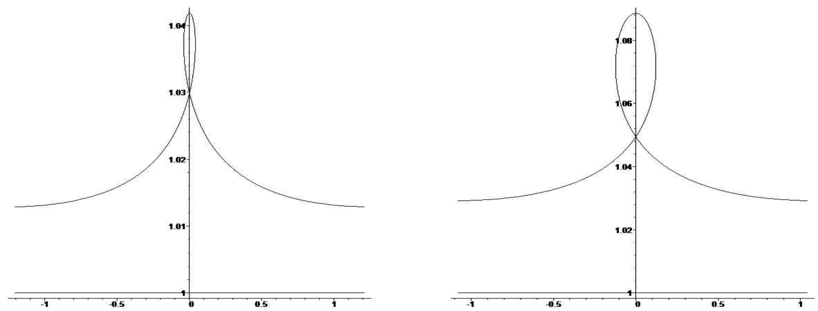


Figure 8. The level curve defined by $H(\phi, y) = h$ for $\beta > 0, \beta < -\frac{2}{3}s$, $h \in (h_2, h_1)$.

where $k_3^2 = \frac{(r_2-\beta)(r_1-r_3)}{(r_1-\beta)(r_2-r_3)}$, $\alpha_3^2 = \frac{r_2-\beta}{r_1-\beta}$ and $\Pi(\cdot, \alpha_3^2, k_3)$ is the elliptic integral of the third kind. χ_{30} satisfies $r_1 + \frac{r_2-r_1}{1-\alpha_3^2 sn^2(\chi_{30}, k_3)} = \beta$.

Also, the expression (3.15) corresponds to the breaking loop wave of system (1.1), see Fig. 9.

From Fig. 2, we easily find the last six figures are similar to the first six figures, respectively. Therefore, the further discussion of the last six cases is similar and omitted here for the limited place.



(a) (b)

Figure 9. Breaking loop waves corresponding to an open branch of $H(\phi, y) = h, h \in (h_2, h_1)$ in Fig .8 for $s = -2, \beta = 1, \alpha = 2$. (a) $h = 0.52$; (b) $h = 0.54$.

4. Conclusions

In this paper, we consider the dynamical behaviors and bifurcations of the DGH equation. Due to the singularity of the equivalent system of DGH equation, we introduce a transformation of the independent variable to obtain the regular system and discuss the dynamics of it. By using the known dynamical behaviors of regular system, we study the wave profiles determined by the bounded solutions of the DGH system. We obtain some new exact explicit solutions of DGH system, such as solitary wave solutions, periodic wave solutions and breaking loop wave solutions analytically and numerically. The results obtained in this paper enrich the analysis of DGH equation.

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References

[1] X. Ai and G. Gui, *On the inverse scattering problem and the low regularity solutions for the Dullin-Gottwald-Holm equation*, Nonlinear Anal.: Real World Appl., 11(2010), 888–894.

[2] A. Biswas and A. Kara, *1-Soliton solution and conservation laws of the generalized Dullin-Gottwald-Holm equation*, Appl. Math. Comput., 217(2010), 929–932.

[3] C. Chen, Y. Li and J. Zhang, *The multi-soliton solutions of the CH-γ equation*, Sci. China (Ser A), 51(2008), 314–320.

[4] O. Christov and S. Hakkaev, *On the inverse scattering approach and action-angle variables for the Dullin-Gottwald-Holm equation*, Physica D, 238(2009), 9–19.

[5] X. Deng, *Exact peaked wave solution of CH-γ equation by the first-integral method*, Appl. Math. Comput., 206(2008), 806–809.

[6] H. Dullin, G. Gottwald and D. Holm, *An integrable shallow water equation with linear and nonlinear dispersion*, Phys. Rev. Lett., 87(2001)194501-1-194501-4.

[7] H. Dullin, G. Gottwald and D. Holm, *Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves*, Fluid Dyn. Res., 33(2003), 73–95.

[8] B. Guo and Z. Liu, *Cusp wave solutions in CH-γ equation*, Sci. China (Ser A), 33(2003), 325–337.

[9] B. Guo and Z. Liu, *Peaked wave solutions in CH-γ equation*, Sci. China (Ser A), 46(2003), 696–709.

[10] Z. Guo and L. Ni, *Wave breaking for the periodic weakly dissipative Dullin-Gottwald-Holm equation*, Nonlinear Anal., 74(2011), 965–973.

[11] T. Ha and H. Liu, *On travelling wave solutions of the θ-equation of disperisve type*, J. Math. Anal. Appl., 421(2015), 399–414.

[12] J. Li, *Singular nonlinear travelling wave equations: Bifurcations and exact solution*, Science press, Beijing, 2013.

[13] J. Li, *Dynamical understanding of loop soliton solution for several nonlinear wave equations*, Sci. China Ser. A: Math., 50(2007), 773–785.

[14] J. Li and G. Chen, *On nonlinear wave equations with breaking loop-solutions*, Int. J. Bifur. Chaos, 20(2010), 519–537.

[15] Y. Liu, *Global existence and blow-up solutions for a nonlinear shallow water equation*, Math. Ann., 335(2006), 717–735.

[16] X. Liu and Z. Yin, *Local well-posedness and stability of peakons for a generalized Dullin-Gottwald-Holm equation*, Nonlinear Anal., 74(2011), 2497–2507.

[17] X. Liu and Z. Yin, *Orbital stability of the sum of N peakons for the Dullin-Gottwald-Holm equation*, Nonlinear Anal.: Real World Appl., 13(2012), 2414–2422.

[18] X. Liu and Z. Yin, *Local well-posedness and stability of solitary waves for the two-component Dullin-Gottwald-Holm system*, Nonlinear Anal., 88(2013), 1–15.

[19] Q. Meng, B. He, Y. Long and Z. Li, *New exact periodic wave solutions for the Dullin-Gottwald-Holm equation*, Appl. Math. Comput., 218(2011), 4533–4537.

[20] R. Naz, I. Naeem and S. Abelman, *Conservation laws for Camassa-Holm equation, Dullin-Gottwald-Holm equation and generalized Dullin-Gottwald-Holm equation*, Nonlinear Anal.: Real World Appl., 10(2009), 3466–3471.

[21] A. Nguetcho, J. Li and J. Bilbault, *Bifurcations of phase portraits of a singular nonlinear equation of the second class*, Commun. Nonlinear Sci. Numer. Simulat., 19(2014), 2590–2601.

[22] T. Rehman, G. Gambino and S. Choudhury, *Smooth and non-smooth travelling wave solutions of some generalized Camassa-Holm equations*, Commun. Nonlinear Sci. Numer. Simulat., 19(2014), 1746–1769.

[23] C. Shen, L. Tian and A. Gao, *Optimal control of the viscous Dullin-Gottwalld-Holm equation*, Nonlinear Anal.: Real World Appl., 11(2010), 480–491.

[24] B. Sun, *Maximum principle for optimal distributed control of the viscous Dullin-Gottwald-Holm equation*, Nonlinear Anal.: Real World Appl., 13(2012), 325–332.

[25] G. Xiao, D. Xian and X. Liu, *Application of Exp-function method to Dullin-Gottwald-Holm equation*, Appl. Math. Comput., 210(2009), 536–541.

[26] K. Yan and Z. Yin, *On the solutions of the Dullin-Gottwald-Holm equation in Besov spaces*, Nonlinear Anal.: Real World Appl., 13(2012), 2580–2592.

[27] J. Yin and L. Tian, *Stumpons and fractal-like wave solutions to the Dullin-Gottwald-Holm equation*, Chaos, Solitons Fract., 42(2009), 643–648.

[28] J. Zhou, L. Tian, W. Zhang and S. Kumar, *Peakon-antipeakon interaction in the Dullin-Gottwald-Holm equation*, Phys. Lett. A, 377(2013), 1233–1238.

[29] M. Zhu and J. Xua, *On the wave-breaking phenomena for the periodic two-component Dullin-Gottwald-Holm system*, J. Math. Anal. Appl., 391(2012), 415–428.