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ON A MULTIVARIABLE CLASS OF MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract The present paper studies and investigates a class of Mittag-Leffler type multivariable functions. We derive the necessary convergence conditions and establish several properties associated with this class and those related with the corresponding class of fractional integral operators. New extensions of the introduced definitions and special cases of some of the results are also pointed out.

Keywords Mittag-Leffler function, generalized Mittag-Leffler function, Euler-Beta transform, fractional integral operator, Hadamard product, Laplace transform, Multidimensional analogue of the Cauchy-Hadamard formula.

MSC(2010) 33E12, 32A05, 26A33, 47B38, 47G10.

1. Introduction and preliminaries

Throughout our present investigation, we use the following notations:

$$\mathbb{N} := \{1, 2, \cdots\}, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{Z}^- := \{-1, -2, \cdots\} \text{ and } \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\},$$

where $\mathbb{Z}$ denotes the set of integers. The symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the set of real, positive real, and complex numbers, respectively. Boldface letters with subscript m denote vectors of dimension m; for instances $\mathbf{k}_m := (k_1, \cdots, k_m) \in \mathbb{N}_0^m$ and $\mathbf{z}_m := (z_1, \cdots, z_m) \in \mathbb{C}^m$. The inner product of two m-dimensional vectors $\mathbf{u}_m$ and $\mathbf{v}_m$ is defined by $\langle \mathbf{u}_m, \mathbf{v}_m \rangle := u_1 v_1 + \cdots + u_m v_m$, and $\langle \mathbf{k}_m \rangle := k_1 + \cdots + k_m$ denotes the length of the vector $\mathbf{k}_m$. For convenience sake, we shall use the simplified notation:

$$\sum_{\mathbf{k}_m=0}^{\infty} \quad \text{for the multiple series} \quad \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty}.$$

The one-parametric Mittag-Leffler function (named after the Swedish mathematician Gösta Magnus Mittag-Leffler (1846–1927)) is an entire function defined by ([13])

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha \in \mathbb{C},\ \Re(\alpha) > 0,\ z \in \mathbb{C}), \tag{1.1}$$

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where $\Gamma(x)$ denotes the familiar Gamma function. A great deal of attention has been paid to the various generalizations of the Mittag-Leffler function by many researchers. The two-parametric Mittag-Leffler function was defined by (see [25,26]; see also [1], [5,6])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad (\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0,\ \Re(\beta)>0),\tag{1.2}$$

and the three-parametric Mittag-Leffler function introduced by Prabhaker [16, Eq. (1.3)] (see also [10]) was defined by

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\qquad (\alpha,\beta,\gamma\in\mathbb{C};\ \Re(\alpha)>0,\ \Re(\beta)>0),\tag{1.3}$$

where $(\lambda)_k$ $(\lambda\in\mathbb{C})$ is the Pochhammer symbol defined as

$$(\lambda)_k=\begin{cases}1 & (k=0;\ \lambda\in\mathbb{C}\setminus\{0\}),\\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases}\tag{1.4}$$

and when k does not belong to $\mathbb{N}$, we adopt the notation

$$[\lambda]_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\nu)},\quad (\lambda,\nu\in\mathbb{C};\ \Re(\nu)>0),$$

instead of the Pochhammer symbol defined by (1.4) (see [11]).

In fact, in Prabhakar's paper, a convolution integral equation involving the function $E^{\gamma}_{\alpha,\beta}(z)$ as its kernel was solved by considering a new fractional integral operator. Kalla *et al.* [8] also introduced a generalized multiparameter function of Mittag-Leffler type. For a comprehensive introduction to Mittag-Leffler type functions of a single variable, one may refer to [4] and the newly published book [3].

Recently, Saxena *et al.* [19] introduced the following multivariable analogue of the generalized Mittag-Leffler type function ([19, p. 536, Eq. (1.14)]):

$$E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m):=\sum_{\mathbf{k}_m=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_m)_{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_m^{k_m}}{k_m!},$$
$$(\lambda,\gamma_j,\rho_j,z_j\in\mathbb{C};\ \ \Re(\rho_j)>0\ \ (j=1,\cdots,m)),\tag{1.5}$$

and studied the boundedness and composition properties of its related fractional integral operator ([19, p. 540, Eq. (4.1)]):

$$\left(E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda;(\boldsymbol{\omega}_m);a+}\varphi\right)(x)$$
$$=\int_a^x(x-t)^{\lambda-1}E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1(x-t)^{\rho_1},\cdots,\omega_1(x-t)^{\rho_m}\right)\varphi(t)\,\mathrm{d}t,$$
$$(x>a;\ \ \lambda,\rho_j,\gamma_j,\omega_j\in\mathbb{C};\ \ \Re(\lambda)>0,\ \Re(\rho_j)>0\ \ (j=1,\cdots,m)),\tag{1.6}$$

on the space $L(a,b)$ of Lebesgue measurable functions:

$$L(a,b)=\left\{f:\|f\|:=\int_a^b|f(t)|\mathrm{d}t<\infty\right\}.\tag{1.7}$$

Garg *et al.* [2] also introduced a Mittag-Leffler-type function of two variables which is defined by ([2, p. 936, Eq. (11)])

$$\begin{aligned} E_1(x,y) &\equiv E_1\left(\begin{array}{c|c} \gamma_1,\alpha_1;\gamma_2,\beta_1 & x \\ \delta_1,\alpha_2,\beta_2;\delta_2,\alpha_3;\delta_3,\beta_3 & y \end{array}\right) \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[\gamma_1]_{\alpha_1 m}[\gamma_2]_{\beta_1 n}}{\Gamma(\delta_1+\alpha_2 m+\beta_2 n)} \frac{x^m}{\Gamma(\delta_2+\alpha_3 m)} \frac{y^n}{\Gamma(\delta_3+\beta_3 n)} \\ &(\gamma_1,\gamma_2,\delta_1,\delta_2,\delta_3,x,y\in\mathbb{C};\quad \min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3\}>0), \end{aligned} \tag{1.8}$$

and also studied its related fractional integral operator defined by ([2, p. 942, Eq. (38)])

$$\begin{aligned} &\left(I^{\rho,(\sigma)}_{(\alpha),(\beta),(\gamma),(\delta),(w);a+}\varphi\right)(x) \\ &:= \int_a^x (x-t)^{\rho-1} E_1\left(w_1(x-t)^{\sigma_1}, w_2(x-t)^{\sigma_2}\right)\varphi(t)\,\mathrm{d}t, \end{aligned} \tag{1.9}$$

where $\rho,\gamma_1,\gamma_2,\delta_2,\delta_3,\sigma_1,\sigma_2,w_1,w_2\in\mathbb{C}$, $\min\{\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\delta_1\}>0$.

A very distinctive approach mentioned in Raina [17] to generalizing the Mittag-Leffler type function suggests the following function:

$$\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}z^k, \quad (\rho,\lambda\in\mathbb{C};\ \Re(\rho)>0,\Re(\lambda)>0;\ |z|<R), \tag{1.10}$$

where $\boldsymbol{\sigma}=\sigma(k)$ $(k\in\mathbb{N}_0)$ is assumed to be an arbitrary bounded sequence. The fractional integral operator containing (1.10) as its kernel is defined by

$$\begin{aligned} \left(\mathcal{J}^{\boldsymbol{\sigma}}_{\rho,\lambda,a+;\omega}\varphi\right)(x) &= \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}\left(\omega(x-t)^{\rho}\right)\varphi(t)\,\mathrm{d}t, \\ &(x>a;\quad \lambda,\rho,\omega\in\mathbb{C};\ \ \Re(\lambda)>0,\ \Re(\rho)>0), \end{aligned} \tag{1.11}$$

and this fractional integral operator was very recently used in defining a certain class of fractional kinetic equations by Luo and Raina [11].

In the present paper, we introduce and develop a theory of the multivariable generalized Mittag-Leffler function defined by

$$\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m) = \sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma(\mathbf{k}_m)}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)} z_1^{k_1}\cdots z_m^{k_m}, \tag{1.12}$$

where $\lambda,\rho_j\in\mathbb{C}$ $(\Re(\rho_j)>0)$, $j=1,\cdots,m$. Here and throughout this paper,

$$\widehat{\boldsymbol{\sigma}} = \sigma(\mathbf{k}_m), \quad (\mathbf{k}_m\in\mathbb{N}_0^m),$$

is a suitably chosen (complex-valued) sequence such that the power series in several complex variables converges absolutely in a polydisk determined by

$$U_R := \{\mathbf{z}_m\in\mathbb{C}^m : |z_j|<R\in\mathbb{R}_+\cup\{\infty\}\ \ (j=1,\cdots,m)\}. \tag{1.13}$$

Evidently, the function (1.5) introduced by Saxena *et al.* in [19] and the function (1.8) due to Garg *et al.* in [2] are special cases of (1.12). Furthermore, it reduces to

Raina's function (1.10) when $m = 1$. We shall present in Section 2 the conditions of convergence of the function (1.14), especially, those which make it an entire function.

With the help of (1.12), we define the following fractional integral operator:

$$\begin{aligned}&\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)\\ &=\int_a^x (x-t)^{\lambda-1}\,\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1(x-t)^{\rho_1},\cdots,\omega_m(x-t)^{\rho_m}\right)\varphi(t)\,\mathrm{d}t\\ &\left(x>a;\quad \lambda,\rho_j,\omega_j\in\mathbb{C};\quad \Re(\lambda)>0,\ \Re(\rho_j)>0\quad (j=1,\cdots,m)\right).\end{aligned}\tag{1.14}$$

If we set

$$\sigma(\mathbf{k}_m)=\sigma^*(\mathbf{k}_m):=\begin{cases}1, & k_1=\cdots=k_m=0,\\ 0, & k_1\geq 1,\cdots,k_m\geq 1,\end{cases}$$

then the operator (1.14) reduces to the following familiar Riemann-Liouville fractional integral operator (see [9, p. 69] and [18, p. 33])

$$\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}^*}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)=\left(I^{\lambda}_{a+}\varphi\right)(x):=\frac{1}{\Gamma(\lambda)}\int_a^x (x-t)^{\lambda-1}\varphi(t)\,\mathrm{d}t\quad (\Re(\lambda)>0),\tag{1.15}$$

and also the operator (1.14) obviously contains as its special cases the operators (1.6) and (1.9).

2. Convergence of the function $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$

In this section, we first give a method to determine the radius of convergence of the function $\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(z)$ defined above by (1.10). We then extend this method to the multidimensional case in order to examine the radius of convergence of the multivariable function defined by (1.12).

Let $f(z)=\sum_{k=0}^{\infty}a_k z^k$ and $g(z)=\sum_{k=0}^{\infty}b_k z^k$ be two power series whose radii of convergence are denoted by R_f and R_g, respectively. Then their Hadamard product is the power series defined by

$$(f*g)(z):=\sum_{k=0}^{\infty}a_k b_k z^k.\tag{2.1}$$

The radius of convergence R of the Hadamard product series $(f*g)(z)$ satisfies $R_f\cdot R_g\leq R$, which can be proved by using the root test and the submultiplicativity of the upper limit. If one of the power series defines an entire function, then the Hadamard product series also defines an entire function (see [7, p. 230]).

The function $\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(z)$ defined by (1.10) can be interpreted as the Hadamard product of two power series given by

$$f(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\lambda)}=E_{\rho,\lambda}(z)\qquad (\rho,\lambda\in\mathbb{C};\ \Re(\rho)>0,\Re(\lambda)>0;\ z\in\mathbb{C}),\tag{2.2}$$

and

$$g(z)=\sum_{k=0}^{\infty}\sigma(k)\,z^k,\qquad (|z|<r),\tag{2.3}$$

that is,

$$\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(z) = (f * g)(z) = (E_{\rho,\lambda} * g)(z)\,. \tag{2.4}$$

From the expression (2.4), we know that if the radius of convergence of $g(z)$ is determined by

$$\overline{\lim_{k\to\infty}} |\sigma(k)|^{\frac{1}{k}} = r^{-1},$$

then $\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(z)$ also defines an entire function on the complex plane $\mathbb{C}$. It may be pointed out that when the function $\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}$ is an entire function, then we need not require that the sequence $\sigma(k)$ is a bounded sequence any more. If $r = 0$, then $\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}(z)$ may not be an entire function and its radius of convergence R can then be found by using the relation that

$$\overline{\lim_{k\to\infty}} \left|\frac{\sigma(k)}{\Gamma(\rho k + \lambda)}\right|^{\frac{1}{k}} = R^{-1}.$$

To consider now the multidimensional case, we first recall the following result in the theory of complex analysis of several variables.

Let

$$\mathcal{P}(\mathbf{z}_m) = \sum_{\mathbf{k}_m=0}^{\infty} \sigma(\mathbf{k}_m)(z_1 - a_1)^{k_1}\cdots(z_m - a_m)^{k_m}$$

be a power series centered at $\mathbf{a}_m$ in $\mathbb{C}^m$. If $\mathcal{P}(\mathbf{z}_m)$ is convergent in the polydisk

$$\Delta : |z_j - a_j| < r_j, \qquad (j = 1, \cdots, m)\,,$$

and is divergent in the product domain

$$|z_j - a_j| > r_j, \qquad (j = 1, \cdots, m)\,,$$

then $\mathbf{r}_m$ is called an associated multiradius of convergence of $\mathcal{P}(\mathbf{z}_m)$.

An associated multiradius of convergence can be determined by the following theorem.

Theorem 2.1. (see [14, p. 9] and [20, p. 32]) *If $\mathbf{r}_m$ is an associated multiradius of convergence of $\mathcal{P}(\mathbf{z}_m)$, then*

$$\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}} \left|\sigma(\mathbf{k}_m)\, r_1^{k_1}\cdots r_m^{k_m}\right|^{\frac{1}{\langle\mathbf{k}_m\rangle}} = 1. \tag{2.5}$$

We shall call this result the multidimensional analogue of the Cauchy-Hadamard formula. If $r_1 = \cdots = r_m = R$, then (2.5) becomes

$$\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}} |\sigma(\mathbf{k}_m)|^{\frac{1}{\langle\mathbf{k}_m\rangle}} = R^{-1}, \tag{2.6}$$

which is more convenient to evaluate the radius of convergence for a given power series in several complex variables.

Let

$$\mathbf{f}(\mathbf{z}_m) = \sum_{\mathbf{k}_m=0}^{\infty} a(\mathbf{k}_m)\, z_1^{k_1}\cdots z_m^{k_m}, \qquad \left(\mathbf{z}_m \in U_{R_f}\right) \tag{2.7}$$

and

$$\mathbf{g}(\mathbf{z}_m)=\sum_{\mathbf{k}_m=0}^{\infty}b(\mathbf{k}_m)\,z_1^{k_1}\cdots z_m^{k_m},\qquad\left(\mathbf{z}_m\in U_{R_g}\right)\tag{2.8}$$

be two power series in m complex variables, where U_R is a polydisk defined by (1.13). The Hadamard product of $\mathbf{f}(\mathbf{z}_m)$ and $\mathbf{g}(\mathbf{z}_m)$ can be defined by

$$(\mathbf{f}*\mathbf{g})(\mathbf{z}_m):=\sum_{\mathbf{k}_m=0}^{\infty}a(\mathbf{k}_m)\,b(\mathbf{k}_m)\,z_1^{k_1}\cdots z_m^{k_m},\qquad(\mathbf{z}_m\in U_R).\tag{2.9}$$

We establish the following result which can be used to determine the radius of convergence R for $(\mathbf{f}*\mathbf{g})(\mathbf{z}_m)$.

Lemma 2.1. *The power series generated by Hadamard product* (2.9) *converges absolutely in the polydisk U_R, where R satisfies*

$$R\geq R_f\cdot R_g,$$

where R_f and R_g denote the radii of convergence of the series (2.7) *and* (2.8), *respectively.*

Proof. Applying the formula (2.6) and using the submultiplicativity of the upper limit, we have

$$\begin{aligned}\frac{1}{R}&=\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}}\,|a(\mathbf{k}_m)\,b(\mathbf{k}_m)|^{\frac{1}{\langle\mathbf{k}_m\rangle}}\\&\leq\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}}\,|a(\mathbf{k}_m)|^{\frac{1}{\langle\mathbf{k}_m\rangle}}\,\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}}\,|b(\mathbf{k}_m)|^{\frac{1}{\langle\mathbf{k}_m\rangle}}=\frac{1}{R_fR_g}.\end{aligned}$$

Hence, we get $R_fR_g\leq R$. □

Lemma 2.2. *For $\Re(\rho_j)>0$ $(j=1,\cdots,m)$, we have*

$$\lim_{\langle\boldsymbol{k}_m\rangle\to\infty}|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|^{\frac{1}{\langle\mathbf{k}_m\rangle}}=1.\tag{2.10}$$

Proof. It follows easily that

$$\begin{aligned}\left(\min_{1\leq j\leq m}\Re(\rho_j)\right)^{\frac{1}{\langle\mathbf{k}_m\rangle}}\langle\mathbf{k}_m\rangle^{\frac{1}{\langle\mathbf{k}_m\rangle}}&\leq|\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle|^{\frac{1}{\langle\mathbf{k}_m\rangle}}\leq|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|^{\frac{1}{\langle\mathbf{k}_m\rangle}}\\&\leq\left(\sum_{j=1}^{m}|\rho_j|\,k_j\right)^{\frac{1}{\langle\mathbf{k}_m\rangle}}\leq\left(\max_{1\leq j\leq m}|\rho_j|\right)^{\frac{1}{\langle\mathbf{k}_m\rangle}}\langle\mathbf{k}_m\rangle^{\frac{1}{\langle\mathbf{k}_m\rangle}}.\end{aligned}$$

By letting $\langle\mathbf{k}_m\rangle\to\infty$ and noting the elementary formula that $\lim_{n\to\infty}n^{\frac{1}{n}}=1$, we arrive at the desired result (2.10). □

In what follows, we write

$$\Upsilon:=\max_{1\leq j\leq m}\Im(\rho_j),\ \Psi_1:=\max_{1\leq j\leq m}\Re(\rho_j)\quad\text{and}\quad\Psi_2:=\min_{1\leq j\leq m}\Re(\rho_j).\tag{2.11}$$

Theorem 2.2. *Let $\sigma(\mathbf{k}_m)$ be a complex-valued sequence such that*

$$\overline{\lim_{\langle\mathbf{k}_m\rangle\to\infty}}\,|\sigma(\mathbf{k}_m)|^{\frac{1}{\langle\mathbf{k}_m\rangle}}=R_h^{-1},\tag{2.12}$$

where $R_h \in \mathbb{R}_+ \cup \{\infty\}$*. Then,* $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ *defines an entire function on* $\mathbb{C}^m$*.*

In general, the associated multiradius of convergence of $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ *can be determined by using the formula that*

$$\overline{\lim_{\langle \mathbf{k}_m\rangle\to\infty}} \left|\frac{\sigma(\mathbf{k}_m)}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\boldsymbol{k}_m\rangle)}\right|^{\frac{1}{\langle\mathbf{k}_m\rangle}} = R^{-1}. \tag{2.13}$$

Proof. In view of the Hadamard product given by (2.9), the multivariable generalized Mittag-Leffler type function defined by (1.12) can be expressed as

$$\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m) = (E_{\boldsymbol{\rho}_m,\lambda} * \mathbf{h})(\mathbf{z}_m), \tag{2.14}$$

where

$$E_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m) = \sum_{\mathbf{k}_m=0}^{\infty} \frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}, \qquad (\mathbf{z}_m\in\mathbb{C}^m), \tag{2.15}$$

is a special case of (1.5) and

$$\mathbf{h}(\mathbf{z}_m) := \sum_{\mathbf{k}_m=0}^{\infty} \sigma(\mathbf{k}_m)\, z_1^{k_1}\cdots z_m^{k_m}, \qquad (\mathbf{z}_m\in U_{R_h}). \tag{2.16}$$

The function $E_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ is an entire function and Saxena *et al.* used the conditions stated by Srivastava and Daoust [22, p. 454] for the generalized Lauricella series in several variables to guarantee the convergence of $E_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$. To prove our result, we shall apply the multidimensional analogue of the Cauchy-Hadamard formula given above by Theorem 2.1.

In fact, by using the Stirling's formula ([15, p. 141, Eq. (5.11.7)])

$$\Gamma(az+b) \sim \sqrt{2\pi}e^{-az}(az)^{az+b-\frac{1}{2}} \quad (a>0, b\in\mathbb{C};\ |\arg z|<\pi),$$

we have

$$\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle) \sim \sqrt{2\pi}e^{-\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle}\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle^{\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle+\lambda-\frac{1}{2}},$$
$$\left(\rho_j,\lambda\in\mathbb{C}\ (\Re(\rho_j)>0),\ j=1,\cdots,m;\ |\arg\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|<\frac{\pi}{2}\right),$$

which gives

$$\overline{\lim_{\langle \mathbf{k}_m\rangle\to\infty}} \frac{1}{|\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)|^{\frac{1}{\langle\mathbf{k}_m\rangle}}} = \overline{\lim_{\langle \mathbf{k}_m\rangle\to\infty}} \frac{e^{\frac{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}{\langle\mathbf{k}_m\rangle}}}{\left|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle^{\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle+\lambda-\frac{1}{2}}\right|^{\frac{1}{\langle\mathbf{k}_m\rangle}}}. \tag{2.17}$$

Since the denominator in the right-hand side of (2.17) can be expressed as

$$\left|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle^{\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle+\lambda-\frac{1}{2}}\right| = |\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|^{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle+\Re(\lambda)-\frac{1}{2}}$$
$$\cdot\, e^{-\langle\Im(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle\arg\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle-\Im(\lambda)\arg\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle},$$

we have

$$\frac{e^{\frac{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}{\langle\mathbf{k}_m\rangle}}}{\left|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle^{\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle+\lambda-\frac{1}{2}}\right|^{\frac{1}{\langle\mathbf{k}_m\rangle}}} = e^{\frac{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}{\langle\mathbf{k}_m\rangle}} \frac{e^{\frac{\langle\Im(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}{\langle\mathbf{k}_m\rangle}\arg\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle+\frac{\Im(\lambda)}{\langle\mathbf{k}_m\rangle}\arg\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle}}{|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|^{\frac{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}{\langle\mathbf{k}_m\rangle}}\,|\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle|^{\frac{2\Re(\lambda)-1}{\langle\mathbf{k}_m\rangle}}}. \tag{2.18}$$

Making use of the notations given in (2.11) and the condition $|\arg\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle| < \frac{\pi}{2}$, we can derive the following inequalities

$$e^{\frac{\langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m\rangle}{\langle \mathbf{k}_m\rangle}} \le e^{\Psi_1}, \tag{2.19}$$

$$e^{\frac{\langle \Im(\boldsymbol{\rho}_m), \mathbf{k}_m\rangle}{\langle \mathbf{k}_m\rangle} \arg\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle + \frac{\Im(\lambda)}{\langle \mathbf{k}_m\rangle} \arg\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle} \le e^{\Upsilon \frac{\pi}{2} + \frac{\Im(\lambda)}{\langle \mathbf{k}_m\rangle} \frac{\pi}{2}}, \tag{2.20}$$

and

$$|\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle|^{\frac{\langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m\rangle}{\langle \mathbf{k}_m\rangle}} \ge |\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle|^{\Psi_2}. \tag{2.21}$$

Then, with the help of the inequalities (2.19)-(2.21), the left-hand side of (2.18) can be estimated in the following manner:

$$\begin{aligned} 0 \le & \frac{e^{\frac{\langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m\rangle}{\langle \mathbf{k}_m\rangle}}}{\left|\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle^{\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle + \lambda - \frac{1}{2}}\right|^{\frac{1}{\langle \mathbf{k}_m\rangle}}} \le e^{\Psi_1 + \frac{\pi}{2}\Upsilon} \frac{e^{\frac{\pi}{2}\frac{\Im(\lambda)}{\langle \mathbf{k}_m\rangle}}}{|\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle|^{\Psi_2} \, |\langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle|^{\frac{1}{\langle \mathbf{k}_m\rangle}\left(\Re(\lambda) - \frac{1}{2}\right)}} \\ & \to 0 \quad (\text{as} \ \ \langle \mathbf{k}_m\rangle \to \infty). \end{aligned}$$

Letting $\langle \mathbf{k}_m\rangle \to \infty$ and using Lemma 2.2, we get

$$\overline{\lim_{\langle \mathbf{k}_m\rangle \to \infty}} \frac{1}{|\Gamma(\lambda + \langle \boldsymbol{\rho}_m, \mathbf{k}_m\rangle)|^{\frac{1}{\langle \mathbf{k}_m\rangle}}} = 0,$$

which, in view of the multidimensional analogue of the Cauchy-Hadamard formula, implies that the radius of convergence of $E_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ is equal to ∞.

Thus, by using Lemma 2.1, we conclude that $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ is an entire function and this proves the first assertion of Theorem 2.2. The second assertion of Theorem 2.2 follows immediately from Theorem 2.1. □

Remark 2.1. If we put

$$\sigma(\mathbf{k}_m) = \frac{(\gamma_1)_{k_1}}{k_1!} \cdots \frac{(\gamma_m)_{k_m}}{k_m!}, \tag{2.22}$$

then (2.16) becomes

$$\mathbf{h}(\mathbf{z}_m) = \prod_{j=1}^{m} \sum_{k_j=0}^{\infty} (\gamma_j)_{k_j} \frac{z^{k_j}}{k_j!}.$$

Obviously, $R_h = 1$, which verifies that the function $E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ defined by (1.5) is an entire function in several complex variables.

3. Basic Properties of the function $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$

In this section, we focus on a special case of (1.12), which could be obtained in the following manner. Let

$$z_j = \omega_j z^{\rho_j}, \quad \left(0 < z < \min_{1 \le j \le m} \left(|\omega_j|^{-1} R\right)^{\frac{1}{\Re(\rho_j)}}; \ \omega_j \in \mathbb{C}, \ \Re(\rho_j) > 0\right),$$

in (1.12), where ω_j and ρ_j $(j = 1, \cdots, m)$ are parameters. The function $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\cdot)$ which was originally defined on $\mathbb{C}^m$ now reduces to the function which is defined on an open interval, and we have

$$\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right)=\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma\left(\mathbf{k}_m\right)\omega_1^{k_1}\cdots\omega_m^{k_m}}{\Gamma\left(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle\right)}z^{\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle}. \tag{3.1}$$

The case when $m=1$ reducing to the single series

$$\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}\left(\omega z^{\rho}\right)=\sum_{k=0}^{\infty}\frac{\sigma\left(k\right)}{\Gamma\left(\lambda+\rho k\right)}\omega^k z^{\rho k}, \tag{3.2}$$

is worthy of note. By setting $\omega=\Lambda^{-\Lambda}$ $(\Lambda=\sum_{i=1}^{r}\lambda_i;\lambda_i>0)$, $\rho=\Lambda$ and

$$\sigma\left(k\right)=\frac{\left(-1\right)^k\Gamma\left(\lambda+\rho k\right)}{\prod_{i=1}^{r}\Gamma\left(1+\mu_i+\lambda_i k\right)},$$

in (3.2), and multiplying the resulting equation by $\left(\frac{z}{\Lambda}\right)^M$ $(M=\sum_{i=1}^{r}\mu_i;\mu_i\in\mathbb{C})$, we get

$$\left(\frac{z}{\Lambda}\right)^M\mathcal{F}^{\boldsymbol{\sigma}}_{\rho,\lambda}\left(\omega z^{\rho}\right)=\left(\frac{z}{\Lambda}\right)^M\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\prod_{i=1}^{r}\Gamma\left(1+\mu_i+\lambda_i k\right)}\left(\frac{z}{\Lambda}\right)^{\Lambda k}=HE^{\lambda_1,\cdots,\lambda_r}_{\mu_1,\cdots,\mu_r}\left(z\right),$$

which is the multiparameter function of Mittag-Leffler type defined by Kalla *et al.* in [8, p. 901, Eq. (1)].

Theorem 3.1. *Let* $\lambda,\rho_j,\omega_j\in\mathbb{C}$, $\Re\left(\rho_j\right)>0$ $(j=1,\cdots,m)$ *and* $n\in\mathbb{N}$. *Then*

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left[z^{\lambda-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right)\right]=z^{\lambda-n-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda-n}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right), \tag{3.3}$$

and

$$\int_0^z\cdots\int_0^z t^{\lambda-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1 t^{\rho_1},\cdots,\omega_m t^{\rho_m}\right)\left(\mathrm{d}t\right)^n$$
$$=z^{\lambda+n-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda+n}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right). \tag{3.4}$$

Proof. The results (3.3) and (3.4) can easily be obtained by standard methods (see, for example [19, p. 538]), and therefore, the details can well be omitted here. □

Theorem 3.2. *Let* $\min\{\Re\left(\lambda\right),\Re\left(\beta\right)\}>0$ *and* $\Re\left(\rho_j\right)>0$ $(j=1,\cdots,m)$. *Then*

$$\mathcal{B}\left\{\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right):\lambda,\beta\right\}=\Gamma\left(\beta\right)\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda+\beta}\left(\omega_1,\cdots,\omega_m\right), \tag{3.5}$$

where $\mathcal{B}\{\ \cdot\ ;\lambda,\beta\}$ *denotes the Euler-Beta transform defined by (see [21])*

$$\mathcal{B}\{f\left(z\right);\alpha,\beta\}=\int_0^1 z^{\alpha-1}\left(1-z\right)^{\beta-1}f\left(z\right)\mathrm{d}z. \tag{3.6}$$

Proof. Using (3.6), we find from (3.1) that

$$\mathcal{B}\left\{\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}\right):\lambda,\beta\right\}$$

$$=\int_0^1 z^{\lambda-1}(1-z)^{\beta-1}\left(\sum_{\mathbf{k}_m=0}^{\infty}\sigma(\mathbf{k}_m)\frac{(\omega_1 z^{\rho_1})^{k_1}\cdots(\omega_m z^{\rho_m})^{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\right)\mathrm{d}z. \tag{3.7}$$

Upon interchanging the order of integration and summation in (3.7) which is permissible in view of the constraints stated with (3.5), we get

$$\begin{aligned}&\mathcal{B}\left\{\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}):\lambda,\beta\right\}\\ =&\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma(\mathbf{k}_m)\,\omega_1^{k_1}\cdots\omega_m^{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\left(\int_0^1 z^{\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle-1}(1-z)^{\beta-1}\mathrm{d}z\right)\\ =&\Gamma(\beta)\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma(\mathbf{k}_m)}{\Gamma(\lambda+\beta+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\omega_1^{k_1}\cdots\omega_m^{k_m},\end{aligned}$$

which in view of (1.12) gives the desired result (3.5). □

Theorem 3.3. *Let* $\min\{\Re(\lambda),\Re(s)\}>0$ *and* $\Re(\rho_j)>0$ $(j=1,\cdots,m)$. *Then*

$$\mathcal{L}\left\{z^{\lambda-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}):s\right\}=\frac{1}{s^{\lambda}}\mathbf{h}\left(\frac{\omega_1}{s^{\rho_1}},\cdots,\frac{\omega_m}{s^{\rho_m}}\right), \tag{3.8}$$

where $\mathbf{h}(\mathbf{z}_m)$ *is given by* (2.16) *and* $\mathcal{L}\{\ \cdot\ ;s\}$ *denotes the Laplace transform defined by*

$$\mathcal{L}\{f(z);s\}=\int_0^{\infty}e^{-sz}f(z)\,\mathrm{d}z. \tag{3.9}$$

Proof. We have

$$\begin{aligned}&\mathcal{L}\left\{z^{\lambda-1}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}):s\right\}\\ =&\int_0^{\infty}z^{\lambda-1}e^{-sz}\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m})\\ =&\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma(\mathbf{k}_m)\,\omega_1^{k_1}\cdots\omega_m^{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\int_0^{\infty}z^{\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle-1}e^{-sz}\mathrm{d}z\\ =&\frac{1}{s^{\lambda}}\sum_{\mathbf{k}_m=0}^{\infty}\sigma(\mathbf{k}_m)\frac{\omega_1^{k_1}}{s^{\rho_1 k_1}}\cdots\frac{\omega_1^{k_m}}{s^{\rho_m k_m}}\\ =&\frac{1}{s^{\lambda}}\mathbf{h}\left(\frac{\omega_1}{s^{\rho_1}},\cdots,\frac{\omega_m}{s^{\rho_m}}\right).\end{aligned}$$

□

Remark 3.1. If we specialize $\sigma(\mathbf{k}_m)$ as in (2.22), then (3.8) becomes

$$\mathcal{L}\left\{z^{\lambda-1}E^{\gamma_m}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 z^{\rho_1},\cdots,\omega_m z^{\rho_m}):s\right\}=\frac{1}{s^{\lambda}}\prod_{j=1}^{m}\left(1-\frac{\omega_j}{s^{\rho_j}}\right)^{-\gamma_j},$$

which was established in [24].

We now consider the Riemann-Liouville fractional integrals and derivatives I_{a+} and D_{a+} of the function $\mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\mathbf{z}_m)$ defined by (1.12).

The Riemann-Liouville fractional integral I_{a+} is given by (1.15), and the Riemann-Liouville fractional derivative D_{a+} of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ is defined by (see [9, p. 70]; see also [18, Sections 2.3 and 2.4])

$$\left(D_{a+}^{\alpha}\varphi\right)(x) := \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \left(I_{a+}^{n-\alpha}\varphi\right)(x)\,, \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0;\ n = [\Re(\alpha)] + 1)\,. \tag{3.10}$$

Theorem 3.4. *Let $a \in (0,\infty)$, $\alpha, \lambda, \rho_j, \gamma_j, \omega_j \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\rho_j) > 0$, $\Re(\lambda) > 0$ $(j = 1, \dots, m)$. Then for $x > a$, there holds the relations:*

$$\begin{aligned}&\left(I_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(t-a)^{\rho_1},\cdots,\omega_m(t-a)^{\rho_m}\right)\right]\right)(x)\\ &= (x-a)^{\lambda+\alpha-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda+\alpha}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(x-a)^{\rho_1},\cdots,\omega_m(x-a)^{\rho_m}\right)\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\left(D_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(t-a)^{\rho_1},\cdots,\omega_m(t-a)^{\rho_m}\right)\right]\right)(x)\\ &= (x-a)^{\lambda-\alpha-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda-\alpha}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(x-a)^{\rho_1},\cdots,\omega_m(x-a)^{\rho_m}\right).\end{aligned} \tag{3.12}$$

Proof. Using (1.12) and (1.15) and applying the formula [9, p. 71, Property 2.1]:

$$\left(I_{a+}^{\alpha}\left[(t-a)^{\beta-1}\right]\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\,, (\alpha,\beta \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0)\,,$$

we get (for $x > a$)

$$\begin{aligned}&\left(I_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(t-a)^{\rho_1},\cdots,\omega_m(t-a)^{\rho_m}\right)\right]\right)(x)\\ &= \left(I_{a+}^{\alpha}\left[\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma(\mathbf{k}_m)\,\omega_1^{k_1}\cdots\omega_m^{k_m}}{\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}(t-a)^{\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle-1}\right]\right)(x)\\ &= (x-a)^{\lambda+\alpha-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda+\alpha}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(x-a)^{\rho_1},\cdots,\omega_m(x-a)^{\rho_m}\right).\end{aligned} \tag{3.13}$$

Next, to establish (3.12), we have upon using (1.12) and (3.10):

$$\begin{aligned}&\left(D_{a+}^{\alpha}\left[(t-a)^{\lambda-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(t-a)^{\rho_1},\cdots,\omega_m(t-a)^{\rho_m}\right)\right]\right)(x)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left(I_{a+}^{n-\alpha}\left[(t-a)^{\lambda-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(t-a)^{\rho_1},\cdots,\omega_m(t-a)^{\rho_m}\right)\right]\right)(x)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left[(x-a)^{\lambda+n-\alpha-1}\,\mathcal{F}_{\boldsymbol{\rho}_m,\lambda+n-\alpha}^{\widehat{\boldsymbol{\sigma}}}\left(\omega_1(x-a)^{\rho_1},\cdots,\omega_m(x-a)^{\rho_m}\right)\right].\end{aligned}$$

Applying now (3.3), we are easily led to the desired result (3.12). □

4. Results involving a class of operators $\mathcal{J}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}^{\widehat{\boldsymbol{\sigma}}}$

In this section, we investigate the boundedness and composition properties of the fractional integral operator defined by (1.14). New examples obtained by making use of the summable hypergeometric functions are also mentioned.

Theorem 4.1. *Let* $\lambda, \rho_j, \gamma_j, \omega_j \in \mathbb{C}$ $(\Re(\rho_j) > 0, \Re(\lambda) > 0)$ $(j = 1, \ldots, m)$ *and* $b > a$. *Suppose that* $\sigma(\mathbf{k}_m)$ *satisfies the condition* (2.12). *Then the integral operator* $\mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}$ *is bounded on the space* $L(a,b)$ *and*

$$\left\| \mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,\boldsymbol{\omega}_m;a+} \varphi \right\|_1 \leq \mathfrak{M} (b-a)^{\Re(\lambda)} \|\varphi\|_1, \tag{4.1}$$

where the constant $\mathfrak{M}$ $(0 < \mathfrak{M} < \infty)$ *is given by*

$$\mathfrak{M} = \sum_{\mathbf{k}_m=0}^{\infty} \frac{|\sigma(\mathbf{k}_m)| |\omega_1|^{k_1} \cdots |\omega_m|^{k_m}}{|\Gamma(\lambda + \langle \boldsymbol{\rho}_m, \mathbf{k}_m \rangle)|} \frac{(b-a)^{\langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m \rangle}}{[\Re(\lambda) + \langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m \rangle]}. \tag{4.2}$$

Proof. First, we note that the series given by (4.2) is convergent.

Now, by the definition of integral operator and $\|.\|$ given by (1.14) and (1.7), respectively, we have

$$\begin{aligned} &\left\| \mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)} \varphi \right\|_1 \\ &= \int_a^b \left| \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda} \left(\omega_1 (x-t)^{\rho_1}, \cdots, \omega_m (x-t)^{\rho_m}\right) \varphi(t) \, \mathrm{d}t \right| \mathrm{d}x, \end{aligned}$$

which upon changing the orders of integrations and using of the substitution $x-t = u$ yields

$$\begin{aligned} &\left\| \mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)} \varphi \right\|_1 \\ &\leq \int_a^b |\varphi(t)| \left[\int_t^b (x-t)^{\Re(\lambda)-1} \left| \mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 (x-t)^{\rho_1}, \cdots, \omega_m (x-t)^{\rho_m}) \right| \mathrm{d}x \right] \mathrm{d}t \\ &\leq \int_a^b |\varphi(t)| \left[\int_0^{b-t} u^{\Re(\lambda)-1} \left| \mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 u^{\rho_1}, \cdots, \omega_m u^{\rho_m}) \right| \mathrm{d}u \right] \mathrm{d}t \\ &\leq \int_a^b |\varphi(t)| \left[\int_0^{b-a} u^{\Re(\lambda)-1} \left| \mathcal{F}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda}(\omega_1 u^{\rho_1}, \cdots, \omega_m u^{\rho_m}) \right| \mathrm{d}u \right] \mathrm{d}t. \end{aligned} \tag{4.3}$$

Using the definition (1.12) and interchanging summation and integration, we find that

$$\begin{aligned} &\left\| \mathcal{J}^{\widehat{\boldsymbol{\sigma}}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)} \varphi \right\|_1 \\ &\leq \sum_{\mathbf{k}_m=0}^{\infty} \frac{|\sigma(\mathbf{k}_m)| |\omega_1|^{k_1} \cdots |\omega_m|^{k_m}}{|\Gamma(\lambda + \langle \boldsymbol{\rho}_m, \mathbf{k}_m \rangle)|} \int_0^{b-a} u^{\Re(\lambda) + \langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m \rangle - 1} \mathrm{d}u \int_a^b |\varphi(t)| \mathrm{d}t. \end{aligned}$$

Expressing now the $t-$integral as $\|\varphi\|_1$ and evaluating the $u-$integral, we are easily lead to the result (4.1). □

Remark 4.1. Applying Theorem 4.1 to the sequence (2.22), we get the following result (see [19, Theorem 4.1])

$$\left\| E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)} \varphi \right\|_1 \leq (b-a)^{\Re(\lambda)} \mathfrak{L} \|\varphi\|_1,$$

where

$$\mathfrak{L} := \sum_{\mathbf{k}_m=0}^{\infty} \frac{|(\gamma_1)_{k_1}|}{k_1!} \cdots \frac{|(\gamma_m)_{k_m}|}{k_m!} \frac{|\omega_1|^{k_1} \cdots |\omega_m|^{k_m}}{|\Gamma(\lambda + \langle \boldsymbol{\rho}_m, \mathbf{k}_m \rangle)|} \frac{(b-a)^{\langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m \rangle}}{[\Re(\lambda) + \langle \Re(\boldsymbol{\rho}_m), \mathbf{k}_m \rangle]}.$$

Theorem 4.2. *If $\Re(\lambda_1) > 0$, $\Re(\lambda_2) > 0$, then*

$$\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x) = \left(\mathcal{J}^{\boldsymbol{\Omega}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x), \tag{4.4}$$

where the sequences $\sigma_1(\mathbf{l}_m)$ $(\mathbf{l}_m \in \mathbb{N}_0^m)$ and $\sigma_2(\mathbf{k}_m)$ $(\mathbf{k}_m \in \mathbb{N}_0^m)$ satisfy the condition (2.12) *and the sequence $\boldsymbol{\Omega} = \Omega(\mathbf{l}_m)$ $(\mathbf{l}_m \in \mathbb{N}_0^m)$ is given by*

$$\boldsymbol{\Omega} = \sum_{\mathbf{k}_m=0}^{\mathbf{l}_m} \sigma_1(\mathbf{l}_m - \mathbf{k}_m)\,\sigma_2(\mathbf{k}_m). \tag{4.5}$$

In addition, the operator $\mathcal{J}^{\boldsymbol{\Omega}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}$ is also bounded on $L(a,b)$.

Proof. Since $\varphi \in L(a,b)$, we know from Theroem 4.1 that $\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)$ $\in L(a,b)$, therefore, upon using Theorem 4.1 again, we have

$$\begin{aligned}\left\|\left(\mathcal{J}^{\boldsymbol{\Omega}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)\right\|_1 &= \left\|\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)\right\|_1\\ &\leq \mathfrak{M}_1(b-a)^{\Re(\lambda_1)}\left\|\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right\|_1\\ &\leq \mathfrak{M}_1\mathfrak{M}_2(b-a)^{\Re(\lambda_1)+\Re(\lambda_2)}\|\varphi\|_1,\end{aligned} \tag{4.6}$$

where

$$\mathfrak{M}_1 = \sum_{\mathbf{l}_m=0}^{\infty}\frac{|\sigma_1(\mathbf{l}_m)||\omega_1|^{k_1}\cdots|\omega_m|^{k_m}}{|\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle)|}\frac{(b-a)^{\langle\Re(\boldsymbol{\rho}_m),\mathbf{l}_m\rangle}}{[\Re(\lambda_1)+\langle\Re(\boldsymbol{\rho}_m),\mathbf{l}_m\rangle]},$$

and

$$\mathfrak{M}_2 = \sum_{\mathbf{k}_m=0}^{\infty}\frac{|\sigma_2(\mathbf{k}_m)||\omega_1|^{k_1}\cdots|\omega_m|^{k_m}}{|\Gamma(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)|}\frac{(b-a)^{\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle}}{[\Re(\lambda_2)+\langle\Re(\boldsymbol{\rho}_m),\mathbf{k}_m\rangle]}.$$

This proves the boundedness of the operator $\mathcal{J}^{\boldsymbol{\Omega}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}$ on $L(a,b)$.

We now prove the composition property (4.4). Using (1.14), we have

$$\begin{aligned}&\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)\\ &= \int_a^x (x-u)^{\lambda_1-1}\,\mathcal{F}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1}\left(\omega_1(x-u)^{\rho_1},\cdots,\omega_m(x-u)^{\rho_m}\right)\\ &\quad\cdot\left[\int_a^u (u-t)^{\lambda_2-1}\,\mathcal{F}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2}\left(\omega_1(u-t)^{\rho_1},\cdots,\omega_m(u-t)^{\rho_m}\right)\varphi(t)\,\mathrm{d}t\right]\mathrm{d}u\\ &= \int_a^x \varphi(t)\,\mathrm{d}t\int_t^x (x-u)^{\lambda_1-1}(u-t)^{\lambda_2-1}\\ &\quad\cdot\mathcal{F}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1}\left(\omega_1(x-u)^{\rho_1},\cdots,\omega_m(x-u)^{\rho_m}\right)\\ &\quad\cdot\mathcal{F}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2}\left(\omega_1(u-t)^{\rho_1},\cdots,\omega_m(u-t)^{\rho_m}\right)\mathrm{d}u.\end{aligned} \tag{4.7}$$

By setting now $v = \frac{x-u}{x-t}$, the inner integral becomes

$$\begin{aligned}&(x-t)^{\lambda_1+\lambda_2-1}\int_0^1 v^{\lambda_1-1}(1-v)^{\lambda_2-1}\\ &\cdot\mathcal{F}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1}\left(\omega_1(x-t)^{\rho_1}v^{\rho_1},\cdots,\omega_m(x-t)^{\rho_m}v^{\rho_m}\right)\end{aligned}$$

$$
\begin{aligned}
&\cdot \mathcal{F}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2}\left(\omega_1 (x-t)^{\rho_1}(1-v)^{\rho_1},\cdots,\omega_m (x-t)^{\rho_m}(1-v)^{\rho_m}\right)\mathrm{d}v\\
&=(x-t)^{\lambda_1+\lambda_2-1}\sum_{\mathbf{l}_m=0}^{\infty}\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma_1(\mathbf{l}_m)\,\sigma_2(\mathbf{k}_m)\,\omega_1^{l_1+k_1}\cdots\omega_m^{l_m+k_m}}{\Gamma(\lambda_1+\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle)\,\Gamma(\lambda_2+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle)}\\
&\quad\cdot(x-t)^{\langle\boldsymbol{\rho}_m,\mathbf{l}_m+\mathbf{k}_m\rangle}\int_0^1 v^{\lambda_1+\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle-1}(1-v)^{\lambda_2+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle-1}\,\mathrm{d}v\\
&=(x-t)^{\lambda_1+\lambda_2-1}\sum_{\mathbf{l}_m=0}^{\infty}\sum_{\mathbf{k}_m=0}^{\infty}\frac{\sigma_1(\mathbf{l}_m)\,\sigma_2(\mathbf{k}_m)\,\omega_1^{l_1+k_1}\cdots\omega_m^{l_m+k_m}}{\Gamma(\lambda_1+\lambda_2+\langle\boldsymbol{\rho}_m,\mathbf{l}_m+\mathbf{k}_m\rangle)}(x-t)^{\langle\boldsymbol{\rho}_m,\mathbf{l}_m+\mathbf{k}_m\rangle}\\
&=(x-t)^{\lambda_1+\lambda_2-1}\sum_{\mathbf{l}_m=0}^{\infty}\left[\sum_{\mathbf{k}_m=0}^{\mathbf{l}_m}\sigma_1(\mathbf{l}_m-\mathbf{k}_m)\,\sigma_2(\mathbf{k}_m)\right]\frac{\omega_1^{l_1}\cdots\omega_m^{l_m}(x-t)^{\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle}}{\Gamma(\lambda_1+\lambda_2+\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle)}\\
&=(x-t)^{\lambda_1+\lambda_2-1}\sum_{\mathbf{l}_m=0}^{\infty}\frac{\Omega(\mathbf{l}_m)\,\omega_1^{l_1}\cdots\omega_m^{l_m}}{\Gamma(\lambda_1+\lambda_2+\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle)}(x-t)^{\langle\boldsymbol{\rho}_m,\mathbf{l}_m\rangle}\\
&=(x-t)^{\lambda_1+\lambda_2-1}\,\mathcal{F}^{\boldsymbol{\Omega}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2}\left(\omega_1(x-t)^{\rho_1},\cdots,\omega_m(x-t)^{\rho_m}\right).
\end{aligned}\tag{4.8}
$$

The result (4.4) follows immediately on using (4.7) and (4.8). □

Remark 4.2. If we choose

$$
\sigma_1(\mathbf{l}_m-\mathbf{k}_m)=\frac{(\gamma_1)_{l_1-k_1}}{(l_1-k_1)!}\cdots\frac{(\gamma_m)_{l_m-k_m}}{(l_m-k_m)!},\quad\text{and}\quad \sigma_2(\mathbf{k}_m)=\frac{(\mu_1)_{k_1}}{k_1!}\cdots\frac{(\mu_m)_{k_m}}{k_m!},
$$

in Theorem 4.2, then, by using ([23, p. 17])

$$
(a)_{l-k}=\frac{(-1)^k(a)_l}{(1-a-l)_k},\quad\text{and}\quad\frac{1}{(l-k)!}=(-1)^k\frac{(-l)_k}{l!},
$$

we have

$$
\begin{aligned}
\Omega(\mathbf{l}_m)&=\sum_{\mathbf{k}_m=0}^{\mathbf{l}_m}\frac{(\gamma_1)_{l_1-k_1}}{(l_1-k_1)!}\cdots\frac{(\gamma_m)_{l_m-k_m}}{(l_m-k_m)!}\frac{(\mu_1)_{k_1}}{k_1!}\cdots\frac{(\mu_m)_{k_m}}{k_m!}\\
&=\frac{(\gamma_1)_{l_1}}{l_1!}\cdots\frac{(\gamma_m)_{l_m}}{l_m!}\prod_{j=1}^{m}\sum_{k_j=0}^{l_j}\frac{(-l_j)_{k_j}(\mu_j)_{k_j}}{(1-\gamma_j-l_j)_{k_j}}\frac{1}{k_j!}\\
&=\frac{(\gamma_1)_{l_1}}{l_1!}\cdots\frac{(\gamma_m)_{l_m}}{l_m!}\prod_{j=1}^{m}{}_2F_1\left[\begin{matrix}-l_j,\ \ \mu_j\\ 1-\gamma_j-l_j\end{matrix};1\right],
\end{aligned}\tag{4.9}
$$

where ${}_2F_1$ denotes the well-known Gauss hypergeometric function (see [23, p. 18, Eq. (17)]). By appealing to the Chu-Vandermonde identity [23, p. 19, Eq. (21)], we obtain

$$
\Omega(\mathbf{l}_m)=\frac{(\gamma_1+\mu_1)_{l_1}}{l_1!}\cdots\frac{(\gamma_m+\mu_m)_{l_m}}{l_m!}.\tag{4.10}
$$

Thus, the relation given by (4.4) can be expressed as

$$
E^{\boldsymbol{\gamma}_m}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(E^{\boldsymbol{\mu}_m}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)=\left(E^{\boldsymbol{\gamma}_m+\boldsymbol{\mu}_m}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x),\tag{4.11}
$$

which is equivalent to the result given in [19, Theorem 6.1].

Theorem 4.3. *Let* $\Re(\lambda_1) > 0$, $\Re(\lambda_2) > 0$, *and let the sequences* $\widehat{\boldsymbol{\sigma}}_1 = \sigma_1(\mathbf{k}_m)$ $(\mathbf{l}_m \in \mathbb{N}_0^m)$ *and* $\widehat{\boldsymbol{\sigma}}_2 = \sigma_2(\mathbf{k}_m)$ $(\mathbf{l}_m \in \mathbb{N}_0^m)$ *satisfy the condition* (2.12). *In addition, if the sequence* $\boldsymbol{\Omega} = \Omega(\mathbf{l}_m)$ $(\mathbf{l}_m \in \mathbb{N}_0^m)$ *is given by*

$$\boldsymbol{\Omega} := \sum_{\mathbf{k}_m=0}^{\mathbf{l}_m} \sigma_1(\mathbf{l}_m - \mathbf{k}_m)\,\sigma_1(\mathbf{k}_m) = \begin{cases} 1, & l_1 = \cdots = l_m = 0, \\ 0, & l_1 \geq 1, \cdots, l_m \geq 1, \end{cases} \tag{4.12}$$

then we get

$$\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_1}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(\mathcal{J}^{\widehat{\boldsymbol{\sigma}}_2}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x) = \left(I^{\lambda_1+\lambda_2}_{a+}\varphi\right)(x), \tag{4.13}$$

where I^{λ}_{a+} *denotes the Riemann-Liouville fractional integral operator defined by* (1.15).

Proof. The theorem follows immediately from the composition property (4.4) and the reduction formula (1.15). □

Next, we specialize Theorem 4.2 by putting

$$\sigma_1(\mathbf{l}_m - \mathbf{k}_m) = \frac{(\gamma_1)_{l_1-k_1}}{(l_1-k_1)!}\cdots\frac{(\gamma_m)_{l_m-k_m}}{(l_m-k_m)!},$$

and

$$\begin{aligned}\sigma_2(\mathbf{k}_m) =& \frac{(\mu_1)_{k_1}}{k_1!}\frac{((f^{(1)}_{r_1}+N^{(1)}_{r_1}))_{k_1}}{((f^{(1)}_{r_1}))_{k_1}}\cdots\frac{(\mu_m)_{k_m}}{k_m!}\frac{((f^{(m)}_{r_m}+N^{(m)}_{r_m}))_{k_m}}{((f^{(m)}_{r_m}))_{k_m}} =: \boldsymbol{\Phi}, \\ &\left(N^{(j)}_1,\cdots,N^{(j)}_{r_j}\in\mathbb{N},\quad \mu_j \neq f^{(j)}_i\ (1\leq i\leq r_j),\quad j=1,\cdots,m\right).\end{aligned} \tag{4.14}$$

Here, we define the product of p Pochhammer symbols by $((a_p))_k \equiv (a_1)_k\cdots(a_p)_k$, which is suggested by Miller (see [12]).

From Remark 4.2, we have

$$\begin{aligned}\Omega(\mathbf{l}_m) =& \sum_{\mathbf{k}_m=0}^{\mathbf{l}_m}\frac{(\gamma_1)_{l_1-k_1}}{(l_1-k_1)!}\cdots\frac{(\gamma_m)_{l_m-k_m}}{(l_m-k_m)!} \\ &\cdot\frac{(\mu_1)_{k_1}}{k_1!}\frac{((f^{(1)}_{r_1}+N^{(1)}_{r_1}))_{k_1}}{((f^{(1)}_{r_1}))_{k_1}}\cdots\frac{(\mu_m)_{k_m}}{k_m!}\frac{((f^{(m)}_{r_m}+N^{(m)}_{r_m}))_{k_m}}{((f^{(m)}_{r_m}))_{k_m}} \\ =& \frac{(\gamma_1)_{l_1}}{l_1!}\cdots\frac{(\gamma_m)_{l_m}}{l_m!}\prod_{j=1}^{m}\sum_{k_j=0}^{l_j}\frac{(-l_j)_{k_j}(\mu_j)_{k_j}}{(1-\gamma_j-l_j)_{k_j}}\frac{((f^{(j)}_{r_j}+N^{(j)}_{r_j}))_{k_j}}{((f^{(j)}_{r_j}))_{k_j}}\frac{1}{k_j!} \\ =& \frac{(\gamma_1)_{l_1}}{l_1!}\cdots\frac{(\gamma_m)_{l_m}}{l_m!}\prod_{j=1}^{m}{}_{r_j+2}F_{r_j+1}\left[\begin{matrix}-l_j, & \mu_j, & (f^{(j)}_{r_j}+N^{(j)}_{r_j}) \\ 1-\gamma_j-l_j, & & (f^{(j)}_{r_j})\end{matrix};1\right],\end{aligned} \tag{4.15}$$

where ${}_pF_q$ denotes the generalized hypergeometric function defined by (see, for instance [23, p. 19, Eq. (23)]).

In general, to sum a generalized hypergeometric function at unit argument is a very difficult task. However, major progress was achieved on this topic by Miller [12] who gave the following interesting summation formula.

Theorem 4.4. ([12, p. 968]) *For nonnegative integer n and positive integers (m_r):*

$$ {}_{r+2}F_{r+1}\left[\begin{matrix} -n, a, \ (f_r+m_r) \\ b, \qquad (f_r) \end{matrix}; 1\right] = \frac{(\lambda)_n}{(b)_n}\frac{(1+\xi_1)_n}{(\xi_1)_n}\cdots\frac{(1+\xi_m)_n}{(\xi_m)_n}, $$

where $m = m_1+\cdots+m_r$, $\lambda = b-a-m$, $(\lambda)_m \neq 0$, $a \neq f_i$ $(1 \le i \le r)$.

The $\xi_1, \cdots, \xi_m$ are nonvanishing zeros of the associated parametric polynomial of degree m given by

$$ Q_m(t) = \sum_{\mathbf{j}_r=0}^{\mathbf{m}_r} s^{(1)}_{m_1-j_1}\cdots s^{(r)}_{m_r-j_r}\sum_{l=0}^{j}\begin{Bmatrix} j \\ l \end{Bmatrix}(a)_l\,(t)_l\,(\lambda-t)_{m-l}\,, $$

where $j = j_1+\cdots+j_r$ and $s^{(i)}_{m_i-j_i}$ $(1 \le i \le r)$ are determined by the generating functions

$$ (f_1+x)_{m_1} = \sum_{j_1=0}^{m_1} s^{(1)}_{m_1-j_1}x^{j_1}, \cdots, \ (f_r+x)_{m_r} = \sum_{j_r=0}^{m_r} s^{(r)}_{m_r-j_r}x^{j_r}. $$

By applying Theorem 4.4, we have

$$ {}_{r_j+2}F_{r_j+1}\left[\begin{matrix} -l_j, \ \ \mu_j, \ \ (f^{(j)}_{r_j}+N^{(j)}_{r_j}) \\ 1-\gamma_j-l_j, \qquad (f^{(j)}_{r_j}) \end{matrix}; 1\right] $$
$$ = \frac{\left(\gamma_j+\mu_j+N^{(j)}\right)_{l_j}}{(\gamma_j)_{l_j}}\frac{(1+\xi^{(j)}_1)_{l_j}}{(\xi^{(j)}_1)_{l_j}}\cdots\frac{(1+\xi^{(j)}_{N^{(j)}})_{l_j}}{(\xi^{(j)}_{N^{(j)}})_{l_j}} \quad \left(N^{(j)} = N^{(j)}_1+\cdots+N^{(j)}_{r_j}\right), $$

where $\xi^{(j)}_1, \cdots, \xi^{(j)}_{N^{(j)}}$ are nonvanishing zeros of the associated parametric polynomial of degree $N^{(j)}$ given by

$$ Q^{(j)}_m(t) = \sum_{i_1=0}^{N^{(j)}_1}\cdots\sum_{i_{r_j}=0}^{N^{(j)}_{r_j}} s^{(1)}_{N^{(j)}_1-i_1}\cdots s^{(r_j)}_{N^{(j)}_{r_j}-i_{r_j}} $$
$$ \cdot\sum_{l=0}^{i_1+\cdots+i_{r_j}}\begin{Bmatrix} i_1+\cdots+i_{r_j} \\ l \end{Bmatrix}(\mu_j)_l\,(t)_l\,(1-\gamma_j-\mu_j-N^{(j)}-l_j-t)_{N^{(j)}-l}, $$

and $s^{(i)}_{m_i-j_i}$ $(1 \le i \le r)$ are determined by the generating functions

$$ \left(f^{(j)}_1+x\right)_{N^{(j)}_1} = \sum_{i_1=0}^{N^{(j)}_1} s^{(1)}_{N^{(j)}_1-i_1}x^{i_1}, \quad \cdots, \quad \left(f^{(j)}_{r_j}+x\right)_{N^{(j)}_{r_j}} = \sum_{i_{r_j}=0}^{N^{(j)}_{r_j}} s^{(r_j)}_{N^{(j)}_{r_j}-i_{r_j}}x^{i_{r_j}}. $$

The sequence $\Omega(\mathbf{l}_m)$ becomes

$$ \Omega(\mathbf{l}_m) = \prod_{j=1}^{m}\frac{(\gamma_j+\mu_j+N^{(j)})_{l_j}}{l_j!}\frac{(1+\xi^{(j)}_1)_{l_j}}{(\xi^{(j)}_1)_{l_j}}\cdots\frac{(1+\xi^{(j)}_{N^{(j)}})_{l_j}}{(\xi^{(j)}_{N^{(j)}})_{l_j}} =: \mathbf{\Theta}. \tag{4.16} $$

Upon using (4.14) and (4.16), the above evaluation suggests the following new functions of Mittag-Leffler type:

$$\mathcal{E}^{\mathbf{\Phi}}_{\boldsymbol{\rho}_m,\lambda}\left(\mathbf{z}_m\right)=\sum_{\mathbf{k}_m=0}^{\infty}\prod_{j=1}^{m}\frac{(\mu_j)_{k_j}}{k_j!}\frac{((f^{(j)}_{r_j}+N^{(j)}_{r_j}))_{k_j}}{((f^{(j)}_{r_j}))_{k_j}}\frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma\left(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle\right)}, \tag{4.17}$$

and

$$\mathcal{E}^{\mathbf{\Theta}}_{\boldsymbol{\rho}_m,\lambda}\left(\mathbf{z}_m\right) =\sum_{\mathbf{l}_m=0}^{\infty}\prod_{j=1}^{m}\frac{(\gamma_j+\mu_j+N^{(j)})_{l_j}}{l_j!}\frac{(1+\xi^{(j)}_1)_{l_j}}{(\xi^{(j)}_1)_{l_j}}\cdots\frac{(1+\xi^{(j)}_{N^{(j)}})_{l_j}}{(\xi^{(j)}_{N^{(j)}})_{l_j}}\frac{z_1^{k_1}\cdots z_m^{k_m}}{\Gamma\left(\lambda+\langle\boldsymbol{\rho}_m,\mathbf{k}_m\rangle\right)}. \tag{4.18}$$

The corresponding fractional integral operators of (4.17) and (4.18) are, respectively, given by

$$\left(\varepsilon\mathcal{J}^{\mathbf{\Phi}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x) =\int_a^x (x-t)^{\lambda-1}\,\mathcal{E}^{\mathbf{\Phi}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1(x-t)^{\rho_1},\cdots,\omega_m(x-t)^{\rho_m}\right)\varphi(t)\,\mathrm{d}t$$

and

$$\left(\varepsilon\mathcal{J}^{\mathbf{\Theta}}_{\boldsymbol{\rho}_m,\lambda,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x) =\int_a^x (x-t)^{\lambda-1}\,\mathcal{E}^{\mathbf{\Theta}}_{\boldsymbol{\rho}_m,\lambda}\left(\omega_1(x-t)^{\rho_1},\cdots,\omega_m(x-t)^{\rho_m}\right)\varphi(t)\,\mathrm{d}t.$$

With these new notations, we have the following corollary.

Corollary 4.1. *If $\Re(\lambda_1)>0$, $\Re(\lambda_2)>0$, then*

$$E^{\gamma_m}_{\boldsymbol{\rho}_m,\lambda_1,a+;(\boldsymbol{\omega}_m)}\left(\varepsilon\mathcal{J}^{\mathbf{\Phi}}_{\boldsymbol{\rho}_m,\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x)=\left(\varepsilon\mathcal{J}^{\mathbf{\Theta}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}\varphi\right)(x),$$

where the sequence $\mathbf{\Phi}=\sigma_2(\mathbf{k}_m)$ $(\mathbf{k}_m\in\mathbb{N}_0^m)$ is given by (4.14) and the sequence $\mathbf{\Theta}=\Omega(\mathbf{l}_m)$ $(\mathbf{k}_m\in\mathbb{N}_0^m)$ is given by (4.16).

In addition, the operator $\varepsilon\mathcal{J}^{\mathbf{\Theta}}_{\boldsymbol{\rho}_m,\lambda_1+\lambda_2,a+;(\boldsymbol{\omega}_m)}$ is also bounded on $L(a,b)$.

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