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THE CENTER-FOCUS PROBLEM AND BIFURCATION OF LIMIT CYCLES IN A CLASS OF 7TH-DEGREE POLYNOMIAL SYSTEMS*

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Abstract By computing singular point values, the center conditions are established for a class of 7th-degree planar polynomial systems with 15 parameters. It is proved that such systems can have 13 small-amplitude limit cycles in the neighborhood of the origin. To the best of our knowledge, this is the first example of a 7th-degree system having non-homogeneous nonlinearities with thirteen limit cycles bifurcated from a fine focus.

Keywords Limit cycle, center variety, singular point value, time-reversibility.

MSC(2010) 34C05, 34C07.

1. Introduction

Consider a planar differential system

$$\frac{du}{dt} = \lambda\, u - v + P_n(u, v), \quad \frac{dv}{dt} = u + \lambda\, v + Q_n(u, v), \tag{1.1}$$

where $P_n(u, v), Q_n(u, v) \in \mathbb{R}[u, v]$ are polynomials of degree n without constants and linear terms. If $\lambda = 0$, it is well-known that the singularity at the origin is a fine focus (surrounded by spirals) or a center (surrounded by closed trajectories). The center-focus problem for system (1.1) is to determine conditions on the coefficients of P_n and Q_n, under which an open neighborhood of the origin is covered by closed trajectories.

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For system (1.1), there exists a unique formal power series

$$H(u,v) = u^2 + v^2 + \sum_{k=3}^{\infty}\left(\sum_{j=0}^{k} B_{k,j}u^{k-j}v^j\right) = u^2 + v^2 + H_3(u,v) + H_4(u,v) + \cdots, \tag{1.2}$$

where $B_{k,k} = 0$ with k even and $H_k(u,v)$ are homogeneous polynomials of degree k, so that

$$\left.\frac{dH}{dt}\right|_{(1.1)} = \sum_{n=0}^{\infty} V_n(u^2+v^2)^{(n+1)}, \tag{1.3}$$

where V_n is called the n-th Poincaré-Liapunov constant of system (1.1) at the origin. The origin is called a center if $V_0 = V_1 = V_2 = \cdots = 0$. The origin is said to be a fine focus of order k if V_k is the first non-zero Poincaré-Liapunov constant. In this case at most k small-amplitude limit cycles can bifurcate from the fine focus.

The classification of centers for system (1.1) starting from quadratic terms can be found in the survey article [23]. The analysis for cubic systems without quadratic terms is given in [17]. Some sufficient center conditions for quartic homogeneous systems are obtained in [3].

Consider the following autonomous system

$$\begin{cases} \dfrac{dx}{dt} = x + \displaystyle\sum_{\alpha+\beta=2}^{\infty} a_{\alpha,\beta}x^\alpha y^\beta = X(x,y), \\ \dfrac{dy}{dt} = -y - \displaystyle\sum_{\alpha+\beta=2}^{\infty} b_{\alpha,\beta}y^\alpha x^\beta = Y(x,y), \end{cases} \tag{1.4}$$

where $X(x,y), Y(x,y) \in C^\infty$, $\alpha \geq 0, \beta \geq 0, a_{\alpha,\beta} = \overline{b_{\alpha,\beta}}$, and $x, y, t \in \mathbb{C}$. By means of transformation

$$x = u + \mathtt{i}v, \quad y = u - \mathtt{i}v, \quad t = \mathtt{i}t_1, \quad \mathtt{i} = \sqrt{-1}, \tag{1.5}$$

system (1.4) can be transformed into the following real system

$$\begin{cases} \dfrac{du}{dt_1} = -v + h.o.t. = U(u,v), \\ \dfrac{dv}{dt_1} = u + h.o.t. = V(u,v), \end{cases} \tag{1.6}$$

which is system (1.4)'s concomitant real system.

For system (1.4), we can derive a formal power series of the form

$$F(x,y) = xy + \sum_{k=3}^{\infty}\sum_{j=0}^{k} B_{k,j}x^{k-j}y^j, \tag{1.7}$$

with $B_{2s,s} = 0,\ s = 2, 3, \cdots$, such that

$$\left.\frac{dF}{dt}\right|_{(1.4)} = \frac{\partial F}{\partial x}X + \frac{\partial F}{\partial y}Y = \sum_{n=1}^{\infty} W_n(xy)^{n+1}, \tag{1.8}$$

where W_n is called the n-th singular point value (also known as 1:-1 resonant focus number) of system (1.4) at the origin. The origin of system (1.4) is called a complex center if and only if $W_1 = W_2 = \cdots = 0$. The ideal $\mathcal{B} := \langle W_1, W_2, \cdots \rangle$ is called the Bautin ideal and its variety $V(\mathcal{B})$ is called the center variety.

Lemma 1.1 (see [16])**.** *Let W_n be the n-th singular point value of system* (1.4) *at the origin, and V_n be the n-th Poincaré-Liapunov constant of system* (1.6) *at the origin. Then we have*

$$W_n \equiv \frac{V_n}{\boldsymbol{i}} \ mod \ \langle V_1, V_2, \cdots, V_{n-1} \rangle.$$

From Lemma 1.1, we have the following result.

Lemma 1.2. *The origin of system* (1.4) *is a complex center if and only if the origin of system* (1.6) *is a center.*

Żołądek [34] generalized the notion of center to the case of a $p : -q$ resonant singular point of the following complex polynomial vector field

$$\begin{cases} \dfrac{dx}{dt} = p\,x + X_m(x, y), \\ \dfrac{dy}{dt} = -q\,y + Y_m(x, y), \end{cases} \tag{1.9}$$

in $\mathbb{C}^2$, where $p, q \in \mathbb{N}, p \leq q, (p, q) = 1$, and

$$X_m(x, y) = \sum_{k=2}^{m} \sum_{j=0}^{k} a_{k,j} x^{k-j} y^j, \qquad Y_m(x, y) = \sum_{k=2}^{m} \sum_{j=0}^{k} b_{k,j} x^{k-j} y^j.$$

Definition 1.1 (see [20])**.** System (1.9) is said to have a $p : -q$ resonant center at the origin if it admits a formal first integral of the form

$$F(x, y) = x^q\, y^p + \sum_{k=p+q+1}^{\infty} \sum_{j=0}^{k} B_{k,j} x^{k-j} y^j. \tag{1.10}$$

For system (1.9), we can derive a formal power series of the form (1.10) with $B_{s(p+q),sp} = 0,\ s = 2, 3, \cdots$, such that

$$\frac{dF}{dt}\Big|_{(1.9)} = \frac{\partial F}{\partial x}(px + X_m) + \frac{\partial F}{\partial y}(-qy + Y_m) = \sum_{n=1}^{\infty} W_n (x^q y^p)^{n+1}, \tag{1.11}$$

where W_n is called the nth order $p : -q$ resonant focus number (or generalized singular point value). For some algorithms to compute these numbers, see [16, 18, 21]. According to Theorem 3.1 of Wang [25], the algorithm of Sang [21] based on pseudo-divisions can be generalized to the situation of three-dimensional polynomial differential systems with two purely imaginary eigenvalues.

A stable limit cycle is the one which attracts all neighboring trajectories. Stable limit cycles model systems that exhibit self-sustained oscillations. Finding limit cycles is of great importance in the theory of nonlinear oscillations and the qualitative theory of dynamical systems, see [9, 14, 27, 29].

The second part of Hilbert's 16th problem is to find an upper bound, called Hilbert number $H(n)$, on the number of limit cycles of all polynomial vector fields with degree n. This problem has not been completely solved even for quadratic systems. The maximum number of bifurcating limit cycles from singular points or from periodic orbits is known for a reduced classes of systems (e.g. see [7,10–13,24,26,28,30–33]).

For a 7th-degree planar system with homogeneous nonlinearities, Giné [7] found a system with 13 small limit cycles by using center bifurcation. In this paper we will use multiple Hopf bifurcation rather than center bifurcation to find the same number of small limit cycles in a 7th-degree system with non-homogeneous nonlinearities. In general for a family of polynomial differential systems, finding the same number of limit cycles by multiple Hopf bifurcation is much more complicated than by center bifurcations, especially for system with non-homogeneous nonlinearities. Consider a class of Z_4-equivariant 7th-degree systems in $\mathbb{C}^2$, i.e.,

$$\begin{cases} \dfrac{dx}{dt} =(1-\mathtt{i}\lambda)x+(a_1+\mathtt{i}b_1)x^5+(a_2+\mathtt{i}b_2)y^2x^3+(a_3+\mathtt{i}b_3)y^4x \\ \qquad +(a_4+\mathtt{i}b_4)yx^6+(a_5+\mathtt{i}b_5)y^3x^4+(a_6+\mathtt{i}b_6)y^5x^2+(a_7+\mathtt{i}b_7)y^7, \\ \dfrac{dy}{dt} =-(1+\mathtt{i}\lambda)y-(a_1-\mathtt{i}b_1)y^5-(a_2-\mathtt{i}b_2)x^2y^3-(a_3-\mathtt{i}b_3)x^4y \\ \qquad -(a_4-\mathtt{i}b_4)xy^6-(a_5-\mathtt{i}b_5)x^3y^4-(a_6-\mathtt{i}b_6)x^5y^2-(a_7-\mathtt{i}b_7)x^7, \end{cases} \tag{1.12}$$

which has 15 independent real parameters. For the definition of Z_n-equivariant complex system, see [6,15]. By means of transformation (1.5), system (1.12) can be transformed into the real system

$$\begin{cases} \dfrac{du}{dt_1} =\lambda u-v+(-b_1-b_2-b_3)u^5+(-5\,a_1-a_2+3\,a_3)vu^4 \\ \qquad +(10\,b_1-2\,b_2+2\,b_3)v^2u^3+(10\,a_1-2\,a_2+2\,a_3)v^3u^2 \\ \qquad +(-5\,b_1-b_2+3\,b_3)v^4u+(-a_1-a_2-a_3)v^5+(-b_5-b_6-b_7-b_4)u^7 \\ \qquad +(-5\,a_4-a_5+3\,a_6+7\,a_7)vu^6+(9\,b_4-3\,b_5+b_6+21\,b_7)v^2u^5 \\ \qquad +(5\,a_4-3\,a_5+5\,a_6-35\,a_7)v^3u^4+(5\,b_4-3\,b_5+5\,b_6-35\,b_7)v^4u^3 \\ \qquad +(9\,a_4-3\,a_5+a_6+21\,a_7)v^5u^2+(-5\,b_4-b_5+3\,b_6+7\,b_7)v^6u \\ \qquad +(-a_4-a_5-a_6-a_7)v^7, \\ \dfrac{dv}{dt_1} =u+\lambda v+(-b_1-b_2-b_3)v^5+(5\,a_1+a_2-3\,a_3)uv^4 \\ \qquad +(10\,b_1-2\,b_2+2\,b_3)u^2v^3+(-10\,a_1+2\,a_2-2\,a_3)u^3v^2 \\ \qquad +(-5\,b_1-b_2+3\,b_3)u^4v+(a_1+a_2+a_3)u^5+(-b_5-b_6-b_7-b_4)v^7 \\ \qquad +(5\,a_4+a_5-3\,a_6-7\,a_7)uv^6+(9\,b_4-3\,b_5+b_6+21\,b_7)u^2v^5 \\ \qquad +(-5\,a_4+3\,a_5-5\,a_6+35\,a_7)u^3v^4+(5\,b_4-3\,b_5+5\,b_6-35\,b_7)u^4v^3 \\ \qquad +(-9\,a_4+3\,a_5-a_6-21\,a_7)u^5v^2+(-5\,b_4-b_5+3\,b_6+7\,b_7)u^6v \\ \qquad +(a_5+a_6+a_7+a_4)u^7. \end{cases} \tag{1.13}$$

Let

$$I_1=\langle b_3+5\,b_1, a_3-5\,a_1, b_4+\frac{1}{3}\,b_6, a_4-\frac{1}{3}\,a_6, b_2, b_5\rangle,$$

$$I_2 = \langle a_1b_3 + a_3b_1, \frac{1}{5}\,b_1b_4 - \frac{3}{5}\,b_1b_6 + b_3b_4 + \frac{1}{5}\,b_3b_6, -\frac{1}{5}\,a_1b_4 + \frac{3}{5}\,a_1b_6 + a_3b_4 + \frac{1}{5}\,a_3b_6,$$
$$a_4b_6 + a_6b_4, \frac{1}{5}\,a_4b_1 + a_4b_3 + \frac{3}{5}\,a_6b_1 - \frac{1}{5}\,a_6b_3, -\frac{1}{5}\,a_1a_4 - \frac{3}{5}\,a_1a_6 + a_3a_4 - \frac{1}{5}\,a_3a_6,$$
$$b_7, a_7, b_2, b_5\rangle,$$
$$I_3 = \langle\frac{1}{2}\,{a_1}^2b_7 + a_1a_7b_1 - \frac{1}{2}\,{b_1}^2b_7, a_1b_6 + a_6b_1, a_1a_6b_7 - 2\,a_1a_7b_6 + b_1b_6b_7,$$
$${a_6}^2b_7 - 2\,a_6a_7b_6 - {b_6}^2b_7, a_1b_3 + a_3b_1, a_1a_3b_7 - 2\,a_1a_7b_3 + b_1b_3b_7, a_3b_6 - a_6b_3,$$
$$a_3a_6b_7 - 2\,a_3a_7b_6 - b_3b_6b_7, {a_3}^2b_7 - 2\,a_3a_7b_3 - {b_3}^2b_7, -a_1b_4 + a_4b_1,$$
$$a_1a_4b_7 + 2\,a_1a_7b_4 - b_1b_4b_7, a_4b_6 + a_6b_4, a_4a_6b_7 - 2\,a_4a_7b_6 + b_4b_6b_7, a_3b_4 + a_4b_3,$$
$$a_3a_4b_7 - 2\,a_4a_7b_3 + b_3b_4b_7, {a_4}^2b_7 + 2\,a_4a_7b_4 - {b_4}^2b_7, b_2, b_5\rangle.$$

Theorem 1.1. *For system* $(1.12)_{\lambda=0}$*, the center variety* $V(\mathcal{B})$ *has three irreducible components:*

$$V(\mathcal{B}) = V(\mathcal{B}_{15}) = V(I_1) \cup V(I_2) \cup V(I_3).$$

Lemma 1.3. *Suppose that*

$$\begin{cases} a_1 = -7\,a_3, \quad b_1 = 7\,b_3, \\ a_2 = -4\left(\dfrac{2}{57}\right)^{\frac{1}{2}}(a_3^2 + b_3^2)^{\frac{1}{2}}, \\ a_4 = 10\left(\dfrac{2}{57}\right)^{\frac{1}{4}} b_3(a_3^2 + b_3^2)^{\frac{1}{4}}, \\ a_6 = -\dfrac{57^{\frac{3}{4}}2^{\frac{5}{4}}b_3(a_3^2 + b_3^2)^{\frac{1}{4}}}{19}, \\ a_7 = \dfrac{16}{3}a_3b_3\left[\dfrac{2}{57}(a_3^2 + b_3^2)\right]^{-\frac{1}{4}}, \\ b_4 = 10\left(\dfrac{2}{57}\right)^{\frac{1}{4}} a_3(a_3^2 + b_3^2)^{\frac{1}{4}}, \\ b_6 = \dfrac{57^{\frac{3}{4}}2^{\frac{5}{4}}a_3(a_3^2 + b_3^2)^{\frac{1}{4}}}{19}, \\ b_7 = -\dfrac{8}{3}\left(\dfrac{57}{2}\right)^{\frac{1}{4}}(a_3^2 - b_3^2)(a_3^2 + b_3^2)^{-\frac{1}{4}}, \quad \lambda = a_5 = b_2 = b_5 = 0, \end{cases} \tag{1.14}$$

then the origin of system (1.13) *is a fifteenth order weak focus.*

Remark 1.1. From the proof of Theorem 1.1 in the next section, it follows that the essential focal basis (see [1]) of Poincaré-Liapunov constants for (1.13) is $V_0, V_2, V_3, \cdots, V_{15}$, hence the highest order of weak focus is fifteen.

Theorem 1.2. *If condition* (1.14) *holds, then there are perturbations of system* (1.13) *yielding 13 small-amplitude limit cycles bifurcating from the origin.*

2. Proofs of the results

Using the algorithm of [21], we compute the first fifteen singular point values $W_1 = \mathtt{i}U_1, W_2 = \mathtt{i}U_2, \cdots, W_{15} = \mathtt{i}U_{15}$ of system $(1.12)_{\lambda=0}$, where the quantity U_k is reduced w.r.t. the Gröbner basis of $\{U_j : j < k\}$, and

$$U_1 = 0,\quad U_2 = 2\,b_2,\quad U_3 = 2\,b_5,\quad U_4 = -2\,a_1b_3 - 2\,a_3b_1,$$
$$U_5 = \frac{1}{2}\,(b_4 - 3\,b_6)a_1 + \frac{1}{2}\,(-5\,b_4 - b_6)a_3 + \frac{1}{2}\,(-a_4 - 3\,a_6)b_1 + \frac{1}{2}\,(-5\,a_4 + a_6)b_3,$$
$$U_6 = -2\,a_4b_6 - 2\,a_6b_4,$$
$$\begin{aligned}U_7 = {}& \frac{5\,{a_1}^2b_7}{16} + \frac{17\,a_1a_3b_7}{8} + \frac{5}{8}\,a_1a_7b_1 - \frac{7\,{a_3}^2b_7}{16} + \frac{17\,a_3a_7b_1}{4} + \frac{7\,a_3a_7b_3}{8}\\ & - \frac{5\,{b_1}^2b_7}{16} + \frac{17\,b_1b_3b_7}{8} + \frac{7\,{b_3}^2b_7}{16},\end{aligned}$$
$$\begin{aligned}U_8 = {}& \frac{5}{8}\,a_1a_4b_7 + \frac{15\,a_1a_6b_7}{8} + 2\,a_1a_7b_4 - 6\,a_1a_7b_6 + \frac{23\,a_3a_4b_7}{8} - \frac{11\,a_3a_6b_7}{8} - 4\,a_3a_7b_4\\ & - \frac{3}{4}\,a_4a_7b_1 - \frac{39\,a_4a_7b_3}{4} - \frac{9}{4}\,a_6a_7b_1 + \frac{11}{4}\,a_6a_7b_3 - \frac{5}{8}\,b_1b_4b_7 + \frac{15\,b_1b_6b_7}{8}\\ & + \frac{23\,b_3b_4b_7}{8} + \frac{11\,b_3b_6b_7}{8}.\end{aligned}$$

The other polynomials for $U_j, 9 \le j \le 15$ are too long to be presented in this paper. However, the interested reader can compute them with the help of computer algebra system `Maple`. According to Lemma 1.1, the first fifteen Poincaré-Liapunov constants of system $(1.13)_{\lambda=0}$ are $V_j = -U_j,\quad 1 \le j \le 15$.

Proof of Theorem 1.1. By the Hilbert Basis Theorem $V(\mathcal{B}) = V(\mathcal{B}_k)$ for some $k \in \mathbb{N}$. Using the Radical Membership Test one can verify that

$$W_2 \notin \sqrt{\langle W_1\rangle}, \cdots, W_{15} \notin \sqrt{\langle W_1, W_2, \cdots, W_{14}\rangle}, W_{16}, W_{17} \in \sqrt{\langle W_1, W_2, \cdots, W_{15}\rangle},$$

which leads us to expect that $V(\mathcal{B}_{15}) = V(\mathcal{B})$. To verify that this is the case we compute the minimal decomposition of the variety of the ideal $\mathcal{B}_{15}$ with `Singular` routine minAssGTZ (see [8]) and find that

$$V(\mathcal{B}_{15}) = V(I_1) \cup V(I_2) \cup V(I_3),$$

thus we have

$$V(\mathcal{B}) \subseteq V(\mathcal{B}_{15}) = V(I_1) \cup V(I_2) \cup V(I_3).$$

Now to prove the inclusion in the other direction, we verify that each point of $V(I_k), k = 1, 2, 3$ corresponds to a system with a complex center at the origin.

System $(1.12)_{\lambda=0}$ that corresponds to component $V(I_1)$ can be written as

$$\begin{cases}\dfrac{dx}{dt} = x + (a_1 + \mathtt{i}b_1)x^5 + a_2y^2x^3 + (5\,a_1 - 5\,\mathtt{i}b_1)y^4x + (a_4 + \mathtt{i}b_4)yx^6 + a_5y^3x^4\\ \qquad + (3\,a_4 - 3\,\mathtt{i}b_4)y^5x^2 + (a_7 + \mathtt{i}b_7)y^7,\\ \dfrac{dy}{dt} = -\,y - (a_1 - \mathtt{i}b_1)y^5 - a_2x^2y^3 - (5\,a_1 + 5\,\mathtt{i}b_1)x^4y - (a_4 - \mathtt{i}b_4)xy^6 - a_5x^3y^4\\ \qquad - (3\,a_4 + 3\,\mathtt{i}b_4)x^5y^2 - (a_7 - \mathtt{i}b_7)x^7.\end{cases} \tag{2.1}$$

System (2.1) is Hamiltonian with Hamiltonian function

$$\begin{aligned}\Phi(x, y) = {}& xy + (a_1 + \mathtt{i}b_1)yx^5 + 1/3\,a_2x^3y^3 + (a_1 - \mathtt{i}b_1)y^5x + (-\mathtt{i}/8b_7 + 1/8\,a_7)x^8\\ & + (\mathtt{i}/2b_4 + 1/2\,a_4)y^2x^6 + 1/4\,a_5x^4y^4 + (-\mathtt{i}/2b_4 + 1/2\,a_4)y^6x^2\\ & + (\mathtt{i}/8b_7 + 1/8\,a_7)y^8,\end{aligned}$$

thus the origin of it is a complex center.

The zero set of the ideal I_2 consists of four solutions:

(i) $a_3 = -3a_1,\ b_3 = 3b_1,\ a_4 = a_7 = b_2 = b_4 = b_5 = b_7 = 0$. In this case, system $(1.12)_{\lambda=0}$ admits an integrating factor of the form $\mu(x,y) = (xy)^{-2}$,

(ii) $a_6 = 5a_4,\ b_6 = -5b_4,\ a_1 = a_7 = b_1 = b_2 = b_5 = b_7 = 0$. In this case, system $(1.12)_{\lambda=0}$ admits an integrating factor of the form $\mu(x,y) = (xy)^{-1}$,

(iii) $b_3 = \frac{(3a_6+a_4)b_1}{a_6-5a_4}, b_4 = -\frac{a_4b_6}{a_6}, a_1 = a_3 = a_7 = b_2 = b_5 = b_7 = 0$. In this case, system $(1.12)_{\lambda=0}$ admits an integrating factor of the form $\mu(x,y) = (xy)^{\frac{2(a_6-3a_4)}{a_4-a_6}}$,

(iv) $b_6 = \frac{(5b_3+b_1)b_4}{3b_1-b_3},\ a_1 = a_3 = a_4 = a_6 = a_7 = b_2 = b_5 = b_7 = 0$. In this case, system $(1.12)_{\lambda=0}$ admits an integrating factor of the form $\mu(x,y) = (xy)^{-\frac{b_3+5b_1}{b_1+b_3}}$.

According to Theorem 4.13 of [5], for any system that corresponds to component $V(I_2)$, there exists a Lyapunov first integral on a neighborhood of the origin, which is thus a center.

For the component $V(I_3)$, using the algorithm from [19], we find that the Zariski closure of all time-reversible systems in the family $(1.12)_{\lambda=0}$, denoted by $\bar{R}$, is the variety of the ideal J_3, where

$$\begin{aligned}
J_3 = \langle & 2\,\mathtt{i}b_2, 2\,\mathtt{i}b_5, \mathtt{i}(2\,a_3b_6 - 2\,a_6b_3), \mathtt{i}(2\,b_7({a_3}^2 - {b_3}^2) - 4\,a_3a_7b_3),\\
& \mathtt{i}(2\,a_3a_6b_7 - 2\,a_3a_7b_6 - 2\,a_6a_7b_3 - 2\,b_3b_6b_7), \mathtt{i}(2\,b_7({a_6}^2 - {b_6}^2) - 4\,a_6a_7b_6),\\
& \mathtt{i}(-2\,a_1b_3 - 2\,a_3b_1), \mathtt{i}(-2\,a_1b_6 - 2\,a_6b_1),\\
& \mathtt{i}(-2\,a_1a_3b_7 + 2\,a_1a_7b_3 - 2\,a_3a_7b_1 - 2\,b_1b_3b_7),\\
& \mathtt{i}(-2\,a_1a_6b_7 + 2\,a_1a_7b_6 - 2\,a_6a_7b_1 - 2\,b_1b_6b_7),\\
& \mathtt{i}(2\,b_7({a_1}^2 - {b_1}^2) + 4\,a_7a_1b_1), \mathtt{i}(2\,a_1b_4 - 2\,a_4b_1),\\
& \mathtt{i}(-2\,a_3b_4 - 2\,a_4b_3), \mathtt{i}(-2\,a_4b_6 - 2\,a_6b_4),\\
& \mathtt{i}(-2\,a_3a_4b_7 - 2\,a_3a_7b_4 + 2\,a_4a_7b_3 - 2\,b_3b_4b_7),\\
& \mathtt{i}(-2\,a_4a_6b_7 + 2\,a_4a_7b_6 - 2\,a_6a_7b_4 - 2\,b_4b_6b_7),\\
& \mathtt{i}(2\,a_1a_4b_7 + 2\,a_1a_7b_4 + 2\,a_4a_7b_1 - 2\,b_1b_4b_7), \mathtt{i}(2\,b_7({a_4}^2 - {b_4}^2) + 4\,a_4a_7b_4)\rangle.
\end{aligned}$$

Because J_3 and I_3 have the same reduced Gröbner basis, then $J_3 = I_3$. Thus every system from $V(I_3)$ admits a first integral of the form (1.7), and therefore has a center. □

Proof of Lemma 1.3. Let condition (1.14) be satisfied, then the first fifteenth order Poincaré-Liapunov constants of system (1.13) are as follows:

$$V_0 = V_1 = \cdots = V_{14} = 0,$$
$$V_{15} = 213920\Big(\frac{2}{57}\Big)^{\frac{3}{4}}(a_3^2 + b_3^2)^{\frac{15}{4}} \neq 0,$$

and thus the origin of the system is a fifteenth order fine focus. □

Proof of Theorem 1.2. Under condition (1.14), the Jacobian matrix of Poincaré-Liapunov constants $V_0, V_2, \cdots, V_{13}$ of system (1.13) with respect to $\lambda, a_5, a_2,$

$b_5, b_2, a_4, b_4, a_6, b_6, a_7, b_7, a_1, b_1$ has full rank, i.e.,

$$\text{rank}\left[\frac{\partial(V_0, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10}, V_{11}, V_{12}, V_{13})}{\partial(\lambda,\ a_5,\ a_2,\ b_5,\ b_2,\ a_4,\ b_4,\ a_6,\ b_6,\ a_7,\ b_7,\ a_1,\ b_1)}\right]_{(1.14)} = 13.$$

Furthermore, we have

$$\text{rank}\left[\frac{\partial(V_0, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10}, V_{11}, V_{12}, V_{13}, V_{14})}{\partial(\lambda,\ a_5,\ a_2,\ b_5,\ b_2,\ a_4,\ b_4,\ a_6,\ b_6,\ a_7,\ b_7,\ a_1,\ b_1)}\right]_{(1.14)} = 13,$$

which implies that the linear parts of $V_0, V_2, \cdots, V_{13}$ at the critical values (1.14) are independent in the parameters and hence by Theorem 1 of [4] only 13 limit cycles can be bifurcated from the origin. □

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