Journal of Applied Analysis and Computation
Volume 6, Number 3, August 2016, 665–683
Website:http://jaac-online.com/
DOI:10.11948/2016044

A FIXED POINT INDEX THEORY FOR NOWHERE NORMAL-OUTWARD COMPACT MAPS AND APPLICATIONS*

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Abstract A fixed point index theory is developed for a class of nowhere normal-outward compact maps defined on a cone which do not necessarily take values in the cone. This class depends on the retractions on the cone and contains self-maps for any retractions, and weakly inward maps and generalized inward maps when the retraction is a continuous metric projection. The new index coincides with the previous fixed point index theories for compact self-maps and generalized inward compact maps. New fixed point theorems are obtained for nowhere normal-outward compact maps and applied to treat some abstract boundary value problems and Sturm-Liouville boundary value problems with nonlinearities changing signs.

Keywords Fixed point index, fixed point theorem, nowhere normal-outward map, boundary value problem, positive solution.

MSC(2010) 47H11, 34B18, 47H10, 47H30.

1. Introduction

The classical fixed point index theories for compact or condensing maps defined on cones [1, 17] require the maps to be self-maps that take values in the cones. Such indices were generalized to weakly inward or generalized inward maps [19], where the maps are allowed to take values outside the cones but satisfy the weakly inward or generalized inward conditions. The definitions of all the indices employ retractions from a Banach space to its cone, but these indices are proved to be independent of the choice of the retractions involved. However, for some maps such as the Hammerstein integral operators $A : P \to C[0, 1]$ defined by

$$Az(x) := \int_0^1 k(x, s)f(s, z(s))\, ds \quad \text{for } x \in [0, 1] \tag{1.1}$$

arising from boundary value problems and some biological models (see Theorem 5.2), it is not easy to verify whether they are weakly inward or generalized inward

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*The first author was supported in part by the National Natural Science Foundation of China under grant No. 11171046, and the second author was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

maps since the nonlinearities f involved satisfy $\lim_{u\to\infty} f(u) = -\infty$, where P is the standard positive cone in $C[0,1]$.

In this paper, we establish a fixed point index theory for a class of r-nowhere normal-outward maps in a Banach space X, where r is a retraction from X to a closed convex set K of X. We first work on the class of r-nowhere normal-outward maps. The concept of a nowhere normal-outward map was first introduced by Halpern and Bergman in [4] when K is a compact convex set in a strictly convex normed linear space. Roughly speaking, a map $A : \Omega \subset K \to X$ is a r-nowhere normal-outward map if Ax is not in the retractive set $r^{-1}(x) \setminus \{x\}$ of x for $x \in \Omega$. We shall provide and prove the criteria for maps to be r-nowhere normal-outward maps. Using the criteria, we show that if A is a self-map, that is, $A(\Omega) \subset K$, then A is a r-nowhere normal-outward map for any retractions r, and if r is a continuous metric projection, a weakly inward or generalized inward map is a r-nowhere normal-outward map. Moreover, we prove that if the continuous metric projection r is unique, then a r-nowhere normal-outward map is a generalized inward map.

Next, we define the fixed point index denoted by $i_{r,K}(A, D_K)$ for r-nowhere normal-outward compact maps $A : \overline{D}_K \to X$ as the fixed point index $i_K(rA, D_K)$ for the self-map rA. The latter was employed by Lan and Webb [13] to define a fixed point index for generalized inward maps of condensing type, where r is a metric projection. The new index coincides with those given in [1] when A is a self-map and in [13] when A is a weakly inward or generalized inward map. The new index $i_{r,K}(A, D_K)$ depends on the retraction r while the indices defined in [1,13] are independent of the choice of the retractions r involved. However, in applications, we always use the same retraction r to compute the indices and the dependence on the retractions r will not affect the computations of the indices. We show that the index has the standard properties such as existence property, normalization, additivity and homotopy properties. Using the index, we prove some fixed point theorems for r-nowhere normal-outward compact maps, some of which involve the first eigenvalues of linear operators.

Using our fixed point theorems, we study the existence of positive solutions in $C[0,1]$ of the following abstract nonlinear equations of the form

$$\mathcal{N}z(x) = Fz(x) \quad \text{for a.e. } x \in [0,1] \tag{1.2}$$

subject to the so-called generalized separated boundary conditions (GSBCs). We shall provide conditions on $\mathcal{N}$ and F, under which the composite map TF is a r-nowhere normal-outward map (see Theorem 4.1), where T is the right inverse of $\mathcal{N}$. One of these conditions requires $\mathcal{N}$ to be a semi-negative operator. We show that the well-known nonlinear operator

$$\mathcal{N}z(x) := -\big(\omega(x)|z'(x)|^{p-2}z'(x)\big)'$$

is a semi-negative operator.

As illustration, we study the existence of positive solutions of Sturm-Liouville boundary value problems

$$-\,(\omega(x)z'(x))' = f(x, z(x)) \quad \text{a.e. on } [0,1] \tag{1.3}$$

subject to the separated BCs, where f is defined on $[0,1]\times\mathbb{R}_+$ and satisfies Carathéodory conditions. The key assumption on f is the positivity condition at 0: $f(x,0) \geq 0$, so f is allowed to take negative values and may have no lower bounds. Hence, f may not satisfy the semi-positone condition: $f(x,u) \geq -\eta$ for some $\eta > 0$, employed in [2,9–11]. We prove that under the positivity condition at 0, the operator A defined in (1.1) with the Green's function k for $-(\omega(x)z'(x))' = 0$ subject to the separated BCs, is a r-nowhere normal-outward map on P although it is not clear whether the operator A is a weakly inward or generalized inward map. Hence, the new index can be applied to study the existence of nonnegative solutions of such Sturm-Liouville boundary value problems (1.3), where the previous index theories and results on semi-positone problems [2,9,15] can not be applied.

In section 2 of this paper, we study r-nowhere normal-outward maps and define the fixed point index for such maps. In section 3, we prove some fixed point theorems. In section 4, we provide a special class of r-nowhere normal-outward maps and study existence of solutions of (1.2) subject to the GSBCs. In section 5, we study the existence of nonzero nonnegative solutions of Sturm-Liouville boundary value problems with nonlinearities satisfying the positivity condition at 0.

2. A fixed point index for nowhere normal-outward compact maps

Let X be a Banach space and K a closed convex set in X. Let $r : X \to K$ be a retraction, that is, r is continuous and satisfies $r(x) = x$ for $x \in K$. The Dugundji extension theorem [3] (also see [5, section 18]) shows that for every closed convex set K in X there exists a retraction from X onto K.

Definition 2.1. A map $A : \Omega \subset K \to X$ is called a r-nowhere normal-outward map on Ω relative to K if

$$Ax \in \big(X \setminus r^{-1}(x)\big) \cup \{x\} \quad \text{for } x \in \Omega. \tag{2.1}$$

The concept of a nowhere normal-outward map was first introduced in [4] when K is a compact convex set in a strictly convex normed linear space.

The following new results provide criteria for a map to be a r-nowhere normal-outward map.

Proposition 2.1. *Assume that $A : \Omega \subset K \to X$ is a map. Then the following assertions are equivalent.*

(H_1) *A is a r-nowhere normal-outward map on Ω relative to K.*

(H_2) *If $x = r(Ax)$ for some $x \in \Omega$, then $x = Ax$.*

(H_3) *If $y = A(r(y))$ for some $y \in r^{-1}(\Omega)$, then $y \in \Omega$.*

Proof. Assume that (H_1) holds and $x = r(Ax)$ for some $x \in \Omega$. By (2.1), $x = Ax$ and (H_2) holds. Assume that (H_2) holds and $y = A(r(y))$ for some $y \in r^{-1}(\Omega)$. Let $x = r(y)$. Then $x \in \Omega$ and $y = Ax$. This implies that $x = r(y) = r(Ax)$. By (H_2), we have $y = Ax = x \in \Omega$ and (H_3) holds. Assume that (H_3) holds and $Ax \notin \big(X \setminus r^{-1}(x)\big) \cup \{x\}$ for some $x \in \Omega$. Then $x \neq Ax$ and $x = r(Ax)$. Let $y = Ax$. Then $y \in r^{-1}(x) \subset r^{-1}(\Omega)$ and $y = Ax = A(r(y))$. By (H_3), $y \in \Omega$. Hence, we have $x = r(Ax) = r(y) = y = Ax$, a contradiction. Hence, (H_1) holds. □

Recall that a map $A : \Omega \subset K \to X$ is called a weakly inward map on Ω relative to K if $Ax \in \overline{I}_K(x)$ for $x \in \Omega$, where $\overline{I}_K(x)$ is the closure of the inward set

$$I_K(x) = \{x + c(z - x) : z \in K \quad \text{for } c \geq 0\}$$

([13]); a generalized inward map on Ω relative to K if $Ax \in G_K(x)$, where

$$G_K(x) = \{y \in X \setminus K : d(y, K) < \|y - x\|\} \cup K$$

is the generalized inward set of x relative to K. It is known that a weakly inward map is a generalized inward map, but the converse is false (see [13]).

By Proposition 2.1 (H_2), we see that if $A : \Omega \subset K \to K$, then A is a r-nowhere normal-outward on Ω relative to K. It is not clear whether a weakly inward or generalized inward map is a r-nowhere normal-outward map if there are no restrictions on the retraction r. However, in the following we prove that it is true if the retraction r is a continuous metric projection, and the converse is true if the metric projection is unique.

Recall that a map $r : X \to K$ is called a metric projection if

$$\|x - r(x)\| = d(x, K) \quad \text{for each } x \in X,$$

where $d(x, K) = \inf\{\|x - y\| : y \in K\}$.

Proposition 2.2. *Assume that $r : X \to K$ is a continuous metric projection. If $A : \Omega \subset K \to X$ is a generalized inward map on Ω relative to K, then A is a r-nowhere normal-outward on Ω relative to K. The converse is true if the continuous metric projection r is unique.*

Proof. We prove that

$$G_K(x) \subset \big(X \setminus r^{-1}(x)\big) \cup \{x\} := N_{r,K}(x) \quad \text{for } x \in K.$$

If the inclusion were false, then there exist $x \in K$, $y \in G_K(x)$ and $y \notin N_{r,K}(x)$. Since $y \notin N_{r,K}(x)$, $x = r(y)$ and $y \neq x$. This implies $y \notin K$. Since $y \in G_K(x)$, $0 < d(y, K) < \|y - x\|$. Since r is a metric projection, we have

$$\|y - x\| = \|y - r(y)\| = d(y, K) < \|y - x\|,$$

a contradiction. For the converse, it suffices to show that

$$N_{r,K}(x) \subset G_K(x) \quad \text{for } x \in K.$$

In fact, if not, there exist $x \in K$ and $y \in N_{r,K}(x)$ such that $y \notin G_K(x)$. The latter implies that $y \notin K$ and $0 < d(y, K) = \|y - x\|$. Since r is unique, $x = r(y)$ and $y \neq x$ and we have $y \notin N_{r,K}(x)$, a contradiction. □

The following example shows that a r-nowhere normal-outward map may not be generalized inward even when r is a metric projection and the sum of two r-nowhere normal-outward maps may not be a r-nowhere normal-outward map.

Example 2.1. Let X be the Banach space $\mathbb{R}^2$ with the maximum norm $\|(x, y)\| = \max\{|x|, |y|\}$, $K = \mathbb{R}^2_+$ and let $r : X \to K$ be the metric projection defined by

$$r(x, y) = (\max\{x, 0\}, \max\{y, 0\}).$$

Define maps $A, B : K \to X$ by

$$A(x, y) = (x - y, 0) \quad \text{and } B(x, y) = (0, y - x).$$

Then the following assertions hold.

(1) A and B are r-nowhere normal-outward maps on K relative to K.
(2) A is not a generalized inward map on K relative to K.
(3) $A + B$ is not a r-nowhere normal-outward map on K relative to K.

Proof. (1) Suppose $(x,y) = Ar(x,y)$ for some $(x,y) \in \mathbb{R}^2 (= r^{-1}(K))$. Then

$$(x,y) = Ar(x,y) = \big(\max\{x,0\} - \max\{y,0\}, 0\}\big)$$

and $x = \max\{x,0\} - \max\{y,0\}$ and $y = 0$. Hence, $x = \max\{x,0\} \geq 0$ and $y = 0$. It follows that $(x,y) = (x,0) \in K$. By Proposition 2.1 (H_3), A is a r-nowhere normal-outward map. A similar proof shows that B is a r-nowhere normal-outward map.

(2) Since $A(0,y) = (-y,0) \notin K$ for $y > 0$ and

$$d(A(0,y),K) = |y| = \|(-y,y)\| = \|A(0,y) - (0,y)\|,$$

A is not a generalized inward map.

(3) Since $(1,-1) = (A+B)r(1,-1)$ and $(1,-1) \notin K$, by Proposition 2.1 (H_3), $A+B$ is not a r-nowhere normal-outward map. □

Example 2.1 shows that in general, when A and B are r-nowhere normal-outward maps, $\alpha A + \beta B$ may not be a r-nowhere normal-outward map for $\alpha, \beta \geq 0$. However, in section 4, we shall show that it is true for some special maps arising in applications, see Corollary 4.1.

The following result shows that if A is a r-nowhere normal-outward map, then the sets of fixed points for the three maps A, rA and Ar are same. Its proof follows directly from Proposition 2.1 and is omitted.

Proposition 2.3. *Assume that $A : \Omega \subset K \to X$ is r-nowhere normal-outward on Ω relative to K. Then*

$$\{x \in \Omega : x = Ax\} = \{x \in \Omega : x = r(Ax)\} = \{y \in r^{-1}(\Omega) : y = A(ry)\}.$$

Recall that a map $A : D \subset X \to X$ is said to be compact if A is continuous and $A(\Omega)$ is relatively compact for each bounded subset Ω of D.

Let D be an open set in X. We denote by $\overline{D}_K$ and ∂D_K the closure and the boundary, respectively, of $D_K = D \cap K$ relative to K. We refer to [7] for properties among these sets.

We define the fixed point index for r-nowhere normal-outward compact maps.

Definition 2.2. Let K be a closed convex set in X and let $r : X \to K$ be a retraction. Let D be a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is compact and is a r-nowhere normal-outward map on ∂D_K relative to K such that $x \neq Ax$ for $x \in \partial D_K$. Then we define a fixed point index of A over D_K relative to r and K as follows:

$$i_{r,K}(A, D_K) = i_K(rA, D_K), \tag{2.2}$$

where $i_K(rA, D_K)$ is the fixed point index defined in [1].

Since $r : X \to K$ is continuous and A is compact, $rA : \overline{D}_K \to K$ is compact. By Proposition 2.3 with $\Omega = \partial D_K$, we see that $x \neq rAx$ for $x \in \partial D_K$ since $x \neq Ax$ for $x \in \partial D_K$. Hence, the index $i_K(rA, D_K)$ is well defined, so is $i_{r,K}(A, D_K)$. If $A(\overline{D}_K) \subset K$, then the index $i_{r,K}(A, D_K)$ coincides with the usual index $i_K(A, D_K)$ given in [1] for any retraction r from X to K.

We remark that the index $i_{r,K}(A, D_K)$ defined in (2.2) depends on the retraction r. This is different from those in [1,13]. In applications, we need to choose a suitable retraction r and always use the same r to compute the fixed point index.

The new fixed point index has most of the properties of fixed point index. Note that the new fixed point index is defined for a map A that is a r-nowhere normal-outward map on ∂D_K relative to K, but we need A to be a r-nowhere normal-outward map on $\overline{D}_K$ relative to K to ensure that a nonzero index implies the existence of fixed points of A.

Theorem 2.1. *Let D be a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a compact map such that $x \neq Ax$ for $x \in \partial D_K$ and A is r-nowhere normal-outward on $\overline{D}_K$ relative to K. Then the index $i_{r,K}(A, D_K)$ has the following properties:*

(P_1) *(Existence property) If $i_{r,K}(A, D_K) \neq 0$, then A has a fixed point in D_K.*

(P_2) *(Normalization) If $x_0 \in D_K$, then $i_{r,K}(\widehat{x_0}, D_K) = 1$, where $\widehat{x_0}(x) = x_0$ for $x \in \overline{D}_K$.*

(P_3) *(Additivity property) If W_1 and W_2 are disjoint relatively open subsets of D_K such that $x \neq Ax$ for $x \in \overline{D}_K \setminus (W_1 \cup W_2)$, then*

$$i_{r,K}(A, D_K) = i_{r,K}(A, W^1) + i_{r,K}(A, W^2).$$

(P_4) *(Homotopy property) If $H : [0,1] \times \overline{D}_K \to X$ is compact such that $x \neq H(t,x)$ for $t \in [0,1]$ and $x \in \partial D_K$, and if $H(t,\cdot)$ is r-nowhere normal-outward on ∂D_K relative to K for each $t \in [0,1]$, then*

$$i_{r,K}(H(0,\cdot), D_K) = i_{r,K}(H(1,\cdot), D_K).$$

Proof. (P_1) If $i_{r,K}(A, D_K) \neq 0$, then by Definition 2.2, $i_K(rA, D_K) \neq 0$. It follows from the fixed point index theory in [1, Thorem 11.1] that rA has a fixed point $x \in D_K$. By Proposition 2.3, x is a fixed point of A.

(P_2) is obvious since $i_{r,K}(\widehat{x_0}, D_K) = i_K(\widehat{r(x_0)}, D_K) = i_K(\widehat{x_0}, D_K) = 1$.

(P_3) and (P_4) follow from the additivity and homotopy properties of the fixed point index theory in [1, Thorem 11.1] and use of Proposition 2.3. □

3. Fixed point theorems

In this section, we obtain some fixed point theorems for r-nowhere normal-outward maps by employing the fixed point index established in section 2.

Theorem 3.1. *Let D be a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a compact map such that the following conditions hold.*

(h_1) *There exists $x_0 \in D_K$ such that $tA + (1-t)\widehat{x_0}$ is r-nowhere normal-outward on $\overline{D}_K$ relative to K for $t \in (0,1]$.*

(LS) *$x \neq tAx + (1-t)x_0$ for $x \in \partial D_K$ and $t \in (0,1)$.*

Then A has a fixed point in $\overline{D}_K$, and if $x \neq Ax$ for $x \in \partial D_K$, then $i_{r,K}(A, D_K) = 1$.

Proof. We assume without loss of generality that $x \neq Ax$ for $x \in \partial D_K$. We define $H : [0,1] \times \overline{D}_K \to X$ by

$$H(t,x) = tAx + (1-t)x_0.$$

It is easy to see that H satisfies all the conditions of Theorem 2.1 (P_4). Hence, by Theorem 2.1 (P_4) and (P_1), we have

$$i_{r,K}(A, D_K) = i_{r,K}(H(1,\cdot), D_K) = i_{r,K}(H(0,x), D_K) = i_{r,K}(\widehat{x_0}, D_K) = 1$$

and A has a fixed point in D_K. □

Remark 3.1. If $A(\overline{D}_K) \subset K$, then A satisfies (h_1). Hence, Theorem 3.1 generalizes [1, Lemma 12.1 (i)], where A takes values in K. The special case of Theorem 3.1 when r is a metric projection, is an improvement of [13, Theorem 3.2], where A is a generalized inward map of contractive type.

The following result gives the conditions under which $i_{r,K}(A, D_K) = 0$.

Lemma 3.1. *Let D be a bounded open set in X such that $D_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a compact map such that the following conditions hold.*

- (h_2) *There exists $e \in K \setminus \{0\}$ such that $A + \lambda\widehat{e}$ is r-nowhere normal-outward on $\overline{D}_K$ relative to K for $\lambda \geq 0$.*
- (E) *$x \neq Ax + \lambda e$ for $x \in \partial D_K$ and $\lambda \geq 0$.*

Then $i_{r,K}(A, D_K) = 0$.

Proof. Let $\lambda_1 > \sup\{\|x - Ax\|\|e\|^{-1} : x \in \overline{D}_K\}$. Then $A + \lambda_1\widehat{e}$ has no fixed points in $\overline{D}_K$ and by Theorem 2.1 (P_1), $i_{r,K}(A + \lambda_1\widehat{e}, D_K) = 0$. It follows from Theorem 2.1 (P_4) that $i_{r,K}(A, D_K) = i_{r,K}(A + \lambda_1\widehat{e}, D_K) = 0$. □

Remark 3.2. Lemma 3.1 generalizes [1, Lemma 12.1 (ii)], where A takes values in K, and improves [13, Theorem 4.1], where A is weakly inward.

Now, we state the following result on existence of nonzero fixed points for r-nowhere normal-outward maps.

Theorem 3.2. *Let D^1, D be bounded open sets in X such that $\overline{D^1_K} \subset D_K$ and $D^1_K \neq \emptyset$. Assume that $A : \overline{D}_K \to X$ is a compact map such that (h_1) and (LS) hold on ∂D_K and (h_2) and (E) hold on ∂D^1_K. Then A has a fixed point in $\overline{D}_K \setminus D^1_K$. The same conclusion holds if (h_1) and (LS) hold on ∂D^1_K and (h_2) and (E) hold on ∂D_K.*

The proof of Theorem 3.2 follows from Theorems 2.1 and 3.1 and Lemma 3.1 and is similar to that of [13, Theorem 4.5], so we omit it.

Recall that a closed convex set K is called a wedge if $\lambda x \in K$ for $x \in K$ and $\lambda \geq 0$. If a wedge K satisfies $K \cap (-K) = \{0\}$, then K is called a cone.

As a special of Theorem 3.2, we obtain the following result which requires K to be a wedge and $Ax \in K$ for $x \in \partial D^1_K$.

Corollary 3.1. *Let K be a wedge in X and let $r : X \to K$ be a retraction. Let D^1, D be bounded open sets in X such that $\overline{D^1_K} \subset D_K$ and $D^1_K \neq \emptyset$. Assume that $A : \overline{D}_K \setminus D^1_K \to X$ is compact such that the following conditions hold.*

- (h'_1) *There exists $x_0 \in D_K$ such that $tA + (1-t)\widehat{x_0}$ is r-nowhere normal-outward on $\overline{D}_K \setminus \overline{D^1_K}$ relative to K for $t \in (0,1]$.*
- (LS) *$x \neq tAx + (1-t)x_0$ for $x \in \partial D_K$ and $t \in (0,1)$.*
- (h'_2) *$Ax \in K$ for $x \in \partial D^1_K$.*

(*E*) *There exists* $e \in K \setminus \{0\}$ *such that* $x \neq Ax + \lambda e$ *for* $x \in \partial D_K^1$ *and* $\lambda > 0$.

Then A *has a fixed point in* $\overline{D}_K \setminus D_K^1$.

Proof. With loss of generality, we assume that $x \neq Ax$ for $x \in \partial D_K \cup \partial D_K^1$. Since $A : \overline{D}_K \setminus D_K^1 \to X$ is compact, by (h_2'), there exists a compact map $A^* : \overline{D_K^1} \to K$ such that $A^*x = Ax$ for $x \in \partial D_K^1$. We define a map $\hat{A} : \overline{D}_K \to X$ by

$$\hat{A}x = \begin{cases} Ax & \text{if } x \in \overline{D}_K \setminus D_K^1, \\ A^*x & \text{if } x \in D_K^1. \end{cases}$$

Since $\hat{A}(\overline{D_K^1}) \subset K$, $\hat{A}$ satisfies (h_2). The result follows from Theorem 3.2. □

Let K be a cone in X. Then K defines a partial order $\leq$ in X by $x \leq y$ if and only if $y - x \geq 0$. A cone K is said to be *reproducing* if $X = K - K$, to be *total* if $X = \overline{K - K}$ and to be *normal* if there exists $\sigma > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \sigma \|y\|$. We refer to [1] for other cones.

Recall that a real number λ is called an eigenvalue of a linear operator $L : X \to X$ if there exists $\varphi \in X \setminus \{0\}$ such that $\lambda\varphi = L\varphi$. The radius of the spectrum of L in X, denoted by $r(L)$, is given by $r(L) = \lim_{m\to\infty} \sqrt[m]{\|L\|^m}$. We write

$$\mu_1(L) = \frac{1}{r(L)}. \tag{3.1}$$

We denote by $\mathcal{L}(K)$ the set of compact linear operators $L : X \to X$ satisfying $L(K) \subset K$ and $r(L) > 0$. By Krein-Rutman theorem (see [1, Theorem 3.1] or [6]), if K is a total cone and $L \in \mathcal{L}(K)$, then there exists an eigenvector $\varphi \in K \setminus \{0\}$ such that

$$\varphi = \mu_1(L) L\varphi. \tag{3.2}$$

Let $\rho > 0$ and let $K_\rho = \{x \in K : \|x\| < \rho\}$, $\overline{K}_\rho = \{x \in K : \|x\| \leq \rho\}$ and $\partial K_\rho = \{x \in K : \|x\| = \rho\}$.

As applications of Corollary 3.1, we obtain the following new result.

Theorem 3.3. *Let* K *be a total and normal cone in* X*. Assume that* $A : K \to X$ *is a* r*-nowhere normal-outward compact map on* K *relative to* K *and satisfies the following conditions:*

(h_1'') *There exists* $x_0 \in K$ *such that* $tA + (1-t)\widehat{x_0}$ *is* r*-nowhere normal-outward on* K *relative to* K *for* $t \in (0, 1]$.

$(LS)_1$ *There exist* $u_1 \in K \setminus \{0\}$, $L_1 \in \mathcal{L}(K)$ *and* $\varepsilon \in (0, \mu_1(L_1))$ *such that*

$$Ax \leq \big(\mu_1(L_1) - \varepsilon\big) L_1(x) + u_1 \quad \text{for } x \in K.$$

$(E)_1$ *There exist* $L \in \mathcal{L}(K)$, $\rho_0 > 0$ *and* $\varepsilon > 0$ *such that*

$$Ax \geq (\mu_1(L) + \varepsilon) L(x) \quad \text{for } x \in \partial K_{\rho_0}.$$

Then A *has a fixed point in* $K \setminus K_{\rho_0}$.

Proof. Since

$$r((\mu_1(L_1) - \varepsilon)L_1) = (\mu_1(L_1) - \varepsilon) r(L_1) < 1,$$

$(I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}$ exists and is a bounded linear operator such that

$$(I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}(K)\subset K.$$

Let σ be the normality constant of K and

$$\rho^* = \max\{\rho_0, \sigma\|(I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}(u_1+x_0)\|\}.$$

Let $\rho > \rho^*$. We prove that

$$x \neq tAx + (1-t)x_0 \quad \text{for } x \in \partial K_\rho \text{ and } t \in (0,1]. \tag{3.3}$$

In fact, if not, there exist $x \in \partial K_\rho$ and $t \in (0,1]$ such that $x = tAx + (1-t)x_0$. This, together with $(LS)_1$, implies

$$\begin{aligned} x = tAx + (1-t)x_0 &\le t[\mu_1(L_1)-\varepsilon)L_1(x) + u_1] + (1-t)x_0 \\ &\le (\mu_1(L_1)-\varepsilon)L_1(x) + u_1 + x_0 \end{aligned}$$

and

$$(I-(\mu_1(L_1)-\varepsilon)L_1)x \le u_1 + x_0.$$

This, together with $(I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}(K)\subset K$, implies

$$x \le (I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}(u_1+x_0).$$

Since K is a normal cone with normality constant σ, it follows that

$$\|x\| \le \sigma\|(I-(\mu_1(L_1)-\varepsilon)L_1)^{-1}(u_1+x_0)\| \le \rho^*.$$

Hence, we have $\rho = \|x\| \le \rho^* < \rho$, a contradiction.

Since K is total and $L \in \mathcal{L}(K)$, it follows from the Krein-Rutman theorem that there exists $\varphi \in K\setminus\{0\}$ such that $\varphi = \mu_1(L)L(\varphi)$. We prove that

$$x \neq Ax + \nu\varphi \quad \text{for } x \in \partial K_{\rho_0} \text{ and } \nu > 0. \tag{3.4}$$

In fact, if not, there exist $x \in \partial K_{\rho_0}$ and $\nu > 0$ such that

$$x = Ax + \nu\varphi. \tag{3.5}$$

Since $Ax \ge (\mu_1(L)+\varepsilon)L(x) \ge 0$ for $x \in \partial K_{\rho_0}$, by (3.5), $x \ge \nu\varphi$. Let

$$\tau_1 = \sup\{\tau > 0 : x \ge \tau\varphi\}. \tag{3.6}$$

Then $0 < \nu \le \tau_1 < \infty$, $x \ge \tau_1\varphi$ and $L(x) \ge \tau_1 L(\varphi)$. By (3.5) and $(E)_1$,

$$x \ge A(x) \ge (\mu_1(L)+\varepsilon)L(x) \ge (\mu_1(L)+\varepsilon)\tau_1 L(\varphi) = (\mu_1(L)+\varepsilon)\tau_1[\mu_1(L)]^{-1}\varphi.$$

Hence, by (3.6) we have $\tau_1 \ge (\mu_1(L)+\varepsilon)\tau_1/\mu_1 > \tau_1$, a contradiction.

By Corollary 3.1, A has a fixed point in $\overline{K}_\rho \setminus K_{\rho_0}$. □

4. Positive solutions of nonlinear equations

In this section, we study existence of positive solutions of the following nonlinear equation

$$\mathcal{N}z(x) = Fz(x) \quad \text{for a.e. } x \in [0,1] \tag{4.1}$$

subject to the generalized separated boundary conditions (GSBCs):

$$\text{if } z(0) < 0, \text{ then } z'(0) \le 0 \text{ and if } z(1) < 0, \text{ then } z'(1) \ge 0, \tag{4.2}$$

where $\mathcal{N}$ and F are suitable maps to be defined later. The GSBCs are new and contain the separated BCs (see (5.2) below). In particular, if the BCs are the Dirichlet BCs: $z(0) = z(1) = 0$, then (4.2) is automatically satisfied.

We denote by $C[0,1]$ the Banach space of continuous functions defined on $[0,1]$ with the maximum norm and by P the standard positive cone in $C[0,1]$, that is,

$$P = C_+[0,1] := \{z \in C[0,1] : z(x) \ge 0 \quad \text{for } x \in [0,1]\}. \tag{4.3}$$

It is known that P is a total and normal cone with normality constant 1. We denote by $L_+(0,1)$ the standard positive cone of the Banach space $L(0,1)$ of Lebesgue integrable functions defined on $[0,1]$. Let W be a nonempty subset of the following set

$$C^{'}[0,1] = \{z \in C[0,1] : z \text{ is differentiable everywhere on } [0,1]\}. \tag{4.4}$$

Note that we denote by $z'(0)$ and $z'(1)$ the right-hand and left-hand derivatives ($z'_+(0)$ and $z'_-(1)$) of z, respectively.

Definition 4.1. A map $\mathcal{N} : W \to L(0,1)$ is said to be right-invertible if there exists a map $T : L(0,1) \to W$ satisfying

$$\mathcal{N}(Tz) = z \quad \text{for } z \in L(0,1).$$

The map T is called a right inverse of $\mathcal{N}$.

Definition 4.2. A map $\mathcal{N} : W \to L(0,1)$ is said to be a semi-negative operator if $\mathcal{N}$ satisfies the following two conditions:

(N_1) Let $z \in W$ and $\varsigma, b \in [0,1)$ with $\varsigma < b$. If $z'(\varsigma) = 0$, $z(b) = 0$ and $z(x) < 0$ for $x \in [\varsigma, b)$, then $\int_\varsigma^{t^0} (\mathcal{N}z)(s)\,ds < 0$ for some $t^0 \in (\varsigma, b)$.

(N_2) Let $z \in W$ and $a, \varsigma \in (0,1]$ with $a < \varsigma$. If $z(a) = 0$, $z(x) < 0$ for $x \in (a, \varsigma]$ and $z'(\varsigma) = 0$, then $\int_{t_0}^{\varsigma} (\mathcal{N}z)(s)\,ds < 0$ for some $t_0 \in (a, \varsigma)$.

Definition 4.3. A map $F : \Omega \subset P \to L(0,1)$ is said to be strongly positive at 0 if for $a, b \in [0,1]$ with $a < b$ and $z \in P$ with $z(x) = 0$ for $x \in [a,b]$,

$$(Fz)(x) \ge 0 \text{ for a.e. } x \in [a,b].$$

We make the following conditions.

(A_1) $\mathcal{N} : W \to L(0,1)$ is a right-invertible and semi-negative operator with right inverse $T : L(0,1) \to W$.

(A_2) T satisfies $T(L_+(0,1)) \subset P$ and Tz satisfies the GSBC for $z \in L(0,1)$.

(A_3) $F : \Omega \subset P \to L(0,1)$ is strongly positive at 0.

We define a map $r : C[0,1] \to P$ by

$$r(y)(x) = \max\{y(x), 0\} \quad \text{for } x \in [0,1]. \tag{4.5}$$

By [13, Example 2.8], r is a continuous metric projection from X to P. Hence, $r : C[0,1] \to P$ is a retraction and $r^{-1}(P) = C[0,1]$.

We first provide the criteria for the composite map TF to be a r-nowhere normal-outward map.

Theorem 4.1. *Under the hypotheses (A_1), (A_2), (A_3) the composite map $TF : \Omega \subset P \to C[0,1]$ is a r-nowhere normal-outward map on Ω relative to P, where r is the same as in* (4.5).

Proof. Assume that there exists $y \in r^{-1}(\Omega) \subset X$ such that

$$y(x) = TF(r(y))(x) \quad \text{for } x \in [0,1]. \tag{4.6}$$

By Proposition 2.1 (H_1) and (H_3), we need to prove $y \in P$. We first prove that there exists $t^* \in (0,1)$ such that $y(t^*) \geq 0$. In fact, if not, then

$$y(x) < 0 \quad \text{for } x \in (0,1). \tag{4.7}$$

Since y is continuous on $[0,1]$, $y(x) \leq 0$ for $x \in [0,1]$, by (4.5) and (A_3), $F(r(y)) = F(0) \in L_+(0,1)$. By (A_2), (4.6) and (4.7) we have

$$0 \leq TF(r(y))(x) = y(x) < 0 \quad \text{for } x \in [0,1],$$

a contradiction.

Next, we prove $y \in P$. Since y is continuous on $[0,1]$, it suffices to show that $y(x) \geq 0$ for each $x \in (0,1)$. In fact, if not, then there exists $t_* \in (0,1)$ such that $y(t_*) < 0$. Since $y(t^*) \geq 0$, $t_* \neq t^*$. We consider the following two cases: $t_* < t^*$ or $t^* < t_*$.

Case 1. If $t_* < t^*$, then let

$$t_1 = \inf\{\underline{t} \in [0, t_*) : y(x) < 0 \text{ for } x \in (\underline{t}, t_*)\}$$

and

$$b = \sup\{\bar{t} \in (t_*, t^*] : y(x) < 0 \text{ for } x \in (t_*, \bar{t})\}.$$

Since $y(t_*) < 0$, both t_1 and b exist. For this t_1, we have $t_1 \in [0, t_*)$, $y(t_1) = 0$ if $t_1 > 0$ and $y(t_1) = y(0) \leq 0$ if $t_1 = 0$. For this b, we have $b \in (t_*, t^*)$, $y(b) = 0$ and $y(x) < 0$ for $x \in (t_1, b)$. Let $\varsigma \in [t_1, b]$ be such that

$$y(\varsigma) = \min\{y(x) : x \in [t_1, b]\}. \tag{4.8}$$

Then $y(\varsigma) \leq y(t_*) < 0$, $\varsigma \in [t_1, b)$ and $y(x) < 0$ for $x \in [\varsigma, b)$. Note that $y \in W$ and $y'(x)$ exists for $x \in [0,1]$. By (4.8), $y'(\varsigma) = 0$ if $\varsigma \in (t_1, b)$ or $\varsigma = t_1$. If $\varsigma = t_1$, then $y(t_1) = y(\varsigma) < 0$. By the definition of t_1, $\varsigma = t_1 = 0$ and $y(0) = y(\varsigma) < 0$. By (4.8), $y(0) \leq y(x)$ for $x \in [0, b]$. It follows that

$$y'_+(\varsigma) = y'_+(0) = \lim_{x \to 0^+} \frac{y(x) - y(0)}{x} \geq 0. \tag{4.9}$$

On the other hand, by (A_2), Tz satisfies the GSBC, where $z = F(r(y))$. Since $TF(r(y))(0) = y(0) < 0$, $y'_+(0) = (TF(r(y)))'(0) \leq 0$. By (4.9), $y'_+(\varsigma) = y'_+(0) = 0$.

Hence, we have showed that $y'(\varsigma) = 0$, $y(b) = 0$ and $y(x) < 0$ for $x \in [\varsigma, b)$. By (N_1), there exists $t^0 \in (\varsigma, b)$ such that

$$\int_{\varsigma}^{t^0} (\mathcal{N}y)(s)\, ds < 0. \tag{4.10}$$

Since $\mathcal{N}$ is a right-invertible operator and $F(r(y))(x) = F(0)(x)$ for $x \in [\varsigma, b]$, it follows from (4.6) and strong positivity of F that

$$(\mathcal{N}y)(x) = \mathcal{N}(TF(r(y)))(x) = F(r(y))(x) = F(0)(x) \geq 0 \text{ for a.e. } x \in [\varsigma, b].$$

Integrating the above inequality and applying (4.10), we have

$$0 \leq \int_{\varsigma}^{t^0} (\mathcal{N}y)(s)\, ds < 0,$$

a contradiction.

Case 2. If $t^* < t_*$, then the proof is similar to that of Case 1. We sketch the proof. Let

$$a = \inf\{\underline{t} \in [t^*, t_*) : y(x) < 0 \text{ for } x \in (\underline{t}, t_*)\}$$

and

$$t_2 = \sup\{\bar{t} \in (t_*, 1] : y(x) < 0 \text{ for } x \in (t_*, \bar{t})\}.$$

Let $\varsigma \in [a, t_2]$ be such that

$$y(\varsigma) = \min\{y(x) : x \in [a, t_2]\}. \tag{4.11}$$

Then $y(a) = 0$, $y(x) < 0$ for $x \in (a, \varsigma]$ and $y'(\varsigma) = 0$. By (N_2), there exists $t_0 \in (a, \varsigma)$ such that

$$\int_{t_0}^{\varsigma} (\mathcal{N}y)(s)\, ds < 0. \tag{4.12}$$

It can be shown that

$$(\mathcal{N}y)(x) = \mathcal{N}(TF(r(y)))(x) = F(r(y))(x) = F(0)(x) \geq 0 \text{ for a.e. } x \in [a, \varsigma].$$

Integrating the above inequality and applying (4.12), we have

$$0 \leq \int_{t_0}^{\varsigma} (\mathcal{N}y)(s)\, ds < 0,$$

a contradiction.

By Proposition 2.1 (H_1) and (H_3), we see that $A : \Omega \to C[0,1]$ is a r-nowhere normal-outward map on Ω relative to P. □

Note that in Theorem 4.1, the right inverse T of $\mathcal{N}$ is not required to be linear. However, if it is linear, we obtain the following more general result.

Corollary 4.1. *Assume that (A_1), (A_2), (A_3) hold and T is linear. Assume that $G : \Omega \subset P \to L(0,1)$ is strongly positive at 0. Then $\alpha TF + \beta TG$ is a r-nowhere normal-outward map on Ω relative to P for $\alpha, \beta \in \mathbb{R}_+$.*

Proof. Since F and G are strongly positive at 0, so is $\alpha F+\beta G$ for $\alpha,\beta\in\mathbb{R}_+$. By Theorem 4.1 and linearity of T, we have $\alpha TF+\beta TG=T(\alpha F+\beta G)$ is a r-nowhere normal-outward map on Ω relative to P for $\alpha,\beta\in\mathbb{R}_+$ □

The following result shows that if $\mathcal{N}$ is a right-invertible operator with right inverse T, then (4.1) can be changed into the fixed point equation:

$$z=TF(z(x)):=Az(x)\quad \text{for } x\in[0,1]. \tag{4.13}$$

Proposition 4.1. *Assume that $\mathcal{N}:W\to L(0,1)$ is right-invertible with a right inverse operator T and $F:\Omega\subset P\to L(0,1)$ is a map. Then if $z\in\Omega$ satisfies* (4.13)*, then z satisfies* (4.1)*.*

Proof. Assume that $z\in\Omega$ satisfies $z=TF(z)$. Since $\mathcal{N}:W\to L(0,1)$ is right-invertible with a linear right inverse T, $\mathcal{N}z=\mathcal{N}T(F(z))=T(F(z))$. The result follows. □

Theorem 4.2. *Assume that (A_1) and (A_2) hold and $T:L(0,1)\to C[0,1]$ is a linear compact operator. Assume that $F:P\to L(0,1)$ is bounded, continuous and strongly positive at 0, and the following conditions hold.*

$(LS)_2$ *There exist $u_1\in P\setminus\{0\}$, $S_1\in\mathcal{L}(P)$ and $\varepsilon\in(0,\mu_1(S_1))$ such that*

$$TF(z)\le(\mu_1(S_1)-\varepsilon)S_1(z)+u_1\quad \text{for } z\in P.$$

$(E)_2$ *There exist $S\in\mathcal{L}(P)$, $\rho_0>0$ and $\varepsilon>0$ such that*

$$TF(z)\ge(\mu_1(S)+\varepsilon)S(z)\quad \text{for } z\in\partial P_{\rho_0}.$$

Then (4.1)-(4.2) *has a solution in $P\setminus P_{\rho_0}$.*

Proof. Since $T:L(0,1)\to C[0,1]$ is compact and $F:P\to L(0,1)$ is bounded and continuous, the map $A:=TF:P\to C[0,1]$ is compact. By Theorem 4.1 and the hypotheses (A_1) and (A_2), we see that $A:P\to C[0,1]$ is a r-nowhere normal-outward map on P relative to P. Let $x_0=0$ and define a map $G:P\to L(0,1)$ by $Gz=0$. By Corollary 4.1,

$$tA=tA+(1-t)G=tA+(1-t)\widehat{x_0}$$

is r-nowhere normal-outward on P relative to P for $t\in(0,1]$. Note that $(E)_2$ and $(LS)_2$ imply $(E)_1$ and $(LS)_1$ of Theorem 3.3. It follows from Theorem 3.3 that (4.13) has a solution z in $P\setminus P_{\rho_0}$. By Proposition 4.1, z is a solution of (4.1)-(4.2). □

We show that equation (4.1) contains the following nonlinear equation

$$-\big(\omega(x)|z'(x)|^{p-2}z'(x)\big)'=f(x,z(x))\quad \text{for a.e. } x\in[0,1]. \tag{4.14}$$

We always assume the following conditions:

(i) $p\in(1,\infty)$.

(ii) $\omega:[0,1]\to(0,\infty)$ is a continuous function.

We denote by $AC[0,1]$ the space of all the absolutely continuous functions defined on $[0,1]$. We define a set

$$W_p=\{z\in C'[0,1]:\omega|z'|^{p-2}z'\in AC[0,1]\} \tag{4.15}$$

and a map $\mathcal{N}:W_p\to L(0,1)$ by

$$\mathcal{N}z(x)=-\big(\omega(x)|z'(x)|^{p-2}z'(x)\big)'. \tag{4.16}$$

Proposition 4.2. *The map $\mathcal{N} : W_p \to L(0,1)$ defined in* (4.16) *is a semi-negative operator.*

Proof. Let $\varsigma, b \in [0,1)$ with $\varsigma < b$ and let $z \in W_p$ satisfy $z'(\varsigma) = 0$, $z(b) = 0$ and $z(x) < 0$ for $x \in [\varsigma, b)$. We prove that there exists $t^0 \in (\varsigma, b)$ such that

$$\int_{\varsigma}^{t^0} (\mathcal{N}z)(s)\, ds < 0. \tag{4.17}$$

In fact, if not, then we have for $x \in [\varsigma, b)$,

$$\begin{aligned}\int_{\varsigma}^{x} (\mathcal{N}z)(s)\, ds &= -\omega(x)|z'(x)|^{p-2}z'(x) + \omega(\varsigma)|z'(\varsigma)|^{p-2}z'(\varsigma)\\ &= -\omega(x)|z'(x)|^{p-2}z'(x) \geq 0\end{aligned}$$

and $z'(x) \leq 0$ for $x \in [\varsigma, b)$. Hence, z is decreasing on $[\varsigma, b)$ and we have $0 = z(b) \leq z(x) < 0$ for $x \in [\varsigma, b)$, a contradiction. It follows that (4.17) holds and $\mathcal{N}$ satisfies (N_1). A similar proof shows that $\mathcal{N}$ satisfies (N_2). □

By Proposition 4.2 and Theorem 4.1, we see that if the map $\mathcal{N}$ defined in (4.16) has a right inverse T which maps $L(0,1)$ to W and satisfies the condition (A_2), then under suitable assumptions on f, TF is a r-nowhere normal-outward map on P relative to P, where $Fz(x) = f(x, z(x))$, and the fixed point theorems in section 3 can be applied to treat existence of nonzero nonnegative solutions of (4.14) subject to the GSBCs. In the following section, we apply Theorem 4.2 to study existence of nonzero nonnegative solutions of the Sturm-Liouville boundary value problem (4.14) with $p = 2$.

5. Sturm-Liouville boundary value problems

In this section, we consider the existence of nonzero nonnegative solutions for the Sturm-Liouville differential equations of the form

$$-\left(\omega(x)z'(x)\right)' = f(x, z(x)) \quad \text{a.e. on } [0,1] \tag{5.1}$$

subject to the separated boundary condition (BCs)

$$\begin{cases} \alpha z(0) - \beta\omega(0)z'(0) = 0 \\ \gamma z(1) + \delta\omega(1)z'(1) = 0, \end{cases} \tag{5.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma \int_0^1 \frac{1}{\omega(\mu)}\, d\mu + \alpha\delta > 0$.

We make the following assumptions on ω and f.

(C_0) $\omega : [0,1] \to (0, \infty)$ is continuous.

(C_1) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies Carathéodory conditions, that is, $f(\cdot, u)$ is measurable for $u \in \mathbb{R}_+$ and $f(x, \cdot)$ is continuous for almost every (a.e.) $x \in [0,1]$, and for each $r > 0$, there exists $g_r \in L(0,1)$ such that

$$|f(x,u)| \leq g_r(x) \;\text{ for a.e } x \in [0,1] \text{ and all } u \in [0,r].$$

(C_2) (Positivity condition at 0) $f(x, 0) \geq 0$ for a.e. $x \in [0,1]$.

Note that (C_2) only requires f to be positive at $(x,0)$ and allows f to take negative values. The function f is not required to have a lower bound, so it may not satisfy the semi-positone condition: $f(x,u) \geq -\eta$ for some $\eta > 0$.

Existence of solutions of(5.1)-(5.2) has been widely studied. For example, under (C_2), (5.1)-(5.2) was studied in [20] by using the topological degree theory, where the nonlinearity is of the form $g(x)f(x,u)$ with $g \in L_+(0,1) \cap C(0,1)$ and ω, f are continuous, and in [2, 9], where f satisfies the semi-pontone condition, and in [18], where f is allowed to change signs.

A function $z : [0,1] \to \mathbb{R}$ is said to be a solution of (5.1)-(5.2) if $z \in W_2$, where W_2 is same as in (4.15) with $p = 2$, namely,

$$W_2 = \{z \in C'[0,1] : \omega z' \in AC[0,1]\}.$$

A solution z of (5.1)-(5.2) is said to be nonnegative if $z \in P$, where P be same as in (4.3).

Let k be the Green's function for the equation $-(\omega(x)z'(x))' = 0$ subject to the BC (5.2). It is well known that $k : [0,1] \times [0,1] \to \mathbb{R}_+$ is given by

$$k(x,s) = \frac{1}{\Gamma}\begin{cases} (\delta + \gamma \int_x^1 \frac{1}{\omega(\mu)}\,d\mu)(\beta + \alpha \int_0^s \frac{1}{\omega(\mu)}\,d\mu) \text{ if } s \leq x, \\ (\beta + \alpha \int_0^x \frac{1}{\omega(\mu)}\,d\mu)(\delta + \gamma \int_s^1 \frac{1}{\omega(\mu)}\,d\mu) \text{ if } x < s. \end{cases} \tag{5.3}$$

We consider the Hammerstein integral equation

$$z(x) = \int_0^1 k(x,s)f(s,z(s))\,ds = (LF)z(x) :\equiv Az(x) \quad \text{for } x \in [0,1], \tag{5.4}$$

where

$$Lu(x) = \int_0^1 k(x,s)u(s)\,ds \quad \text{for } x \in [0,1] \tag{5.5}$$

and the Nemytskii operator $F : P \to L(0,1)$ is defined by

$$(Fz)(x) = f(x,z(x)). \tag{5.6}$$

Note that for each $y \in L(0,1)$,

$$Ly(s) := \int_0^1 k(x,s)y(s)\,ds < \infty \quad \text{for } x \in [0,1]$$

and $Ly \in W_2$. Hence, the operator L maps $L(0,1)$ to W_2. It is easy to verify that if $z \in P$ is a solution of (5.4), then z is a solution of (5.1)-(5.2).

We define a map $\mathcal{N} : W_2 \to L(0,1)$ by

$$\mathcal{N}z(x) = -(\omega(t)z'(t))'. \tag{5.7}$$

Proposition 5.1. *(i) The map $\mathcal{N} : W_2 \to L(0,1)$ is a right-invertible and semi-negative operator with right inverse L.*

(ii) The linear operator $L : L(0,1) \to W_2$ satisfies $L(L_+(0,1)) \subset P$, $L : L(0,1) \to C[0,1]$ is compact, and Lz satisfies the GSBC for each $z \in L(0,1)$.

(iii) Under (C_1) and (C_2), $F : P \to L(0,1)$ is bounded, continuous and strongly positive at 0.

Proof. (i) It is known (or directly verified) that $\mathcal{N}(Lz) = z$ for $z \in L(0,1)$, so $\mathcal{N} : W_2 \to L(0,1)$ is a right-invertible operator with right inverse L defined in (5.5). By Proposition 4.2, $\mathcal{N} : W_2 \to L(0,1)$ is a semi-negative operator.

(ii) It is obvious that $L(L_+(0,1)) \subset P$ since $k(x,s) \geq 0$ for $x, s \in [0,1]$. By a similar proof to those of [8, Theorem 2.1] or [12, Lemma 3.3] or [16, Proposition 3.4, p. 167], $L : L(0,1) \to C[0,1]$ is compact. By the equivalence of (5.1)-(5.2) and (5.4) (or a direct verification), for $z \in (0,1)$, Lz satisfies the BCs (5.2), that is,

$$\begin{cases} \alpha(Lz)(0) - \beta\omega(0)(Lz)'(0) = 0, \\ \gamma(Lz)(1) + \delta\omega(1)(Lz)'(1) = 0. \end{cases} \tag{5.8}$$

This implies that Lz satisfies the GSBC for each $z \in L(0,1)$.

(iii) Under (C_1), $F : P \to L(0,1)$ is bounded and continuous. Let $a, b \in [0,1]$ with $a < b$, $z \in P$ with $z(x) = 0$ for $x \in [a,b]$. By (C_2),

$$(Fz)(x) = f(x, z(x)) = f(x, 0) \geq 0 \text{ for a.e. } x \in [a,b].$$

By Definition 4.3, $F : P \to L(0,1)$ is strongly positive at 0. □

By Proposition 5.1 and Theorem 4.1, we obtain

Corollary 5.1. *Under (C_0), (C_1) and (C_2), the operator A defined in (5.4) maps P to $C[0,1]$ and is r-nowhere normal-outward on P relative to P.*

Remark 5.1. We remark that under the hypotheses (C_0), (C_1) and (C_2), it is not clear whether the integral operator A defined in (5.4) is a weakly inward or generalized inward map. Hence, the index theories established in [13] have not been yet applied to treat boundary value problems like (5.1)-(5.2).

Lemma 5.1. *Assume that $g \in L(0,1)$ with $\int_0^1 g(s)\,ds > 0$. Then there exists $\varphi_g \in P \setminus \{0\}$ such that*

$$\varphi_g = \mu_1(L_g)L_g(\varphi_g),$$

where $L_g : C[0,1] \to W_2 \subset C[0,1]$ is defined by

$$L_g u(x) = \int_0^1 k(x,s)g(s)u(s)\,ds \quad \textit{for } x \in [0,1].$$

Proof. By $\int_0^1 g(s)\,ds > 0$, there exist $a, b \in (0,1)$ with $a < b$ such that $\int_a^b g(s)\,ds > 0$. Let $\Phi(s) = k(s,s)$ for $s \in [0,1]$. since $\Phi(s) > 0$ for $s \in (0,1)$, $\int_a^b \Phi(s)g(s)\,ds > 0$. The result follows from [19, Theorem 2.6]. □

Theorem 5.1. *Assume that (C_0), (C_1) and the following conditions hold.*

(i) There exist $r_0 > 0$, $\phi_{r_0} \in L_+(0,1)$ with $\int_0^1 \phi_{r_0}(s)\,ds > 0$ and $\varepsilon \in (0, \mu_1(L_{\phi_{r_0}}))$ such that

$$f(x,u) \leq \big(\mu_1(L_{\phi_{r_0}}) - \varepsilon\big)\phi_{r_0}(x)u \quad \textit{for a.e. } x \in [0,1] \textit{ and } u \in [r_0, \infty). \tag{5.9}$$

(ii) There exist ρ_0, $\varepsilon > 0$ and $\psi_{\rho_0} \in L^1_+(0,1)$ with $\int_0^1 \psi_{\rho_0}(s)\,ds > 0$ such that

$$f(x,u) \geq \big(\mu_1(L_{\psi_{\rho_0}}) + \varepsilon\big)\psi_{\rho_0}(x)u \quad \textit{for a.e. } x \in [0,1] \textit{ and } u \in [0, \rho_0]. \tag{5.10}$$

Then (5.1)-(5.2) has a solution in $P \setminus P_{\rho_0}$.

Proof. By (C_1), there exists $g_{r_0} \in L^1_+(0,1)$ such that

$$|f(x,u)| \le g_{r_0}(x) \quad \text{for a.e. } x \in [0,1] \text{ and all } u \in [0, r_0].$$

This, together with (5.9), implies

$$f(x,u) \le g_{r_0}(x) + \big(\mu_1(L_{\phi_{r_0}}) - \varepsilon\big)\phi_{r_0}(x)u \quad \text{for a.e. } x \in [0,1] \text{ and all } u \in \mathbb{R}_+.$$

Let $u_1(x) = \int_0^1 k(x,s)g_{r_0}(s)\,ds$ for $x \in [0,1]$. Then

$$Az(x) \le u_1(x) + \big(\mu_1(L_{\phi_{r_0}}) - \varepsilon\big)L_{\phi_{r_0}}z(x) \quad \text{for } x \in [0,1].$$

By (5.10), we have for $z \in \partial P_{\rho_0}$ and $x \in [0,1]$,

$$Az(x) \ge \big(\mu_1(L_{\psi_{\rho_0}}) + \varepsilon\big)\int_0^1 k(t,s)\psi_{\rho_0}(s)z(s)\,ds = \big(\mu_1(L_{\psi_{\rho_0}}) + \varepsilon\big)L_{\psi_{\rho_0}}z(x).$$

Note that the condition (ii) implies that f satisfies the condition (C_2). By Proposition 5.1 and Theorem 4.2, (5.1)-(5.2) has a solution z in $P \setminus P_{\rho_0}$. □

Let E be a fixed subset of $[0,1]$ of measure zero. Let

$$\underline{f}(u) = \inf_{x\in[0,1]\setminus E} f(x,u), \quad (f)_0 = \liminf_{u\to 0+} \underline{f}(u)/u,$$
$$\overline{f}(u) = \sup_{x\in[0,1]\setminus E} f(x,u), \quad f^\infty = \limsup_{u\to\infty} \overline{f}(u)/u.$$

Corollary 5.2. *Assume that (C_0), (C_1) and the following condition holds.*

$$-\infty \le f^\infty < \mu_1(L) < (f)_0 \le \infty. \tag{5.11}$$

Then (5.1)-(5.2) *has a solution in* $P \setminus \{0\}$.

Proof. It is easy to see that by (5.11), (5.9) with $\phi_{r_0} \equiv 1$ and (5.10) with $\psi_{\rho_0} \equiv 1$ hold for some $\varepsilon > 0$ and ρ_0, r_0 with $0 < \rho_0 < r_0 < \infty$. The result follows from Theorem 5.1. □

Corollary 5.2 improves Theorem 3.2 in [20], where the nonlinearity is a product of $g \in L^1_+(0,1) \cap C(0,1)$ and a continuous function f, and the Leray-Schauder degree theory is used.

As an application of Corollary 5.2, we study existence of nonzero nonnegative solutions of the following Sturm-Liouville boundary value problem

$$-(\omega(x)z'(x))' = \mu\Big[z(x)(1-z(x)) - \frac{a(x)z^2(x)}{1+z^2(x)}\Big] \quad \text{for } x \in (0,1) \tag{5.12}$$

subject to (5.2).

Equation (5.12) with $\omega(x) \equiv 1$ is the steady-state equation of the reaction diffusion population models of spruce budworm [14, p.235, (6.1)] under suitable changes of variables.

By Corollary 5.2, we obtain the following result.

Theorem 5.2. *Assume that $a \in L^\infty_+(0,1)$ with $a_* := \inf_{x\in[0,1]\setminus E}\{a(x) : x \in [0,1]\} > 0$ for a fixed subset E of $[0,1]$ of measure zero and $\mu \in (\mu_1(L), \infty)$. Then* (5.12) *has a solution in* $P \setminus \{0\}$.

Proof. Let $\mu \in (\mu_1(L), \infty)$ and define a function $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ by

$$f(x,u) = \mu\big[u(1-u) - \frac{a(x)u^2}{1+u^2}\big]. \tag{5.13}$$

Then

$$\overline{f}(u) = \sup_{x\in[0,1]\setminus E} f(x,u) = \mu u\big[1 - u - \frac{a_* u}{1+u^2}\big],$$

and

$$f^\infty = \limsup_{u\to\infty} \frac{\overline{f}(u)}{u} = \mu \limsup_{u\to\infty}\big[1 - u - \frac{a_* u}{1+u^2}\big] = -\infty.$$

Moreover,

$$\underline{f}(u) = \inf_{x\in[0,1]\setminus E} f(x,u) = \mu u\big[1 - u - \frac{\|a\|_\infty u}{1+u^2}\big],$$

and

$$f_0 = \limsup_{u\to 0^+} \frac{\underline{f}(u)}{u} = \mu \limsup_{u\to 0^+}\big[1 - u - \frac{\|a\|_\infty u}{1+u^2}\big] = \mu > \mu_1(L).$$

It follows that $-\infty \le f^\infty < \mu_1(L) < (f)_0 \le \infty$. The result follows from Corollary 5.2. □

We remark that since $\lim_{u\to\infty} f(x,u) = -\infty$ for $x \in [0,1]$, the function f defined in (5.13) has no lower bounds. Hence, f does not satisfies the semi-position condition and the results obtained in [2,9] can not be applied to treat Theorem 5.2. Also, if $a(x)$ is not a continuous function, then Theorem 5.2 can not be treated by Theorem 3.1 in [20], where f is continuous. Our method is different from that used in [20], where the topological degree theory is used and the nonlinearity is of the form $g(x)f(x,u)$ with $g \in L_+(0,1) \cap C(0,1)$ and ω, f are continuous.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces,* SIAM Rev., 18(1976), 620–709.

[2] V. Anuradha, D. D. Hai and R. Shivaji, *Existence results for superlinear semi-positone BVP,* Proc. Amer. Math. Soc., 124(1996), 757–763.

[3] J. Dugundji, *An extension of Tietze's theorem,* Pacific J. Math., 1(1951), 353–367.

[4] B. Halpern and G. Bergman, *A fixed point theorem for inward and outward maps,* Trans. Amer. Math. Soc., 130(1968), 353–358.

[5] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometric methods of nonlinear analysis,* Springer Verlag, Berlin, 1984.

[6] M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space,* Amer. Math. Soc. Transl., 10(1962), 199–325.

[7] K. Q. Lan, *A fixed point theory for weakly inward S-contractive maps,* Nonlinear Anal., 45(2001), 189–201.

[8] K.Q. Lan, *Multiple positive solutions of semilinear differential equations with singularities,* J. London Math. Soc., 63(2001)(2), 690–704.

[9] K. Q. Lan, *Multiple positive solutions of semi-positone Sturm-Liouvilie boundary value problems,* Bull. London Math. Soc., 38(2006), 283–293.

[10] K. Q. Lan, *Positive solutions of semi-positone Hammerstein integral equations and applications,* Commun. Pure Appl. Anal., 6(2007)(2), 441–451.

[11] K. Q. Lan, *Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems,* Nonlinear Anal., 71(2009)(12), 5979–5993.

[12] K. Q. Lan, *Existence of nonzero positive solutions of systems of second order elliptic boundary value problems,* J. Appl. Anal. Comput., 1(2011)(1), 21–31.

[13] K. Q. Lan and J. R. L. Webb, *A fixed point index for generalized inward mappings of condensing type,* Trans. Amer. Math. Soc., 349(1997), 2175–2186.

[14] D. Ludwig, D. C. Aronson and H. F. Weinberger, *Spatial patterning of the spruce budworm,* J. Math. Biol., 8(1979), 217–258.

[15] R. Ma, *Existence of positive solutions for superlinear semipositone m-point boundary-value problems,* Proc. Edinburgh Math. Soc., 46(2003), 279–292.

[16] R. H. Martin, *Nonlinear operators and differential equations in Banach spaces,* Wiley, New York, 1976.

[17] R. D. Nussbaum, *The fixed point index for local condensing maps,* Ann. Mat. Pura Appl., 84(1971), 217–258.

[18] B. P. Rynne, *Second order, Sturm-Liouville problems with asymmetric, superlinear nonlinearities II,* Nonlinear Anal., 57(2004), 905–916.

[19] J. R. L. Webb and K. Q. Lan, *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type,* Topol. Methods Nonlinear Anal., 27(2006)(1), 91–116.

[20] G. C. Yang and P. F. Zhou, *A new existence result of positive solutions for the Sturm-Liouville boundary value problems,* Appl. Math. Lett., 27(2014), 1401–1406.