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STATISTICAL ANALYSIS OF SERIES SYSTEM FOR MASKED DATA UNDER SUCCESSIVE CENSORED LIFE TEST

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Abstract In this paper, considering series system of masked data under simple successive censored and multiple successive censored life test, the likelihood function and maximum likelihood estimate are respectively proposed for series system composed of two units under two kinds of situations. One is the series system composed of two units with constant failure rate, and the other is the series system composed of two units with linear failure rate through the origin. The approximate interval estimates of parameters are given by using the method of likelihood ratio. Besides, the examples show the feasibility of the methods through Monte-Carlo simulations.

Keywords Masked data, series system, successive censored, maximum likelihood estimate, approximate interval estimate.

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1. Introduction

In reliability analysis, we often estimate unknown parameters of the life distribution of the system components through the analysis of the systems data. System life test data includes two aspects, one is the failure time and the other is the failure reason. Ideally, system life data should include both the failure time of the system and the information of the specific unit which causes the entire unit and system to fail. But most of the time, the unit that results in the system failure is not able to be accurately identified, and people can only attribute the cause of system failure to a collection of certain units, thus the real reason of system failure is masked. In real life, since the cost of fault diagnosis and fault detection is expensive, especially modular design is increasingly used in the modern system, the exact unit that results in system failure is often unknown. We also encounter similar problems when we study on the reliability of system in computer or integrated circuit and so on. There are various reasons that result in masked data, such as inadequate funding, time constraints, record errors, and lack of diagnostic tools and so on. Therefore the statistical analysis of masked data becomes one of the hot topics in recent years, and many scholars have done a very good job and made a series of studies, specifically seeing [1–5, 7–19, 22–24].

In this paper, considering series system of masked data under simple successive censored and multiple successive censored life test, the likelihood function and

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maximum likelihood estimate are respectively proposed for series system composed of two units under two kinds of situations. One is the series system composed of two units with constant failure rate, and the other is the series system composed of two units with linear failure rate through the origin. The approximate interval estimates of parameters are given by using the method of likelihood ratio. Besides, the examples show the feasibility of the methods through Monte - Carlo simulations.

2. Likelihood Function of Series System for Masked Data

In order to establish models, we usually give following basic assumptions:

Assumption 1: The system is composed of $J(\geq 1)$ independent units in series.

Assumption 2: The occurrence of masking is independent of the failure reason and time.

Assumption 3: In system i, the lifetime of the j^{th}unit is denoted by T_{ij}, and its corresponding density function, distribution function, failure rate function and reliability function are respectively $f_{ij}(t), F_{ij}(t), h_{ij}(t)$ and $\bar{F}_{ij}(t)$.

Since the random variable T_{ij} does not depend on i, above density function, distribution function, failure rate function and reliability function can be respectively denoted by $f_j(t), F_j(t), h_j(t)$ and $\bar{F}_j(t)$.

Considering that n series systems are put in the life test, each system has J units. Let the random variable T_{ij} be the lifetime of the j^{th} unit in the i^{th} system, and its observation value is denoted by t_{ij}, $i = 1, 2, \cdots, n, j = 1, 2, \cdots, J$. Then the lifetime T_i of the i^{th} system is $T_i = \min(T_{i1}, T_{i2}, \cdots, T_{iJ})$, and its observation value is denoted by $t_i, i = 1, \ldots, n$. Let S_i be the set of units which cause the failure of system i and s_i is the realization of S_i. Then the observation data include $(t_1, s_1), (t_2, s_2), \cdots, (t_n, s_n)$. If the set s_i contains only one element or it is composed of a single element, it indicates that the unit which causes the failure of system i is known. If the set s_i contains more than one element, it indicates that the life data of units which cause the failure of system i are masked.

Let K_i be the exact unit that causes the failure of system i. The density function, distribution function, reliability function and failure rate function of the j^{th} unit life are respectively denoted by $f_j(t), F_j(t), \bar{F}_j(t)$ and $h_j(t)$. Assume that the life distributions of units are independent of each other. Take $J_j = \{1, 2, \cdots, j-1, j+1, \cdots, J\}$.

Therefore, according to [19], the simplified likelihood function of n series systems can be denoted by

$$L(\text{data}) = \prod_{i=1}^{n} \left\{ \sum_{j \in s_i} \left[f_j(t_i) \prod_{l \in J_j} \bar{F}_l(t_i) \right] \right\}.$$

Since $f_j(t) = h_j(t)\bar{F}_j(t)$, we have

$$L(\text{data}) = \prod_{i=1}^{n} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\}.$$

3. Statistical Analysis of Series System for Masked Data under Simple Successive Censored Life Test

3.1. Likelihood function of series system for masked data under simple successive censored life test

Considering that n series systems composed of two units are put in simple successive censored life test, the failure reason of each system can be summed up in three categories: $s_1 = \{1\}$, $s_2 = \{2\}$, $s_{12} = \{1,2\}$. When k_1 systems are failed, n_1 systems randomly selected from $n - k_1$ systems that are not failed are evacuated from testing site, and remaining $n - k_1 - n_1$ systems continue the test. When other k_2 systems are failed, $n - k_1 - n_1 - k_2$ systems that are not failed are all evacuated from testing site. Thus the order failure time of all failure systems are $\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}$. Among k_1 failure systems in the first test, r_1 systems belong to class s_1 and their failure time are $t_1, t_2, \cdots, t_{r_1}$; r_2 systems belong to class s_2 and their failure time are $t_{r_1+1}, t_{r_1+2}, \cdots, t_{r_1+r_2}$; r_3 systems belong to classs_{12} and their failure time are $t_{r_1+r_2+1}, t_{r_1+r_2+2}, \cdots, t_{r_1+r_2+r_3}$, where $r_1+r_2+r_3 = k_1$. Among k_2 failure systems in the second test, r_4 systems belong to class s_1 and their failure time are $t_{k_1+1}, t_{k_1+2}, \cdots, t_{k_1+r_4}$; r_5 systems belong to class s_2 and their failure time are $t_{k_1+r_4+1}, t_{k_1+r_4+2}, \cdots, t_{k_1+r_4+r_5}$; r_6 systems belong to classs_{12} and their failure time are $t_{k_1+r_4+r_5+1}, t_{k_1+r_4+r_5+2}, \cdots, t_{k_1+r_4+r_5+r_6}$, where $r_4 + r_5 + r_6 = k_2$. The data form is shown in Table 1, where A denotes the number of failure system in each test and B denotes system failure time according to the classification of failure reason.

Table 1. Data Form of Series System for Masked Data under Simple Successive Censored Life Test

B A	Failure time that belongs to class s_1	Failure time that belongs to class s_2	Failure time that belongs to class s_{12}
k_1	$t_1, t_2, \cdots, t_{r_1}$	$t_{r_1+1}, t_{r_1+2}, \cdots, t_{r_1+r_2}$	$t_{r_1+r_2+1}, t_{r_1+r_2+2}, \cdots, t_{r_1+r_2+r_3}$
k_2	$t_{k_1+1}, t_{k_1+2}, \cdots, t_{k_1+r_4}$	$t_{k_1+r_4+1}, t_{k_1+r_4+2}, \cdots, t_{k_1+r_4+r_5}$	$t_{k_1+r_4+r_5+1}, t_{k_1+r_4+r_5+2}, \cdots, t_{k_1+r_4+r_5+r_6}$

Theorem 3.1. *Supposed that density function of the system is $q(\tau)$ and survival function is $R(\tau)$, the density function, distribution function, reliability function and failure rate function of the j^{th} unit life in each series system are respectively $f_j(t), F_j(t), \bar{F}_j(t)$ and $h_j(t)$, $j = 1, 2, \cdots, J$. Besides, assuming that the life distributions of units are mutually independent, n series systems composed of two units are put in simple successive censored life test and the data form is shown as Table 1. Then the likelihood function of this situation is*

$$\begin{aligned}
L(data) =& C_2^+ \prod_{i=1}^{k_1} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \\
& \times [R(\tau_{k_1})]^{n_1} [R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2} \\
=& C_2^+ \prod_{i=1}^{k_1+k_2} \left[\prod_{l=1}^{2} \bar{F}_l(t_i) \right] \prod_{i=1}^{r_1} \left[\sum_{j\in s_1} h_j(t_i) \right] \prod_{i=k_1+1}^{k_1+r_4} \left[\sum_{j\in s_1} h_j(t_i) \right]
\end{aligned}$$

$$\times \prod_{i=r_1+1}^{r_1+r_2} \left[\sum_{j \in s_2} h_j(t_i) \right] \prod_{i=k_1+r_4+1}^{k_1+r_4+r_5} \left[\sum_{j \in s_2} h_j(t_i) \right] \prod_{i=r_1+r_2+1}^{k_1} \left[\sum_{j \in s_{12}} h_j(t_i) \right]$$

$$\times \prod_{i=k_1+r_4+r_5+1}^{k_1+k_2} \left[\sum_{j \in s_{12}} h_j(t_i) \right] [R(\tau_{k_1})]^{n_1} [R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2},$$

where C_2^+ is a positive constant.

Proof. Supposed that $C_1^+, C_1'^+, C_2^+$ are positive constants, the joint density function of $\tau_1, \tau_2, \cdots, \tau_{k_1}$ is

$$g(\tau_1, \tau_2, \cdots, \tau_{k_1}) = C_1^+ \prod_{i=1}^{k_1} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1})]^{n-k_1}.$$

When $\tau_1, \tau_2, \cdots, \tau_{k_1}$ are given, $\tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}$ are the first k_2 observation values from the censored distribution with sample size $n - k_1 - n_1$, which follow a left censored distribution and its density function $q_1(\tau)$ and survival function $R_1(\tau)$ are respectively

$$q_1(\tau) = \frac{q(\tau)}{R(\tau_{k_1})}, R_1(\tau) = \frac{R(\tau)}{R(\tau_{k_1})}, \tau \geq \tau_{k_1}.$$

Then we have

$$\begin{aligned}
& g(\tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2} | \tau_1, \tau_2, \cdots, \tau_{k_1}) \\
=& C_1'^+ \prod_{i=k_1+1}^{k_1+k_2} [q_1(\tau_i)] [R_1(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2} \\
=& C_1'^+ \frac{\prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}}{[R(\tau_{k_1})]^{k_2} [R(\tau_{k_1})]^{n-k_1-n_1-k_2}} \\
=& C_1'^+ \frac{\prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}}{[R(\tau_{k_1})]^{n-k_1-n_1}} \\
& g(\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}) \\
=& g(\tau_1, \tau_2, \cdots, \tau_{k_1}) g(\tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2} | \tau_1, \tau_2, \cdots, \tau_{k_1}) \\
=& C_2^+ \prod_{i=1}^{k_1} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1})]^{n_1} \\
& \times \prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}.
\end{aligned}$$

Therefore, under the situation that n series systems composed of two units are put in simple successive censored life test, the likelihood function is

$$\begin{aligned}
&L(\text{data})\\
=&C_2^+\prod_{i=1}^{k_1}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}\prod_{i=k_1+1}^{k_1+k_2}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}\\
&\times[R(\tau_{k_1})]^{n_1}\,[R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}\\
=&C_2^+\prod_{i=1}^{r_1}\left\{\left[\sum_{j\in s_1}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\prod_{i=r_1+1}^{r_1+r_2}\left\{\left[\sum_{j\in s_2}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\\
&\times\prod_{i=r_1+r_2+1}^{k_1}\left\{\left[\sum_{j\in s_{12}}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\prod_{i=k_1+1}^{k_1+r_4}\left\{\left[\sum_{j\in s_1}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\\
&\times\prod_{i=k_1+r_4+1}^{k_1+r_4+r_5}\left\{\left[\sum_{j\in s_2}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\prod_{i=k_1+r_4+r_5+1}^{k_1+k_2}\left\{\left[\sum_{j\in s_{12}}h_j(t_i)\right]\prod_{l=1}^{2}\bar{F}_l(t_i)\right\}\\
&\quad\times[R(\tau_{k_1})]^{n_1}\,[R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}\\
=&C_2^+\prod_{i=1}^{k_1+k_2}\left[\prod_{l=1}^{2}\bar{F}_l(t_i)\right]\prod_{i=1}^{r_1}\left[\sum_{j\in s_1}h_j(t_i)\right]\prod_{i=k_1+1}^{k_1+r_4}\left[\sum_{j\in s_1}h_j(t_i)\right]\prod_{i=r_1+1}^{r_1+r_2}\left[\sum_{j\in s_2}h_j(t_i)\right]\\
&\times\prod_{i=k_1+r_4+1}^{k_1+r_4+r_5}\left[\sum_{j\in s_2}h_j(t_i)\right]\prod_{i=r_1+r_2+1}^{k_1}\left[\sum_{j\in s_{12}}h_j(t_i)\right]\\
&\times\prod_{i=k_1+r_4+r_5+1}^{k_1+k_2}\left[\sum_{j\in s_{12}}h_j(t_i)\right][R(\tau_{k_1})]^{n_1}\,[R(\tau_{k_1+k_2})]^{n-k_1-n_1-k_2}.
\end{aligned}$$

□

3.2. Statistical analysis of series system composed of two units with constant failure rate

Supposed that the life of unit 1 is X and its failure rate is the constant α_1 and the life of unit 2 is Y and its failure rate is the constant α_2, X, Y are mutually independent and the life of series system is denoted by T, $T=\min(X,Y)$. Such n series systems are put in simple successive censored life test. For $t\geq 0$, we have

$$\begin{aligned}
P(T\leq t)&=1-P(T>t)=1-P(X>t,Y>t)\\
&=1-P(X>t)P(Y>t)=1-e^{-\alpha_1 t}e^{-\alpha_2 t}=1-e^{-(\alpha_1+\alpha_2)t},\\
P(T>t)&=e^{-(\alpha_1+\alpha_2)t}.
\end{aligned}$$

The likelihood function is $L(\text{data},\alpha_1,\alpha_2)$,

$$\begin{aligned}
L(\text{data},\alpha_1,\alpha_2)=&C_2^+\prod_{i=1}^{k_1+k_2}\left[e^{-(\alpha_1+\alpha_2)t_i}\right]\alpha_1^{r_1+r_4}\alpha_2^{r_2+r_5}(\alpha_1+\alpha_2)^{r_3+r_6}\\
&\times e^{-n_1(\alpha_1+\alpha_2)\tau_{k_1}}e^{-(n-k_1-n_1-k_2)(\alpha_1+\alpha_2)\tau_{k_1+k_2}}
\end{aligned}$$

$$
\begin{aligned}
&= C_2^{+} e^{-(\alpha_1+\alpha_2)\sum\limits_{i=1}^{k_1+k_2} t_i} \alpha_1^{r_1+r_4} \alpha_2^{r_2+r_5} (\alpha_1+\alpha_2)^{r_3+r_6} \\
&\quad \times e^{-n_1(\alpha_1+\alpha_2)\tau_{k_1}} e^{-(n-k_1-n_1-k_2)(\alpha_1+\alpha_2)\tau_{k_1+k_2}}, \\
\ln L(\text{data}, \alpha_1, \alpha_2) &= \ln C_2^{+} - (\alpha_1+\alpha_2) \sum_{i=1}^{k_1+k_2} t_i + (r_1+r_4)\ln\alpha_1 \\
&\quad + (r_2+r_5)\ln\alpha_2 + (r_3+r_6)\ln(\alpha_1+\alpha_2) \\
&\quad - n_1(\alpha_1+\alpha_2)\tau_{k_1} - (n-k_1-n_1-k_2)(\alpha_1+\alpha_2)\tau_{k_1+k_2}, \\
\frac{\partial \ln L(\text{data}, \alpha_1, \alpha_2)}{\partial \alpha_1} &= -\sum_{i=1}^{k_1+k_2} t_i + \frac{r_1+r_4}{\alpha_1} + \frac{r_3+r_6}{\alpha_1+\alpha_2} - n_1\tau_{k_1} \\
&\quad - (n-k_1-n_1-k_2)\,\tau_{k_1+k_2}, \\
\frac{\partial \ln L(\text{data}, \alpha_1, \alpha_2)}{\partial \alpha_2} &= -\sum_{i=1}^{k_1+k_2} t_i + \frac{r_2+r_5}{\alpha_2} + \frac{r_3+r_6}{\alpha_1+\alpha_2} - n_1\tau_{k_1} \\
&\quad - (n-k_1-n_1-k_2)\,\tau_{k_1+k_2}.
\end{aligned}
$$

Take $\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial \alpha_1} = 0$, $\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial \alpha_2} = 0$, and we obtain following equation set

$$
\begin{cases}
-\sum\limits_{i=1}^{k_1+k_2} t_i + \frac{r_1+r_4}{\alpha_1} + \frac{r_3+r_6}{\alpha_1+\alpha_2} - n_1\tau_{k_1} - (n-k_1-n_1-k_2)\,\tau_{k_1+k_2} = 0, \\
-\sum\limits_{i=1}^{k_1+k_2} t_i + \frac{r_2+r_5}{\alpha_2} + \frac{r_3+r_6}{\alpha_1+\alpha_2} - n_1\tau_{k_1} - (n-k_1-n_1-k_2)\,\tau_{k_1+k_2} = 0.
\end{cases}
$$

By solving above equation set, the maximum likelihood estimate $\hat{\alpha}_1, \hat{\alpha}_2$ of parameters α_1, α_2 are respectively

$$
\begin{aligned}
\hat{\alpha}_1 &= \frac{r_1+r_4}{r_1+r_4+r_2+r_5} \frac{k_1+k_2}{\sum\limits_{i=1}^{k_1+k_2} t_i + n_1\tau_{k_1} + (n-k_1-n_1-k_2)\,\tau_{k_1+k_2}}, \\
\hat{\alpha}_2 &= \frac{r_2+r_5}{r_1+r_4+r_2+r_5} \frac{k_1+k_2}{\sum\limits_{i=1}^{k_1+k_2} t_i + n_1\tau_{k_1} + (n-k_1-n_1-k_2)\,\tau_{k_1+k_2}}.
\end{aligned}
$$

When $\alpha_1 = \alpha_2 = \alpha$, the likelihood function is

$$
\begin{aligned}
L(\text{data}, \alpha) &= C_2^{+} e^{-2\alpha \sum\limits_{i=1}^{k_1+k_2} t_i} \alpha^{r_1+r_4+r_2+r_5} (2\alpha)^{r_3+r_6} \\
&\quad \times e^{-2\alpha n_1\tau_{k_1}} e^{-2\alpha(n-k_1-n_1-k_2)\tau_{k_1+k_2}} \\
&= C_2^{+} e^{-2\alpha \sum\limits_{i=1}^{k_1+k_2} t_i} 2^{r_3+r_6} \alpha^{k_1+k_2} e^{-2\alpha n_1\tau_{k_1}} e^{-2\alpha(n-k_1-n_1-k_2)\tau_{k_1+k_2}}, \\
\ln L(\text{data}, \alpha) &= \ln C_2^{+} - 2\alpha \sum_{i=1}^{k_1+k_2} t_i + (r_3+r_6)\ln 2 + (k_1+k_2)\ln\alpha - 2\alpha n_1\tau_{k_1} \\
&\quad - 2\alpha\,(n-k_1-n_1-k_2)\,\tau_{k_1+k_2},
\end{aligned}
$$

$$\frac{d\ln L(\text{data},\alpha)}{d\alpha}=-2\sum_{i=1}^{k_1+k_2}t_i+\frac{k_1+k_2}{\alpha}-2n_1\tau_{k_1}-2\left(n-k_1-n_1-k_2\right)\tau_{k_1+k_2}.$$

Take $\frac{d\ln L(\text{data},\alpha)}{d\alpha}=0$, and we obtain following equation

$$-2\sum_{i=1}^{k_1+k_2}t_i+\frac{k_1+k_2}{\alpha}-2n_1\tau_{k_1}-2\left(n-k_1-n_1-k_2\right)\tau_{k_1+k_2}=0.$$

By solving above equation, the maximum likelihood estimate $\hat{\alpha}$ of parameter α is

$$\hat{\alpha}=\frac{1}{2}\frac{k_1+k_2}{\sum\limits_{i=1}^{k_1+k_2}t_i+n_1\tau_{k_1}+\left(n-k_1-n_1-k_2\right)\tau_{k_1+k_2}}.$$

Then we use the method of likelihood ratio to construct interval estimate. That is, under $H_0:\theta=\theta_0$, the asymptotic distribution of $\Lambda=-2\ln\left[\frac{L(\theta_0)}{L(\hat{\theta})}\right]$ is $\chi^2(k)$, where θ is k-dimension parameter.

If θ is divided into $\theta=(\theta_1,\theta_2)'$, considering $H_0:\theta_1=\theta_{10}$, the asymptotic distribution of $\Lambda=-2\ln\left[\frac{L(\theta_{10},\tilde{\theta}_2(\theta_{10}))}{L(\hat{\theta}_1,\hat{\theta}_2)}\right]$ is $\chi^2(p)$, where θ_1 is p- dimension vector and $\tilde{\theta}_2(\theta_{10})$ is the maximum likelihood estimate of θ_2 under H_0. Further, if parameter θ is one-dimensional, under the hypothesis $H_0:\theta=\theta_0$, the likelihood ratio statistic $\Lambda=-2\ln\left[\frac{L(\theta_0)}{L(\hat{\theta})}\right]$ approximately follows $\chi^2(1)$, where $L(\theta)$ is likelihood function. Significance test regards Λ as $\chi^2(1)$, and the larger value of Λ will result in rejecting H_0. The confidence interval of parameter θ can be obtained by reversing this test. The confidence interval of parameter θ is the set of θ_0 which satisfies $\Lambda\le x^2_\alpha(1)$. In many situations, it indicates that it is very good to approximate Λ by using χ^2, even in the small sample situation. Such confidence interval is very close to the coverage probability. For the given confidence level, the approximate interval estimate can be obtained by using the method of likelihood estimate.

Example 3.1. Take a sample with sample size $n=30$, $k_1=15, r_1=3, r_2=8$, $n_1=3$, $k_2=10, r_4=2, r_5=5$ The failure rates of two units are respectively $\alpha_1=0.3, \alpha_2=0.7$. The failure data generated by Monte-Carlo simulation are shown in Table 2.

Table 2. Failure Data of Example 3.1 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i=\{1\}$	0.4889,0.1987,0.2535
	$s_i=\{2\}$	0.0045,0.0215,0.3567,0.4677,0.4378,0.1902 0.3636,0.3712
	$s_i=\{1,2\}$	0.7105,0.2068,0.0943,0.6200
2	$s_i=\{1\}$	0.7760,1.3733
	$s_i=\{2\}$	1.2968,0.9885,1.9497,2.0954,1.2176
	$s_i=\{1,2\}$	1.9114,1.7256,1.0921

Take $\tau_1=0.7105$ and $\tau_2=2.0954$. We can obtain $\hat{\alpha}_1=0.2720$ and $\hat{\alpha}_2=0.7071$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of α_1 is [0.1788,0.3929], and the approximate interval estimate of α_2 is [0.4649,1.0216].

Example 3.2. Take a sample with sample size $n = 50$, $k_1 = 20, r_1 = 12, r_2 = 3$, $n_1 = 4$, $k_2 = 25, r_4 = 16, r_5 = 4$. The failure rates of two units are respectively $\alpha_1 = 0.8, \alpha_2 = 0.2$. The failure data generated by Monte-Carlo simulation are shown in Table 3.

Table 3. Failure Data of Example 3.2 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.1424,0.0903,0.3951,0.1396,0.0945,0.0227 0.3921,0.2103,0.1130,0.1443,0.1327,0.3304
	$s_i = \{2\}$	0.5540,0.4054,0.2833
	$s_i = \{1,2\}$	0.1694,0.2428,0.0015,0.3621,0.0885
2	$s_i = \{1\}$	1.0298,0.6336,1.2707,1.9890,0.7929,0.7839 1.9089,2.7638,1.8484,1.3079,0.9072,1.9543 1.4315,2.1746,0.8482,1.5311
	$s_i = \{2\}$	1.5467,1.1269,1.0540,2.1112
	$s_i = \{1,2\}$	1.0447,0.8464,1.1456,1.0617,1.9926

Take $\tau_1 = 0.5540$ and $\tau_2 = 2.7638$. We can obtain $\hat{\alpha}_1 = 0.8108$ and $\hat{\alpha}_2 = 0.2027$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of α_1 is [0.5964,1.0713], and the approximate interval estimate of α_2 is [0.1491,0.2678].

3.3. Statistical analysis of series system composed of two units with linear failure rate (through the origin)

Supposed that the life of unit 1 is X and its failure rate is the linear function $\beta_1 t$ of time t and the life of unit 2 is Y and its failure rate is the linear function $\beta_2 t$ of time t, X, Y are mutually independent and the life of series system is denoted by T, $T = \min(X, Y)$. Such n series systems are put in simple successive censored life test. For $t \geq 0$, we have

$$\begin{aligned} P(T \leq t) &= 1 - P(T > t) = 1 - P(X > t, Y > t) = 1 - P(X > t)P(Y > t) \\ &= 1 - e^{-\frac{1}{2}\beta_1 t^2} e^{-\frac{1}{2}\beta_2 t^2} = 1 - e^{-\frac{1}{2}(\beta_1+\beta_2)t^2}, \\ P(T > t) &= e^{-\frac{1}{2}(\beta_1+\beta_2)t^2}. \end{aligned}$$

The likelihood function is

$$\begin{aligned} L(\text{data}, \beta_1, \beta_2) =& C_2^+ \prod_{i=1}^{k_1+k_2} t_i \prod_{i=1}^{k_1+k_2} \left[e^{-\frac{1}{2}(\beta_1+\beta_2)t_i^2}\right] \beta_1^{r_1+r_4} \beta_2^{r_2+r_5} (\beta_1+\beta_2)^{r_3+r_6} \\ & \times e^{-\frac{1}{2}n_1(\beta_1+\beta_2)\tau_{k_1}^2} e^{-\frac{1}{2}(n-k_1-n_1-k_2)(\beta_1+\beta_2)\tau_{k_1+k_2}^2} \\ =& C_2^+ \prod_{i=1}^{k_1+k_2} t_i \beta_1^{r_1+r_4} \beta_2^{r_2+r_5} (\beta_1+\beta_2)^{r_3+r_6} e^{-\frac{1}{2}(\beta_1+\beta_2)\sum\limits_{i=1}^{k_1+k_2} t_i^2} \\ & \times e^{-\frac{1}{2}n_1(\beta_1+\beta_2)\tau_{k_1}^2} e^{-\frac{1}{2}(n-k_1-n_1-k_2)(\beta_1+\beta_2)\tau_{k_1+k_2}^2}, \end{aligned}$$

$$\begin{aligned}\ln L(\text{data}, \beta_1, \beta_2) = &\ln C_2^+ + \sum_{i=1}^{k_1+k_2} \ln t_i + (r_1 + r_4) \ln \beta_1 + (r_2 + r_5) \ln \beta_2 \\ &+ (r_3 + r_6) \ln(\beta_1 + \beta_2) - \frac{1}{2} (\beta_1 + \beta_2) \sum_{i=1}^{k_1+k_2} t_i^2 - \frac{1}{2} n_1 (\beta_1 + \beta_2) \tau_{k_1}^2 \\ &- \frac{1}{2} (n - k_1 - n_1 - k_2) (\beta_1 + \beta_2) \tau_{k_1+k_2}^2.\end{aligned}$$

Take $\frac{\partial \ln L(\text{data}, \beta_1, \beta_2)}{\partial \beta_1} = 0$, $\frac{\partial \ln L(\text{data}, \beta_1, \beta_2)}{\partial \beta_2} = 0$, and we obtain following equation set

$$\begin{cases} \frac{r_1+r_4}{\beta_1} + \frac{r_3+r_6}{\beta_1+\beta_2} - \frac{1}{2} \sum\limits_{i=1}^{k_1+k_2} t_i^2 - \frac{1}{2} n_1 \tau_{k_1}^2 - \frac{1}{2} (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2 = 0, \\ \frac{r_2+r_5}{\beta_2} + \frac{r_3+r_6}{\beta_1+\beta_2} - \frac{1}{2} \sum\limits_{i=1}^{k_1+k_2} t_i^2 - \frac{1}{2} n_1 \tau_{k_1}^2 - \frac{1}{2} (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2 = 0. \end{cases}$$

By solving above equation set, the maximum likelihood estimate $\hat{\beta}_1, \hat{\beta}_2$ of parameters β_1, β_2 are respectively

$$\hat{\beta}_1 = \frac{2 (r_1 + r_4) (k_1 + k_2)}{(r_1 + r_4 + r_2 + r_5) \left[\sum\limits_{i=1}^{k_1+k_2} t_i^2 + n_1 \tau_{k_1}^2 + (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2 \right]},$$

$$\hat{\beta}_2 = \frac{2 (r_2 + r_5) (k_1 + k_2)}{(r_1 + r_4 + r_2 + r_5) \left[\sum\limits_{i=1}^{k_1+k_2} t_i^2 + n_1 \tau_{k_1}^2 + (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2 \right]}.$$

When $\beta_1 = \beta_2 = \beta$, the likelihood function is

$$\begin{aligned}L(\text{data}, \beta) = &C_2^+ \prod_{i=1}^{k_1+k_2} t_i \beta^{r_1+r_4} \beta^{r_2+r_5} (2\beta)^{r_3+r_6} e^{-\beta \sum\limits_{i=1}^{k_1+k_2} t_i^2} \\ &\times e^{-n_1 \beta \tau_{k_1}^2} e^{-(n-k_1-n_1-k_2)\beta \tau_{k_1+k_2}^2},\end{aligned}$$

$$\begin{aligned}\ln L(\text{data}, \beta) = &\ln C_2^+ + \sum_{i=1}^{k_1+k_2} \ln t_i + (r_1 + r_4) \ln \beta + (r_2 + r_5) \ln \beta \\ &+ (r_3 + r_6) \ln 2 + (r_3 + r_6) \ln \beta - \beta \sum_{i=1}^{k_1+k_2} t_i^2 \\ &- n_1 \beta \tau_{k_1}^2 - (n - k_1 - n_1 - k_2) \beta \tau_{k_1+k_2}^2.\end{aligned}$$

Take $\frac{d \ln L(\text{data}, \beta)}{d\beta} = 0$, and we obtain following equation

$$\frac{1}{\beta} \sum_{i=1}^{6} r_i - \left[\sum_{i=1}^{k_1+k_2} t_i^2 + n_1 \tau_{k_1}^2 + (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2 \right] = 0.$$

By solving above equation, the maximum likelihood estimate $\hat{\beta}$ of parameter β is

$$\hat{\beta} = \frac{\sum\limits_{i=1}^{6} r_i}{\sum\limits_{i=1}^{k_1+k_2} t_i^2 + n_1 \tau_{k_1}^2 + (n - k_1 - n_1 - k_2) \tau_{k_1+k_2}^2}.$$

Similarly, for the given confidence level, the approximate interval estimate can be obtained by using the method of likelihood estimate.

Example 3.3. Take a sample with sample size $n = 30$, $k_1 = 15, r_1 = 3, r_2 = 8$, $n_1 = 3$, $k_2 = 10, r_4 = 2, r_5 = 5$. The failure rates of two units are respectively $\beta_1 = 0.3, \beta_2 = 0.7$. The failure data generated by Monte-Carlo simulation are shown in Table 4.

Table 4. Failure Data of Example 3.3 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.4953 0.4180 0.2192
	$s_i = \{2\}$	0.6564,0.2372,0.7541,0.7249,1.1956,0.3422 0.9769,0.8106
	$s_i = \{1, 2\}$	1.1977,0.8176,0.6060,0.4481
2	$s_i = \{1\}$	1.4837,1.5913
	$s_i = \{2\}$	1.8266,1.2948,1.3446,1.2467,2.2476
	$s_i = \{1, 2\}$	2.1972,1.9985,1.3539

Take $\tau_1 = 1.1977$ and $\tau_2 = 2.2476$. We can obtain $\hat{\beta}_1 = 0.2718$ and $\hat{\beta}_2 = 0.7067$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of β_1 is [0.1787,0.3927], and the approximate interval estimate of β_2 is [0.4646,1.0210].

Example 3.4. Take a sample with sample size $n = 50$, $k_1 = 20, r_1 = 12, r_2 = 3$, $n_1 = 4$, $k_2 = 25, r_4 = 16, r_5 = 4$ The failure rates of two units are respectively $\beta_1 = 0.8, \beta_2 = 0.2$. The failure data generated by Monte-Carlo simulation are shown in Table 5.

Table 5. Failure Data of Example 3.4 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.6736,0.5309,0.8719,0.8313,0.9640,0.2261 0.3309,0.4422,0.6061,0.7097,0.3752,0.8517
	$s_i = \{2\}$	0.7862,0.9475,0.8827
	$s_i = \{1, 2\}$	0.6745,0.4890,1.0233,0.9490,0.4627
2	$s_i = \{1\}$	2.7380,1.5915,1.0630,2.0555,1.5788,1.6056 2.0498,1.4826,1.4386,1.3401,1.5821,1.4145 1.3194,1.5537,1.0352,1.0766
	$s_i = \{2\}$	1.4232,1.2368,1.5040,1.0554
	$s_i = \{1, 2\}$	1.3536,2.9902,1.8923,1.3090,1.5705

Take $\tau_1 = 1.0233$ and $\tau_2 = 2.9902$. We can obtain $\hat{\beta}_1 = 0.7950$ and $\hat{\beta}_2 = 0.1987$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of β_1 is [0.5848,1.0504], and the approximate interval estimate of β_2 is [0.1462,0.2626].

4. Statistical Analysis of Series System for Masked Data under Multiple Successive Censored Life Test

4.1. Likelihood function of series system for masked data under multiple successive censored life test

Considering that n series systems composed of two units are put in multiple successive censored life test, when k_1 systems are failed, n_1 systems randomly selected from $n-k_1$ systems that are not failed are evacuated from testing site, and remaining $n-k_1-n_1$ systems continue the test. When other k_2 systems are failed, n_2 systems randomly selected from $n-k_1-n_1-k_2$ systems that are not failed are evacuated from testing site, and remaining $n-k_1-n_1-k_2-n_2$ systems continue the test. The test is going on until k_m systems are failed for the m^{th} time, then the test is stopped and the remaining systems that are not failed are all evacuated from testing site. Thus, the order failure time of all failure systems are $\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}, \cdots, \tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots \tau_{\sum_{\eta=0}^{m} k_\eta}$. The test data is shown as Table 6, where $k_0=0, \tau_0=0, k_i \geq 1, n_i \geq 0, i=1,2,\cdots,m$ and n_m is the number of systems that are not failed when the test is over at last.

Table 6. Series System for Masked Data under Multiple Successive Censored Life Test

Sample size of test	Order failure time	Failure number	Sample size of evacuation
n	$\tau_1, \tau_2, \cdots, \tau_{k_1}$	k_1	n_1
$n-k_1-n_1$	$\tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}$	k_2	n_2
$n-k_1-n_1$ $-k_2-n_2$	$\tau_{k_1+k_2+1}, \tau_{k_1+k_2+2}, \cdots,$ $\tau_{k_1+k_2+k_3}$	k_3	n_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta)$	$\tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2},$ $\cdots, \tau_{\sum_{\eta=0}^{m} k_\eta}$	k_m	$n_m = n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m$

Among k_1 failure systems in the first test, r_1 systems belong to class s_1 and their failure time are $t_1, t_2, \cdots, t_{r_1}$; r_2 systems belong to class s_2 and their failure time are $t_{r_1+1}, t_{r_1+2}, \cdots, t_{r_1+r_2}$; r_3 systems belong to classs_{12} and their failure time are $t_{r_1+r_2+1}, t_{r_1+r_2+2}, \cdots, t_{r_1+r_2+r_3}$, where $r_1+r_2+r_3=k_1$. Among k_2 failure systems in the second test, r_4 systems belong to class s_1 and their failure time are $t_{k_1+1}, t_{k_1+2}, \cdots, t_{k_1+r_4}$; r_5 systems belong to class s_2 and their failure time are $t_{k_1+r_4+1}, t_{k_1+r_4+2}, \cdots, t_{k_1+r_4+r_5}$; r_6 systems belong to classs_{12} and their failure time are $t_{k_1+r_4+r_5+1}, t_{k_1+r_4+r_5+2}, \cdots, t_{k_1+r_4+r_5+r_6}$, where $r_4+r_5+r_6=k_2$. Among k_m failure systems in the m^{th} test, r_{3m-2} systems belong to class s_1 and their failure time are $t_{\sum_{\eta=0}^{m-1} k_\eta+1}, t_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}}$; r_{3m-1} systems belong to class s_2 and their failure time are $t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+1}, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+2}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+r_{3m-1}}$;

r_{3m} systems belong to classs_{12} and their failure time are $t_{\sum_{\eta=0}^{m-1} k_\eta + r_{3m-2}+r_{3m-1}+1}$, $t_{\sum_{\eta=0}^{m-1} k_\eta + r_{3m-2}+r_{3m-1}+2}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta + r_{3m-2}+r_{3m-1}+r_{3m}}$, where $r_{3m-2}+r_{3m-1}+r_{3m} = k_m$. The data form is shown in Table 7, where A denotes the number of failure system in each test and B denotes system failure time according to the classification of failure reason.

Table 7. Data Form of Series System for Masked Data under Multiple Successive Censored Life Test

B A	Failure time that belongs to class s_1	Failure time that belongs to class s_2	Failure time that belongs to class s_{12}
k_1	$t_1, t_2, \cdots, t_{r_1}$	$t_{r_1+1}, t_{r_1+2}, \cdots, t_{r_1+r_2}$	$t_{r_1+r_2+1}, t_{r_1+r_2+2}, \cdots, t_{r_1+r_2+r_3}$
k_2	$t_{k_1+1}, t_{k_1+2}, \cdots, t_{k_1+r_4}$	$t_{k_1+r_4+1}, t_{k_1+r_4+2}, \cdots, t_{k_1+r_4+r_5}$	$t_{k_1+r_4+r_5+1}, t_{k_1+r_4+r_5+2}, \cdots, t_{k_1+r_4+r_5+r_6}$
$\vdots$	$\vdots$		
k_m	$t_{\sum_{\eta=0}^{m-1} k_\eta+1}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}}$	$t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+1}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+r_{3m-1}}$	$t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+r_{3m-1}+1}, \cdots, t_{\sum_{\eta=0}^{m-1} k_\eta+r_{3m-2}+r_{3m-1}+r_{3m}}$

Theorem 4.1. *Supposed that density function of the system is $q(\tau)$ and survival function is $R(\tau)$, the density function, distribution function, reliability function and failure rate function of the j^{th} unit life in each series system are respectively $f_j(t), F_j(t), \bar{F}_j(t)$ and $h_j(t)$. Besides, assuming that the life distributions of units are mutually independent and $J_j = \{1, 2, \cdots, j-1, j+1, \cdots, J\}$, n series systems composed of two units are put in multiple successive censored life test and the data form is shown as Table 7. Then the likelihood function of this situation is*

$$
\begin{aligned}
L(data) =& C_m^+ \prod_{i=1}^{k_1} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1})]^{n_1} \prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \\
& \times [R(\tau_{k_1+k_2})]^{n_2} \cdots \prod_{i=\sum_{\eta=0}^{m-2} k_\eta+1}^{\sum_{\eta=0}^{m-2} k_\eta + k_{m-1}} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \left[R(\tau_{\sum_{\eta=0}^{m-1} k_\eta}) \right]^{n_{m-1}} \\
& \times \prod_{i=\sum_{\eta=0}^{m-1} k_\eta+1}^{\sum_{\eta=0}^{m-1} k_\eta + k_m} \left\{ \sum_{j\in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \left[R(\tau_{\sum_{\eta=0}^{m} k_\eta}) \right]^{n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m},
\end{aligned}
$$

where C_m^+ is a positive constant.

Proof. When $m = 2$, the theorem can be proved by Theorem 3.1.

Supposed that the theorem is true for $m \geq 3$, we need prove that the theorem is also true for the $m+1^{th}$ test.

Since the joint density function of $\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}, \cdots,$ $\tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots \tau_{\sum_{\eta=0}^{m} k_\eta}$ is

$$\begin{aligned}
&g(\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}, \cdots, \tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots \tau_{\sum_{\eta=0}^{m} k_\eta}) \\
&= C_m^+ \prod_{i=1}^{k_1} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} [R(\tau_{k_1})]^{n_1} \prod_{i=k_1+1}^{k_1+k_2} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \\
&\times [R(\tau_{k_1+k_2})]^{n_2} \cdots \prod_{i=\sum_{\eta=0}^{m-1} k_\eta+1}^{\sum_{\eta=0}^{m-1} k_\eta+k_m} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \\
&\times \left[R(\tau_{\sum_{\eta=0}^{m} k_\eta}) \right]^{n-\sum_{\eta=0}^{m-1} (k_\eta+n_\eta)-k_m},
\end{aligned}$$

the life of remaining systems that are not failed in the test follow a left censored distribution when $\tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}, \cdots, \tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots,$ $\tau_{\sum_{\eta=0}^{m} k_\eta}$ are given, and its density function $q_m(\tau)$ and survival function $R_m(\tau)$ are respectively

$$q_m(\tau) = \frac{q(\tau)}{R(\tau_{\sum_{\eta=0}^{m} k_\eta})}, R_m(\tau) = \frac{R(\tau)}{R(\tau_{\sum_{\eta=0}^{m} k_\eta})}, \tau \geq \tau_{\sum_{\eta=0}^{m} k_\eta}.$$

Then $\tau_{\sum_{\eta=0}^{m} k_\eta+1}, \tau_{\sum_{\eta=0}^{m} k_\eta+2}, \cdots, \tau_{\sum_{\eta=0}^{m} k_\eta+k_{m+1}}$ are the first k_{m+1} observation values from this censored distribution with sample size $n - \sum_{\eta=0}^{m} (k_\eta + n_\eta)$. Thus, we have

$$\begin{aligned}
&g\Bigg(\tau_{\sum_{\eta=0}^{m} k_\eta+1}, \tau_{\sum_{\eta=0}^{m} k_\eta+2}, \cdots, \tau_{\sum_{\eta=0}^{m} k_\eta+k_{m+1}} | \tau_1, \tau_2, \cdots, \tau_{k_1}, \tau_{k_1+1}, \tau_{k_1+2}, \cdots, \tau_{k_1+k_2}, \cdots, \\
&\tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \tau_{\sum_{\eta=0}^{m-1} k_\eta+2}, \cdots \tau_{\sum_{\eta=0}^{m} k_\eta} \Bigg) \\
&= C_m'^+ \prod_{i=\sum_{\eta=0}^{m} k_\eta+1}^{\sum_{\eta=0}^{m} k_\eta+k_{m+1}} \left\{ \sum_{j \in s_i} [h_j(t_i)] \prod_{l=1}^{J} \bar{F}_l(t_i) \right\} \left[R(\tau_{\sum_{\eta=0}^{m+1} k_\eta}) \right]^{n-\sum_{\eta=0}^{m} (k_\eta+n_\eta)-k_{m+1}}
\end{aligned}$$

and

$$g(\tau_1, \cdots, \tau_{k_1}, \tau_{k_1+1}, \cdots, \tau_{k_1+k_2}, \cdots, \tau_{\sum_{\eta=0}^{m-1} k_\eta+1}, \cdots \tau_{\sum_{\eta=0}^{m} k_\eta}, \tau_{\sum_{\eta=0}^{m} k_\eta+1}, \cdots \tau_{\sum_{\eta=0}^{m+1} k_\eta})$$

$$=C_{m+1}^{+}\prod_{i=1}^{k_1}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}[R(\tau_{k_1})]^{n_1}\prod_{i=k_1+1}^{k_1+k_2}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}$$

$$\times[R(\tau_{k_1+k_2})]^{n_2}\cdots\prod_{i=\sum_{\eta=0}^{m-1}k_\eta+1}^{\sum_{\eta=0}^{m-1}k_\eta+k_m}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}\left[R(\tau_{\sum_{\eta=0}^{m}k_\eta})\right]^{n_m}$$

$$\times\prod_{i=\sum_{\eta=0}^{m}k_\eta+1}^{\sum_{\eta=0}^{m}k_\eta+k_{m+1}}\left\{\sum_{j\in s_i}[h_j(t_i)]\prod_{l=1}^{J}\bar{F}_l(t_i)\right\}\left[R(\tau_{\sum_{\eta=0}^{m+1}k_\eta})\right]^{n-\sum_{\eta=0}^{m}(k_\eta+n_\eta)-k_{m+1}},$$

where $C_m'^{+}$ and C_{m+1}^{+} are positive constants.

According to the induction method, it can be proved that Theorem 4.1 is true. □

4.2. Statistical analysis of series system composed of two units with constant failure rate

Supposed that the life of unit 1 is X and its failure rate is the constant α_1 and the life of unit 2 is Y and its failure rate is the constant α_2, X, Y are mutually independent and the life of series system is denoted by T, $T=\min(X,Y)$. Such n series systems are put in multiple successive censored life test. For $t\geq 0$, we have

$$P(T\leq t)=1-P(T>t)=1-P(X>t,Y>t)=1-P(X>t)P(Y>t)$$
$$=1-e^{-\alpha_1 t}e^{-\alpha_2 t}=1-e^{-(\alpha_1+\alpha_2)t},$$
$$P(T>t)=e^{-(\alpha_1+\alpha_2)t}.$$

The likelihood function is $L(\text{data},\alpha_1,\alpha_2)$

$$L(\text{data},\alpha_1,\alpha_2)$$

$$=C_m^{+}\prod_{i=1}^{\sum_{\eta=0}^{m}k_\eta}\left[e^{-(\alpha_1+\alpha_2)t_i}\right]\alpha_1^{r_1+r_4+\cdots+r_{3m-2}}\alpha_2^{r_2+r_5+\cdots+r_{3m-1}}(\alpha_1+\alpha_2)^{r_3+r_6+\cdots+r_{3m}}$$

$$\times e^{-n_1(\alpha_1+\alpha_2)\tau_{k_1}}e^{-n_2(\alpha_1+\alpha_2)\tau_{k_1+k_2}}\cdots e^{-\left[n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right](\alpha_1+\alpha_2)\tau_{\sum_{\eta=0}^{m}k_\eta}}$$

$$=C_m^{+}e^{-(\alpha_1+\alpha_2)\sum_{i=1}^{\sum_{\eta=0}^{m}k_\eta}t_i}\alpha_1^{r_1+r_4+\cdots+r_{3m-2}}\alpha_2^{r_2+r_5+\cdots+r_{3m-1}}(\alpha_1+\alpha_2)^{r_3+r_6+\cdots+r_{3m}}$$

$$\times e^{-n_1(\alpha_1+\alpha_2)\tau_{k_1}}e^{-n_2(\alpha_1+\alpha_2)\tau_{k_1+k_2}}\cdots e^{-\left[n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right](\alpha_1+\alpha_2)\tau_{\sum_{\eta=0}^{m}k_\eta}},$$

$$\ln L(\text{data},\alpha_1,\alpha_2)$$

$$=\ln C_m^{+}-(\alpha_1+\alpha_2)\sum_{i=1}^{\sum_{\eta=0}^{m}k_\eta}t_i+(r_1+r_4+\cdots+r_{3m-2})\ln\alpha_1$$

$$
\begin{aligned}
&+(r_2+r_5+\cdots+r_{3m-1})\ln\alpha_2+(r_3+r_6+\cdots+r_{3m})\ln(\alpha_1+\alpha_2)\\
&-n_1(\alpha_1+\alpha_2)\tau_{k_1}-n_2(\alpha_1+\alpha_2)\tau_{k_1+k_2}-\cdots\\
&-\left[n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right](\alpha_1+\alpha_2)\tau_{\sum\limits_{\eta=0}^{m}k_\eta},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial\alpha_1}\\
=&-\sum_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta} t_i+\frac{r_1+r_4+\cdots+r_{3m-2}}{\alpha_1}+\frac{r_3+r_6+\cdots+r_{3m}}{\alpha_1+\alpha_2}\\
&-n_1\tau_{k_1}-n_2\tau_{k_1+k_2}-\cdots-\left[n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right]\tau_{\sum\limits_{\eta=0}^{m}k_\eta},\\
&\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial\alpha_2}\\
=&-\sum_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta} t_i+\frac{r_2+r_5+\cdots+r_{3m-1}}{\alpha_2}+\frac{r_3+r_6+\cdots+r_{3m}}{\alpha_1+\alpha_2}\\
&-n_1\tau_{k_1}-n_2\tau_{k_1+k_2}-\cdots-\left[n-\sum_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right]\tau_{\sum\limits_{\eta=0}^{m}k_\eta}.
\end{aligned}
$$

Take $\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial\alpha_1}=0$, $\frac{\partial \ln L(\text{data},\alpha_1,\alpha_2)}{\partial\alpha_2}=0$, and we obtain following equation set

$$
\begin{cases}
-\sum\limits_{i=1}^{\sum_{\eta=0}^{m}k_\eta} t_i+\frac{r_1+r_4+\cdots+r_{3m-2}}{\alpha_1}+\frac{r_3+r_6+\cdots+r_{3m}}{\alpha_1+\alpha_2}\\
-n_1\tau_{k_1}-n_2\tau_{k_1+k_2}-\cdots-\left[n-\sum\limits_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right]\tau_{\sum_{\eta=0}^{m}k_\eta}=0,\\
-\sum\limits_{i=1}^{\sum_{\eta=0}^{m}k_\eta} t_i+\frac{r_2+r_5+\cdots+r_{3m-1}}{\alpha_2}+\frac{r_3+r_6+\cdots+r_{3m}}{\alpha_1+\alpha_2}\\
-n_1\tau_{k_1}-n_2\tau_{k_1+k_2}-\cdots-\left[n-\sum\limits_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right]\tau_{\sum_{\eta=0}^{m}k_\eta}=0.
\end{cases}
$$

By solving above equation set, the maximum likelihood estimate $\hat\alpha_1,\hat\alpha_2$ of parameters α_1,α_2 are respectively

$$
\begin{aligned}
\hat\alpha_1=&\frac{r_1+r_4+\cdots+r_{3m-2}}{r_1+r_4+\cdots+r_{3m-2}+r_2+r_5+\cdots+r_{3m-1}}\\
&\times\frac{\sum\limits_{\eta=0}^{m}k_\eta}{\sum\limits_{i=1}^{\sum_{\eta=0}^{m}k_\eta} t_i+n_1\tau_{k_1}+n_2\tau_{k_1+k_2}+\cdots+\left[n-\sum\limits_{\eta=0}^{m-1}(k_\eta+n_\eta)-k_m\right]\tau_{\sum_{\eta=0}^{m}k_\eta}},
\end{aligned}
$$

$$\hat{\alpha}_2 = \frac{r_2 + r_5 + \cdots + r_{3m-1}}{r_1 + r_4 + \cdots + r_{3m-2} + r_2 + r_5 + \cdots + r_{3m-1}} \times \frac{\sum_{\eta=0}^{m} k_\eta}{\sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i + n_1\tau_{k_1} + n_2\tau_{k_1+k_2} + \cdots + \left[n - \sum_{\eta=0}^{m-1}(k_\eta + n_\eta) - k_m\right]\tau_{\sum_{\eta=0}^{m} k_\eta}}.$$

When $\alpha_1 = \alpha_2 = \alpha$, the likelihood function is

$$\begin{aligned} L(\text{data}, \alpha) ={}& C_m^{+} 2^{r_3+r_6+\cdots+r_{3m}} e^{-2\alpha \sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i} \alpha^{\sum_{i=1}^{3m} r_i} e^{-2n_1\alpha\tau_{k_1}} \\ &\times e^{-2n_2\alpha\tau_{k_1+k_2}} \cdots e^{-2\left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right]\alpha\tau_{\sum_{\eta=0}^{m} k_\eta}}, \end{aligned}$$

$$\begin{aligned} \ln L(\text{data}, \alpha) ={}& \ln C_m^{+} + (r_3 + r_6 + \cdots + r_{3m})\ln 2 - 2\alpha \sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i + \sum_{i=1}^{3m} r_i \ln\alpha \\ &- 2n_1\alpha\tau_{k_1} - 2n_2\alpha\tau_{k_1+k_2} - \cdots - 2\left[n - \sum_{\eta=0}^{m-1}(k_\eta + n_\eta) - k_m\right]\alpha\tau_{\sum_{\eta=0}^{m} k_\eta}. \end{aligned}$$

Take $\frac{d\ln L(\text{data},\alpha)}{d\alpha} = 0$, and we obtain following equation

$$\begin{aligned} &-2\sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i + \frac{1}{\alpha}\sum_{i=1}^{3m} r_i - 2n_1\tau_{k_1} - 2n_2\tau_{k_1+k_2} - \cdots \\ &-2\left[n - \sum_{\eta=0}^{m-1}(k_\eta + n_\eta) - k_m\right]\tau_{\sum_{\eta=0}^{m} k_\eta} = 0. \end{aligned}$$

By solving above equation, the maximum likelihood estimate $\hat{\alpha}$ of parameter α is

$$\hat{\alpha} = \frac{1}{2}\frac{\sum_{i=1}^{3m} r_i}{\sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i + n_1\tau_{k_1} + n_2\tau_{k_1+k_2} + \cdots + \left[n - \sum_{\eta=0}^{m-1}(k_\eta + n_\eta) - k_m\right]\tau_{\sum_{\eta=0}^{m} k_\eta}}.$$

Similarly, for the given confidence level, the approximate interval estimate can be obtained by using the method of likelihood estimate.

Example 4.1. Take a sample with sample size $n = 50$, $k_1 = 15, r_1 = 2, r_2 = 8$, $n_1 = 2$, $k_2 = 10, r_4 = 1, r_5 = 4$, $n_2 = 2$, $k_3 = 20, r_7 = 3, r_8 = 12$. The failure rates of two units are respectively $\alpha_1 = 0.2, \alpha_2 = 0.8$. The failure data generated by Monte-Carlo simulation are shown in Table 8.

Table 8. Failure Data of Example 4.1 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.1846,0.0120
	$s_i = \{2\}$	0.1219,0.0351,0.0060,0.0130,0.0098,0.0443 0.1302,0.1197
	$s_i = \{1, 2\}$	0.1558,0.1481,0.2104,0.0812,0.2008
2	$s_i = \{1\}$	0.3222
	$s_i = \{2\}$	0.4962,0.2909,0.4560,0.5013
	$s_i = \{1, 2\}$	0.3301,0.4680,0.4236,0.3954,0.3805
3	$s_i = \{1\}$	1.9587,2.8490,2.5376
	$s_i = \{2\}$	0.6373,0.8952,3.1753,0.8103,3.9137,0.9385 2.7328,3.6605,1.5347,1.5345,1.0900,0.8458
	$s_i = \{1, 2\}$	0.7411,1.5444,0.5300,0.9174,1.6165

Take $\tau_1 = 0.2104$, $\tau_2 = 0.5013$ and $\tau_3 = 3.9137$. We can obtain $\hat{\alpha}_1 = 0.1985$ and $\hat{\alpha}_2 = 0.7940$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of α_1 is [0.1460,0.2623], and the approximate interval estimate of α_2 is [0.5841,1.0491].

Example 4.2. Take a sample with sample size $n = 90$, $k_1 = 30, r_1 = 8, r_2 = 12$, $n_1 = 2$, $k_2 = 25, r_4 = 6, r_5 = 9$, $n_2 = 2$, $k_3 = 30, r_7 = 10, r_8 = 15$. The failure rates of two units are respectively $\alpha_1 = 1, \alpha_2 = 1.5$. The failure data generated by Monte-Carlo simulation are shown in Table 9.

Table 9. Failure Data of Example 4.2 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.0030,0.0451,0.0524,0.0867,0.0852,0.0248 0.0015,0.1595
	$s_i = \{2\}$	0.1268,0.0669,0.1185,0.0708,0.1262,0.0333 0.1065,0.1806,0.0556,0.0910,0.0384,0.0698
	$s_i = \{1, 2\}$	0.1118,0.1131,0.0029,0.1357,0.1656,0.0457 0.1139,0.0695,0.0823,0.1770
2	$s_i = \{1\}$	0.3748,0.3479,0.2692,0.2473,0.3738,0.3598
	$s_i = \{2\}$	0.1922,0.3915,0.3658,0.2778,0.2731,0.3162 0.2321,0.2625,0.2024
	$s_i = \{1, 2\}$	0.2083,0.3913,0.1913,0.2998,0.3429,0.4557 0.2606,0.3999,0.2469,0.3633
3	$s_i = \{1\}$	0.6252,1.3080,0.9998,1.0296,0.7108,0.5500 0.5395,0.7954,0.5979,0.7488
	$s_i = \{2\}$	0.5306,0.5319,0.6702,0.6060,1.0256,0.4831 0.4558,0.6885,0.6298,1.4121,0.9763,0.4572 1.0035,1.1605,0.6728
	$s_i = \{1, 2\}$	1.0485,0.5651,1.0927,0.6417,0.4948

Take $\tau_1 = 0.1806$, $\tau_2 = 0.4557$ and $\tau_3 = 1.4121$. We can obtain $\hat{\alpha}_1 = 0.9459$ and $\hat{\alpha}_2 = 1.4189$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of α_1 is [0.7589,1.1615], and the approximate interval estimate of α_2 is [1.1383,1.7423].

4.3. Statistical analysis of series system composed of two units with linear failure rate(through the origin)

Supposed that the life of unit 1 is X and its failure rate is the linear function $\beta_1 t$ of time t and the life of unit 2 is Y and its failure rate is the linear function $\beta_2 t$ of time t, X, Y are mutually independent and the life of series system is denoted by T, $T = \min(X, Y)$. Such n series systems are put in multiple successive censored life test. For $t \geq 0$, we have

$$
\begin{aligned}
P(T \leq t) &= 1 - P(T > t) = 1 - P(X > t, Y > t) = 1 - P(X > t)P(Y > t) \\
&= 1 - e^{-\frac{1}{2}\beta_1 t^2} e^{-\frac{1}{2}\beta_2 t^2} = 1 - e^{-\frac{1}{2}(\beta_1+\beta_2)t^2}, \\
P(T > t) &= e^{-\frac{1}{2}(\beta_1+\beta_2)t^2}.
\end{aligned}
$$

The likelihood function is $L(\text{data}, \beta_1, \beta_2)$

$$
\begin{aligned}
L(\text{data}, \beta_1, \beta_2) =& C_m^+ \prod_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i \prod_{i=1}^{\sum_{\eta=0}^{m} k_\eta} \left[e^{-\frac{1}{2}(\beta_1+\beta_2)t_i^2} \right] \beta_1^{r_1+r_4+\cdots+r_{3m-2}} \beta_2^{r_2+r_5+\cdots+r_{3m-1}} \\
& \times (\beta_1+\beta_2)^{r_3+r_6+\cdots+r_{3m}} e^{-\frac{1}{2}n_1(\beta_1+\beta_2)\tau_{k_1}^2} \\
& \times e^{-\frac{1}{2}n_2(\beta_1+\beta_2)\tau_{k_1+k_2}^2} \cdots e^{-\frac{1}{2}\left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right](\beta_1+\beta_2)\tau^2_{\sum_{\eta=0}^{m} k_\eta}} \\
=& C_m^+ \prod_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i \beta_1^{r_1+r_4+\cdots+r_{3m-2}} \beta_2^{r_2+r_5+\cdots+r_{3m-1}} (\beta_1+\beta_2)^{r_3+r_6+\cdots+r_{3m}} \\
& \times e^{-\frac{1}{2}(\beta_1+\beta_2)\sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i^2} e^{-\frac{1}{2}n_1(\beta_1+\beta_2)\tau_{k_1}^2} \\
& \times e^{-\frac{1}{2}n_2(\beta_1+\beta_2)\tau_{k_1+k_2}^2} \cdots e^{-\frac{1}{2}\left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right](\beta_1+\beta_2)\tau^2_{\sum_{\eta=0}^{m} k_\eta}},
\end{aligned}
$$

$$
\begin{aligned}
\ln L(\text{data}, \beta_1, \beta_2) =& \ln C_m^+ + \sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} \ln t_i + (r_1 + r_4 + \cdots + r_{3m-2}) \ln \beta_1 \\
& + (r_2 + r_5 + \cdots + r_{3m-1}) \ln \beta_2 + (r_3 + r_6 + \cdots + r_{3m}) \ln(\beta_1 + \beta_2) \\
& - \frac{1}{2}(\beta_1+\beta_2) \sum_{i=1}^{\sum_{\eta=0}^{m} k_\eta} t_i^2 - \frac{1}{2}n_1(\beta_1+\beta_2)\tau_{k_1}^2 - \frac{1}{2}n_2(\beta_1+\beta_2)\tau_{k_1+k_2}^2 \\
& - \cdots - \frac{1}{2}\left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right](\beta_1+\beta_2)\tau^2_{\sum_{\eta=0}^{m} k_\eta}.
\end{aligned}
$$

Take $\frac{\partial \ln L(\text{data},\beta_1,\beta_2)}{\partial \beta_1} = 0, \frac{\partial \ln L(\text{data},\beta_1,\beta_2)}{\partial \beta_2} = 0$, and we obtain following two equations

$$\frac{r_1 + r_4 + \cdots + r_{3m-2}}{\beta_1} + \frac{r_3 + r_6 + \cdots + r_{3m}}{\beta_1 + \beta_2} - \frac{1}{2} \sum_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i^2 - \frac{1}{2} n_1 \tau_{k_1}^2 - \frac{1}{2} n_2 \tau_{k_1+k_2}^2$$
$$- \cdots - \frac{1}{2} \left[n - \sum_{\eta=0}^{m-1} (k_\eta + n_\eta) - k_m \right] \tau^2_{\sum\limits_{\eta=0}^{m} k_\eta} = 0,$$

$$\frac{r_2 + r_5 + \cdots + r_{3m-1}}{\beta_2} + \frac{r_3 + r_6 + \cdots + r_{3m}}{\beta_1 + \beta_2} - \frac{1}{2} \sum_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i^2 - \frac{1}{2} n_1 \tau_{k_1}^2 - \frac{1}{2} n_2 \tau_{k_1+k_2}^2$$
$$- \cdots - \frac{1}{2} \left[n - \sum_{\eta=0}^{m-1} (k_\eta + n_\eta) - k_m \right] \tau^2_{\sum\limits_{\eta=0}^{m} k_\eta} = 0.$$

By solving above equations, the maximum likelihood estimate $\hat{\beta}_1, \hat{\beta}_2$ of parameters β_1, β_2 are respectively

$$\hat{\beta}_1 = \frac{2\left(r_1 + r_4 + \cdots + r_{3m-2}\right)}{\left(r_1 + \cdots + r_{3m-2} + r_2 + \cdots + r_{3m-1}\right)} \times \frac{\sum\limits_{\eta=0}^{m} k_\eta}{\left[\sum\limits_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i^2 + n_1 \tau_{k_1}^2 + \cdots + \left[n - \sum\limits_{\eta=0}^{m-1} (k_\eta + n_\eta) - k_m \right] \tau^2_{\sum\limits_{\eta=0}^{m} k_\eta} \right]},$$

$$\hat{\beta}_2 = \frac{2\left(r_2 + r_5 + \cdots + r_{3m-1}\right)}{\left(r_1 + \cdots + r_{3m-2} + r_2 + \cdots + r_{3m-1}\right)} \times \frac{\sum\limits_{\eta=0}^{m} k_\eta}{\left[\sum\limits_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i^2 + n_1 \tau_{k_1}^2 + \cdots + \left[n - \sum\limits_{\eta=0}^{m-1} (k_\eta + n_\eta) - k_m \right] \tau^2_{\sum\limits_{\eta=0}^{m} k_\eta} \right]}.$$

When $\beta_1 = \beta_2 = \beta$, the likelihood function is $L(\text{data}, \beta)$

$$L(\text{data}, \beta) = C_m^+ \prod_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i \beta^{r_1 + r_4 + \cdots + r_{3m-2}} \beta^{r_2 + r_5 + \cdots + r_{3m-1}} (2\beta)^{r_3 + r_6 + \cdots + r_{3m}}$$
$$\times e^{-\beta \sum\limits_{i=1}^{\sum\limits_{\eta=0}^{m} k_\eta} t_i^2} e^{-n_1 \beta \tau_{k_1}^2} e^{-n_2 \beta \tau_{k_1+k_2}^2} \cdots$$
$$\times e^{-\left[n - \sum\limits_{\eta=0}^{m-1} (k_\eta + n_\eta) - k_m \right] \beta \tau^2_{\sum\limits_{\eta=0}^{m} k_\eta}},$$

$$\begin{aligned}\ln L(\text{data},\beta) &= \ln C_m^+ + (r_3+r_6+\cdots+r_{3m})\ln 2 + \sum_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta}\ln t_i + \sum_{i=1}^{3m} r_i \ln\beta \\ &\quad - \beta\sum_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta} t_i^2 - n_1\beta\tau_{k_1}^2 - n_2\beta\tau_{k_1+k_2}^2 - \cdots \\ &\quad - \left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right]\beta\tau^2_{\sum\limits_{\eta=0}^{m}k_\eta}.\end{aligned}$$

Take $\frac{d\ln L(\text{data},\beta)}{d\beta} = 0$, and we obtain following equation

$$\frac{1}{\beta}\sum_{i=1}^{3m} r_i - \sum_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta} t_i^2 - n_1\tau_{k_1}^2 - n_2\tau_{k_1+k_2}^2 - \cdots - \left[n - \sum_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right]\tau^2_{\sum\limits_{\eta=0}^{m}k_\eta} = 0.$$

By solving above equation, the maximum likelihood estimate $\hat{\beta}$ of parameter β is

$$\hat{\beta} = \frac{\sum\limits_{i=1}^{3m} r_i}{\sum\limits_{i=1}^{\sum\limits_{\eta=0}^{m}k_\eta} t_i^2 + n_1\tau_{k_1}^2 + n_2\tau_{k_1+k_2}^2 + \cdots + \left[n - \sum\limits_{\eta=0}^{m-1}(k_\eta+n_\eta) - k_m\right]\tau^2_{\sum\limits_{\eta=0}^{m}k_\eta}}.$$

Similarly, for the given confidence level, the approximate interval estimate can be obtained by using the method of likelihood estimate.

Example 4.3. Take a sample with sample size $n = 50$, $k_1 = 15, r_1 = 2, r_2 = 8$, $n_1 = 2$, $k_2 = 10, r_4 = 1, r_5 = 4$, $n_2 = 2$, $k_3 = 20, r_7 = 3, r_8 = 12$. The failure rates of two units are respectively $\beta_1 = 0.2, \beta_2 = 0.8$. The failure data generated by Monte-Carlo simulation are shown in Table 10.

Table 10. Failure Data of Example 4.3 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.1938,0.7600
	$s_i = \{2\}$	0.7212,0.2612,0.3687,0.7175,0.7395,0.7201 0.3781,0.6429
	$s_i = \{1,2\}$	0.5810,0.5940,0.3937,0.8475,0.7733
2	$s_i = \{1\}$	1.0007
	$s_i = \{2\}$	0.9745,1.0284,1.3319,0.9477
	$s_i = \{1,2\}$	1.2863,0.9654,0.9172,1.2315,1.1587
3	$s_i = \{1\}$	1.5038,2.1023,1.8286
	$s_i = \{2\}$	1.4024,1.4239,2.1741,2.1165,1.3600,1.6100 1.6754,2.2375,1.4245,2.0463,1.5817,1.4172
	$s_i = \{1,2\}$	1.5727,1.7111,1.9628,1.8020,2.1309

Take $\tau_1 = 0.8475$, $\tau_2 = 1.3319$ and $\tau_3 = 2.2375$. We can obtain $\hat{\beta}_1 = 0.1981$ and $\hat{\beta}_2 = 0.7926$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of β_1 is [0.1457,0.2618], and the approximate interval estimate of β_2 is [0.5830,1.0472].

Example 4.4. Take a sample with sample size $n = 90$, $k_1 = 30, r_1 = 8, r_2 = 12$, $n_1 = 2$, $k_2 = 25, r_4 = 6, r_5 = 9$, $n_2 = 2$, $k_3 = 30, r_7 = 10, r_8 = 15$. The failure rates of two units are respectively $\beta_1 = 1, \beta_2 = 1.5$. The failure data generated by Monte-Carlo simulation are shown in Table 11.

Table 11. Failure Data of Example 4.4 Generated by Monte-Carlo Simulation

Test sequence	Unit set causing system failure	System failure time
1	$s_i = \{1\}$	0.1762,0.4655,0.4000,0.3229,0.1830,0.4451 0.4401,0.4510
	$s_i = \{2\}$	0.2987,0.4164,0.4385,0.0654,0.1946,0.4461 0.3597,0.0582,0.4614,0.2980,0.4915,0.4012
	$s_i = \{1,2\}$	0.2553,0.1579,0.4901,0.3960,0.4408,0.3044 0.1350,0.4905,0.4295,0.1735
2	$s_i = \{1\}$	0.9038,0.6017,0.6754,0.5660,0.7231,0.8329
	$s_i = \{2\}$	0.5245,0.8517,0.7625,0.6666,0.5831,0.6102 0.6465,0.8800,0.5530
	$s_i = \{1,2\}$	0.5431,0.9016,0.7068,0.6961,0.6531,0.8605 0.8849,0.6156,0.5019,0.7685
3	$s_i = \{1\}$	0.9945,0.9383,1.0221,1.0142,1.1148,1.3042 1.1923,1.0089,1.6280,1.1317
	$s_i = \{2\}$	1.0803,1.2400,1.2909,0.9228,0.9356,1.0826 1.4332,1.6597,1.0117,1.0644,1.4115,1.4543 1.1966,1.5895,0.9049
	$s_i = \{1,2\}$	1.1811,1.5671,1.4879,1.7217,1.3354

Take $\tau_1 = 0.4915$, $\tau_2 = 0.9038$ and $\tau_3 = 1.7217$. We can obtain $\hat{\beta}_1 = 0.9875$ and $\hat{\beta}_2 = 1.4812$ by using the method presented in this paper. For the given confidence level 0.95, the approximate interval estimate of β_1 is [0.7921,1.2125], and the approximate interval estimate of β_2 is [1.1882,1.8187].

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