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HE'S VARIATIONAL ITERATION METHOD FOR THE SOLUTION OF NONLINEAR NEWELL-WHITEHEAD-SEGEL EQUATION

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Abstract In this paper, we apply He's Variational iteration method (VIM) for solving nonlinear Newell-Whitehead-Segel equation. By using this method three different cases of Newell-Whitehead-Segel equation have been discussed. Comparison of the obtained result with exact solutions shows that the method used is an effective and highly promising method for solving different cases of nonlinear Newell-Whitehead-Segel equation.

Keywords Newell-Whitehead-Segel equation, variational iteration method, nonlinear differential equations.

MSC(2010) 44A99, 35Q99.

1. Introduction

We have to face many real-life time mathematical models for semi-analytical solution of nonlinear differential equations in day-today life. We also know that most of the nonlinear differential equations do not have an analytical solution. But by using He's variational iteration method (VIM), we can solve nonlinear differential equations, which was first envisioned by Professor Ji-He for solving a wide range of problems whose mathematical model yields differential equation or system of differential equations [2–7]. Later on He's variational iteration method (VIM) which is a semi-analytical method is applied to solve the nonlinear non-homogeneous partial differential equations [8–22]. Newell-Whitehead-Segel equation is solved by using the Adomian decomposition and multi-quadric quasi-interpolation methods [1] and Homotopy perturbation method [20].

In this article, we apply He's variational iteration method to obtain the solution of the non-linear Newell-Whitehead-Segel equation. Three different cases of nonlinear Newell- Whitehead-Segel equations are solved by using this method. The Newell-Whitehead-Segel equation models the interaction of the effect of the diffusion term with the nonlinear effect of the reaction term. The Newell- Whitehead-Segel equation is written as

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + au - bu^q, \tag{1.1}$$

where a, b and k are real numbers with $k > 0$ and q is a positive integer. In equation (1.1) the first term on the left hand side, $\frac{\partial u}{\partial t}$, expresses the variations of $u(x,t)$ with time at a fixed location, the first term on the right hand side, $\frac{\partial^2 u}{\partial x^2}$,

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expresses the variations of $u(x,t)$ with spatial variable x at a specific time and the remaining terms on the right hand side $au - bu^q$, takes into account the effect of the source term. In equation (1.1), $u(x,t)$ is a function of the spatial variable x and the temporal variable t with $x \in R$ and $t \geq 0$. The function $u(x,t)$ may be thought of as the (nonlinear) distribution of temperature in an infinitely thin and long rod or as the flow velocity of a fluid in an infinitely long pipe with small diameter. The Newell-Whitehead-Segel equations have wide applicability in mechanical and chemical engineering, ecology, biology and bio-engineering.

2. Proposed He's Variational iteration method for the Newell-Whitehead- Segel equation

Consider the differential equation

$$Lu + Nu = g(x), \tag{2.1}$$

where L and N are Linear and nonlinear operators respectively and $g(x)$ is inhomogeneous function. J. H. He suggested the variational iteration method where a correction functional for equation (2.1) can be written as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt, \tag{2.2}$$

where λ is Lagrange's multiplier, which can be identified optimally by the variational theory and $\tilde{u}_n(t)$ as restricted variation that means $\delta\tilde{u}_n = 0$. It is to be noted that Lagrange multiplier λ can be taken as a constant or function.

There are two steps in variational iteration method, first to find Lagrange's multiplier that can be find out optimally via integration by parts and by using the restricted variation, should be used for the determination of the successive approximations $u_{n+1}(x), n \geq 0$ of the solution $u(x)$. The zeroth approximation u_0 can be any selective function or can be any other initial condition that can be used at the initial stage or using the initial values $u(0), u'(0)$ and $u''(0)$ are preferably used for the selective zeroth approximation u_0 as will be seen later. Finally the solution is given by $u(x) = \lim_{n\to\infty} u_n(x)$. Then $u(x)$ will be the solution of given differential equation.

3. The Newell-Whitehead- Segel equation

To illustrate the capability and reliability of this method, three different cases of nonlinear Newell-Whitehead-Segel equation are presented.

3.1. Case I.

In equation (1.1), if $a = 2, b = 3, k = 1$ and $q = 2$ the Newell-Whitehead-Segel equation is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u - 3u^2, \tag{3.1}$$

subject to a constant initial condition $u(x,0) = 1$.
Now by using variational iteration method, we have

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u_n}(x,\xi)}{\partial x^2} - 2u_n(x,\xi) + 3\tilde{u_n}^2(x,\xi))d\xi, \tag{3.2}$$

where $\tilde{u_n}$ is restricted variation so $\delta\tilde{u_n} = 0$.
Then applying variation on both sides and integrating we obtain equations

$$1 + \lambda(\xi,t)|_{\xi=t} = 0, \frac{\lambda'}{\lambda} = 2. \tag{3.3}$$

Now from these equations, we get

$$\lambda = -e^{-2(t-\xi)}. \tag{3.4}$$

Then by VIM we can construct a sequence. Initially with the given initial condition we can take
$u(x,0) = u_0(x,t) = 1$ and by VIM $\lambda = -e^{-2(t-\xi)}$ and then

$$\begin{aligned}
u_0(x,t) &= 1,\\
u_1(x,t) &= u_0(x,t) - \int_0^t e^{-2(t-\xi)}(\frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial^2 u_0(x,\xi)}{\partial x^2} - 2u_0(x,\xi) + 3u_0^2(x,\xi))d\xi\\
&= 1 - t + t^2 - \frac{2t^3}{3} + \frac{t^4}{3} - \frac{2t^5}{15} + \frac{2t^6}{45} + \cdots\\
u_2(x,t) &= u_1(x,t) - \int_0^t e^{-2(t-\xi)}(\frac{\partial u_1(x,\xi)}{\partial \xi} - \frac{\partial^2 u_1(x,\xi)}{\partial x^2} - 2u_1(x,\xi) + 3u_1^2(x,\xi))d\xi\\
&= 1 - t + 2t^2 - 3t^3 + \frac{10t^4}{3} - \frac{43t^5}{15} + 2t^6 + \cdots\\
u_3(x,t) &= u_2(x,t) - \int_0^t e^{-2(t-\xi)}(\frac{\partial u_2(x,\xi)}{\partial \xi} - \frac{\partial^2 u_2(x,\xi)}{\partial x^2} - 2u_2(x,\xi) + 3u_2^2(x,\xi))d\xi\\
&= 1 - t + 2t^2 - \frac{11t^3}{3} + \frac{19t^4}{3} - \frac{148t^5}{15} + \frac{611t^6}{45} + \cdots\\
u_4(x,t) &= u_3(x,t) - \int_0^t e^{-2(t-\xi)}(\frac{\partial u_3(x,\xi)}{\partial \xi} - \frac{\partial^2 u_2(x,\xi)}{\partial x^2} - 2u_3(x,\xi) + 3u_3^2(x,\xi))d\xi\\
&= 1 - t + 2t^2 - \frac{11t^3}{3} + \frac{20t^4}{3} - 12t^5 + \frac{943t^6}{45} + \cdots
\end{aligned}$$

And also we know that Taylor's series expansion of

$$\begin{aligned}
u(x,t) =& \frac{\frac{-2}{3}e^{2t}}{\frac{1}{3} - e^{2t}}\\
=& 1 - t + 2t^2 - \frac{11t^3}{3} + \frac{20t^4}{3} - \frac{182t^5}{15} + \frac{994t^6}{45} - \frac{12667t^7}{315}\\
&+ \frac{4612t^8}{63} - \frac{377822t^9}{2835} + \cdots
\end{aligned}$$

So as n tends ∞, $u_n(x,t)$ tends to

$$u(x,t) = \frac{\frac{-2}{3}e^{2t}}{\frac{1}{3} - e^{2t}}, \tag{3.5}$$

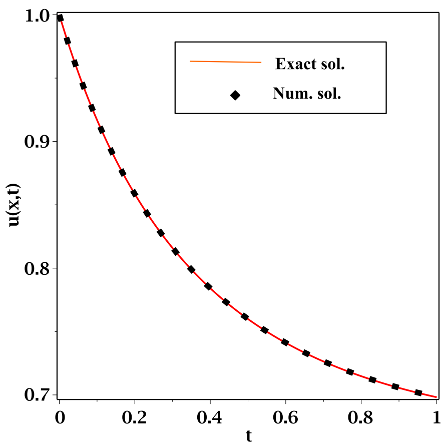

Figure 1. Comparison between approximate solution and exact solution.

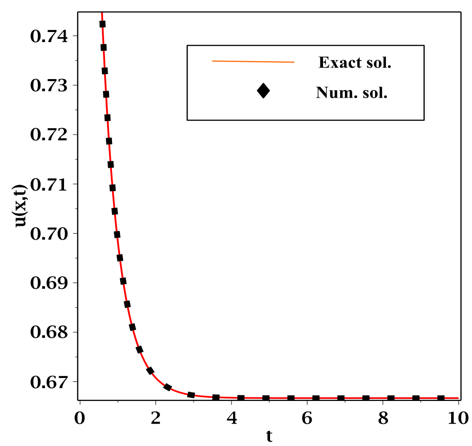

Figure 2. Comparison between approximate solution and exact solution.

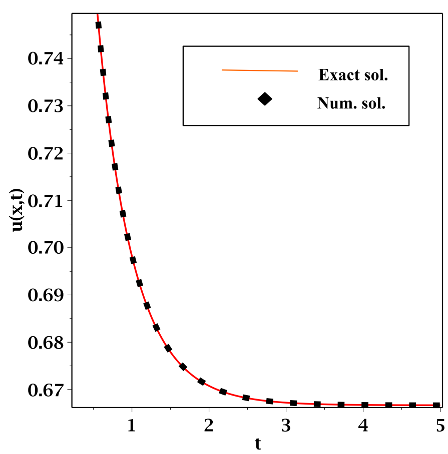

Figure 3. Comparison between approximate solution and exact solution.

which is the exact solution.

Fig. $(1-3)$ shows the comparison between exact solution and fourth order approximate solution for different intervals of time. It can be observed from Fig. $(1-3)$ that this method is effective in a narrow band of time.

3.2. Case II.

In equation (1.1), if $a = 1, b = 1, k = 1$ and $q = 2$, then the Newell-Whitehead-Segel equation is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2, \tag{3.6}$$

subject to the initial condition

$$u_0(x,t) = u(x,0) = \frac{1}{\left(1 + e^{\frac{x}{\sqrt{6}}}\right)^2}.$$

Table 1. Comparison table between exact and approximate solution

t	Apr. solution	Exact solution	Absolute Error	Percentage Relative Error
	u_4	$u(x,t)$	$\|u-u_4\|$	$\frac{\|u-u_4\|}{\|u(x,t)\|}\times 100$
0	1	1	0	0
0.2	0.8584870380025241	0.8584870194709139	1.9×10^{-8}	2.2×10^{-6}
0.4	0.7841077390140231	0.7841073865912336	3.5×10^{-7}	4.5×10^{-6}
0.6	0.7410695902444964	0.7410685158701519	1.1×10^{-6}	1.4×10^{-4}
0.8	0.7147714997899617	0.7147698478137199	1.7×10^{-6}	2.3×10^{-4}
1	0.6981637964113238	0.6981619832493625	1.8×10^{-6}	2.6×10^{-4}

Then by using variational iteration method, we have

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\frac{\partial u_n(x,\xi)}{\partial \xi}-\frac{\partial^2 \tilde{u_n}(x,\xi)}{\partial x^2})-u_n(x,\xi)+\tilde{u_n}^2(x,\xi))d\xi, \quad (3.7)$$

where $\tilde{u_n}$ is restricted variation so $\delta\tilde{u_n}=0$.

Then applying variation on both sides and integrating we obtain equations

$$1+\lambda(\xi,t)|_{\xi=t}=0, \frac{\lambda'}{\lambda}=1. \quad (3.8)$$

Now from these equations, we get $\lambda=-e^{-(t-\xi)}$

Then by VIM we can construct a sequence. Initially with the given initial condition we can take

$$\begin{aligned}
u_0(x,t)=&\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2},\\
u_1(x,t)=&u_0(x,t)-\int_0^t e^{-(t-\xi)}(\frac{\partial u_0(x,\xi)}{\partial \xi}-\frac{\partial^2 u_0(x,\xi)}{\partial x^2}-u_0(x,\xi)+u_0^2(x,\xi))d\xi,\\
=&\frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}+\frac{5e^{\frac{x}{\sqrt{6}}}t}{3(1+e^{\frac{x}{\sqrt{6}}})^3}-\frac{5e^{\frac{x}{\sqrt{6}}}t^2}{6(1+e^{\frac{x}{\sqrt{6}}})^3}+\frac{5e^{\frac{x}{\sqrt{6}}}t^3}{18(1+e^{\frac{x}{\sqrt{6}}})^3}\\
&-\frac{5e^{\frac{x}{\sqrt{6}}}t^4}{72(1+e^{\frac{x}{\sqrt{6}}})^3}+\cdots\\
u_2(x,t)=&u_1(x,t)-\int_0^t e^{-(t-\xi)}(\frac{\partial u_1(x,\xi)}{\partial \xi}-\frac{\partial^2 u_1(x,\xi)}{\partial x^2}-u_1(x,\xi)+u_1^2(x,\xi))d\xi\\
=&\frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}+\frac{5e^{\frac{x}{\sqrt{6}}}t}{3(1+e^{\frac{x}{\sqrt{6}}})^3}+\frac{25e^{\frac{x}{\sqrt{6}}}(-1+2e^{\frac{x}{\sqrt{6}}})t^2}{36(1+e^{\frac{x}{\sqrt{6}}})^4}\\
&-\frac{5(e^{\frac{x}{\sqrt{6}}}(-2+24e^{\sqrt{\frac{2}{3}}x}+13e^{\sqrt{\frac{3}{2}}x}+19e^{\frac{x}{\sqrt{6}}}))t^3}{54(1+e^{\frac{x}{\sqrt{6}}})^6}+\cdots\\
u_3(x,t)=&u_2(x,t)-\int_0^t e^{-(t-\xi)}(\frac{\partial u_2(x,\xi)}{\partial \xi}-\frac{\partial^2 u_2(x,\xi)}{\partial x^2}-u_2(x,\xi)+u_2^2(x,\xi))d\xi\\
=&\frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}+\frac{5e^{\frac{x}{\sqrt{6}}}t}{3(1+e^{\frac{x}{\sqrt{6}}})^3}+\frac{25e^{\frac{x}{\sqrt{6}}}(-1+2e^{\frac{x}{\sqrt{6}}})t^2}{36(1+e^{\frac{x}{\sqrt{6}}})^4}
\end{aligned}$$

$$+\frac{125e^{\frac{x}{\sqrt{6}}}(1+4e^{\sqrt{\frac{2}{3}}x}-7e^{\frac{x}{\sqrt{6}}})t^3}{648(1+e^{\frac{x}{\sqrt{6}}})^5}+\cdots$$

$$\begin{aligned}u_4(x,t)=&u_3(x,t)-\int_0^t e^{-(t-\xi)}(\frac{\partial u_3(x,\xi)}{\partial \xi}-\frac{\partial^2 u_3(x,\xi)}{\partial x^2}-u_3(x,\xi)+u_3^2(x,\xi))d\xi\\ =&\frac{1}{(1+e^{\frac{x}{\sqrt{6}}})^2}+\frac{5e^{\frac{x}{\sqrt{6}}}t}{3(1+e^{\frac{x}{\sqrt{6}}})^3}+\frac{25e^{\frac{x}{\sqrt{6}}}(-1+2e^{\frac{x}{\sqrt{6}}})t^2}{36(1+e^{\frac{x}{\sqrt{6}}})^4}\\ &+\frac{125e^{\frac{x}{\sqrt{6}}}(1+4e^{\sqrt{\frac{2}{3}}x}-7e^{\frac{x}{\sqrt{6}}})t^3}{648(1+e^{\frac{x}{\sqrt{6}}})^5}\\ &+\frac{625e^{\frac{x}{\sqrt{6}}}(-1-33e^{\sqrt{\frac{2}{3}}x}+8e^{\sqrt{\frac{3}{2}}x}+18e^{\frac{x}{\sqrt{6}}})t^4}{15552(1+e^{\frac{x}{\sqrt{6}}})^6}+\cdots\end{aligned}$$

So as n tends ∞, $u_n(x,t)$ tends to

$$u(x,t)=\frac{1}{(1+e^{\frac{-5t}{6}+\frac{x}{\sqrt{6}}})^2}, \tag{3.9}$$

which is the exact solution.

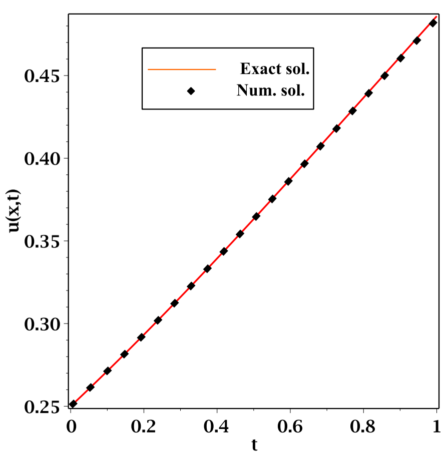

Figure 4. Comparison between approximate solution and exact solution.

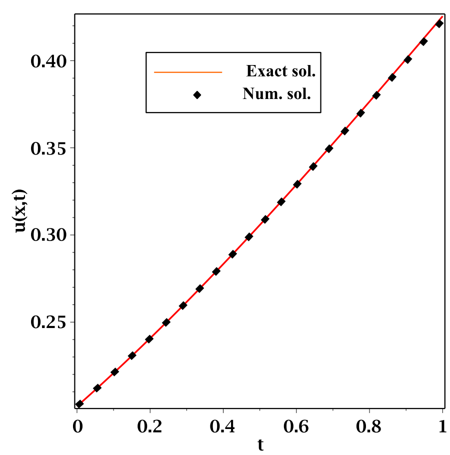

Figure 5. Comparison between approximate solution and exact solution.

Fig. $(4-6)$ shows the comparison between exact solution and fourth order approximate solution for different value of space coordinates $x = 0, 0.5, 1$ respectively. It can be observed from Fig. $(4-6)$ that this method is effective in a narrow band of time and space.

3.3. Case III.

In equation (1.1), if $a = 1, b = 1, k = 1$ and $q = 4$, the Newell-Whitehead-Segel equation is written as:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+u-u^4, \tag{3.10}$$

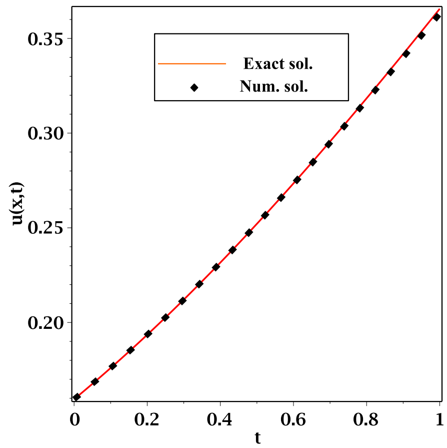

Figure 6. Comparison between approximate solution and exact solution.

Table 2. Comparison table between exact and approximate solution

t	x	Apr. solution	Exact solution	Absolute Error	Percentage Relative Error
		u_4	$u(x,t)$	$\lvert u-u_4\rvert$	$\frac{\lvert u-u_4\rvert}{\lvert u(x,t)\rvert}\times 100$
0	1	0.1594662239430331	0.1594662239430331	0	0
0.2	1	0.1935069770846908	0.1935090374311784	2.1×10^{-6}	1.1×10^{-3}
0.4	1	0.2315774000547367	0.2316304528410767	5.3×10^{-5}	2.3×10^{-2}
0.6	1	0.2731249796768434	0.2734472604166732	3.2×10^{-4}	1.2×10^{-1}
0.8	1	0.3172944314206540	0.3183751893128759	1.1×10^{-2}	3.4×10^{-1}
1	1	0.3630484492815107	0.3656613805363442	2.6×10^{-2}	7.1×10^{-1}

subject to the initial condition

$$u(x,0) = u_0(x,0) = \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}}. \tag{3.11}$$

Now using variational iteration method, we have

$$u_{n+1}(x,t) = u_n(x,t)+\int_0^t \lambda\left(\frac{\partial u_n(x,\xi)}{\partial \xi}-\frac{\partial^2 \tilde{u_n}(x,\xi)}{\partial x^2}-u_n(x,\xi)+\tilde{u_n}^4(x,\xi)\right)d\xi, \tag{3.12}$$

where $\tilde{u_n}$ is restricted variation so $\delta\tilde{u_n} = 0$.

Then applying variation on both sides and integrating we obtain equations

$$1+\lambda(\xi,t)|_{\xi=t} = 0, \frac{\lambda'}{\lambda} = 1. \tag{3.13}$$

Now from these equations, we get $\lambda = -e^{-(t-\xi)}$.

Then by VIM we can construct a sequence. Initially with the given initial condition we can take

$$u_0(x,t) = \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}},$$

$$u_1(x,t) = u_0(x,t) - \int_0^t e^{-(t-\xi)}(\frac{\partial u_0(x,\xi)}{\partial \xi} - \frac{\partial^2 u_0(x,\xi)}{\partial x^2} - u_0(x,\xi) + u_0^4(x,\xi))d\xi$$
$$= (\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{2/3} + \frac{7}{5}e^{\frac{3x}{\sqrt{10}}}(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{5/3}t + \cdots$$
$$u_2(x,t) = u_1(x,t) - \int_0^t e^{-(t-\xi)}(\frac{\partial u_1(x,\xi)}{\partial \xi} - \frac{\partial^2 u_1(x,\xi)}{\partial x^2} - u_1(x,\xi) + u_1^4(x,\xi))d\xi$$
$$= (\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{2/3} + \frac{7}{5}e^{\frac{3x}{\sqrt{10}}}(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{5/3}t + \frac{49}{100}e^{\frac{3x}{\sqrt{10}}}(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{8/3}(-3+2e^{\frac{3x}{\sqrt{10}}})t^2 - \cdots$$
$$u_3(x,t) = u_2(x,t) - \int_0^t e^{-(t-\xi)}\left(\frac{\partial u_2(x,\xi)}{\partial \xi} - \frac{\partial^2 u_2(x,\xi)}{\partial x^2} - u_2(x,\xi) + u_2^4(x,\xi)\right)d\xi$$
$$= (\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{2/3} + \frac{7}{5}e^{\frac{3x}{\sqrt{10}}}(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{5/3}t + \frac{49}{100}e^{\frac{3x}{\sqrt{10}}}(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{8/3}(-3+2e^{\frac{3x}{\sqrt{10}}})t^2 + \frac{343e^{\frac{3x}{\sqrt{10}}}(9+4e^{3\sqrt{\frac{2}{5}}x} - 27e^{\frac{3x}{\sqrt{10}}})(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}})^{11/3}t^3}{3000} + \cdots$$

So as n tends ∞, $u_n(x,t)$ tends to

$$u(x,t) = ((\frac{1}{2})\text{Tanh}(-\frac{3}{2\sqrt{10}}(x - \frac{7}{\sqrt{10}}t)) + \frac{1}{2})^{\frac{2}{3}}, \tag{3.14}$$

which is the exact solution.

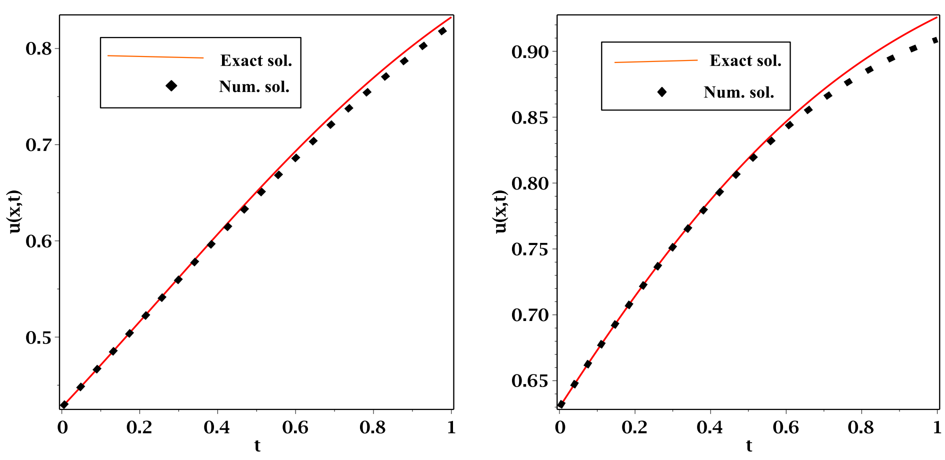

Figure 7. Comparison between approximate solution and exact solution.

Figure 8. Comparison between approximate solution and exact solution.

Fig. $(7-9)$ shows the comparison between exact solution and fourth order approximate solution for different value of space coordinates $x = 0, 0.5, 1$ respectively. It can be observed from Fig. $(7-9)$ that this method is effective in a narrow band of time and space.

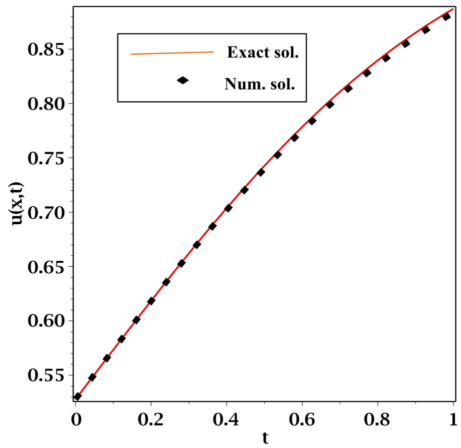

Figure 9. Comparison between approximate solution and exact solution.

Table 3. Comparison table between exact and approximate solution

t	x	Apr. solution	Exact solution	Absolute Error	Percentage Relative Error
		u_3	$u(x,t)$	$\|u-u_3\|$	$\frac{\|u-u_3\|}{\|u(x,t)\|}\times 100$
0	1	0.4271280654486842	0.4271280654486848	6.1×10^{-16}	1.42×10^{-13}
0.2	1	0.5160942139459634	0.5161531564367771	5.9×10^{-5}	1.37×10^{-3}
0.4	1	0.6061850440226317	0.6069494677384791	7.6×10^{-4}	1.79×10^{-3}
0.6	1	0.6900184308626889	0.6932462609512966	3.2×10^{-3}	2.08×10^{-3}
0.8	1	0.7607348582093733	0.7695607252303125	8.8×10^{-3}	7.72×10^{-2}
1	1	0.8133927927151118	0.8326215520144739	1.9×10^{-2}	1.01×10^{-1}

4. Conclusion

In the present work, He's variational iteration method (VIM) has been successfully applied to obtain numerical solution for various types of Newell-Whitehead-Segel nonlinear diffusion equation. It is apparently seen that VIM is a very efficient and powerful numerical method to obtain the approximate solution. It is shown that He's variational iteration method is a promising tool for treating nonlinear equations and in some cases yields exact solution in a few iteration. The method is used in a direct way without using adomain polynomial, linearization, perturbation or restrictive assumptions. Therefore, FVIM is easier and more convenient than other methods.

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References

[1] R. Ezzati and K. Shakibi, *Using adomians decomposition and multi-quadric quasi-interpolation methods for solving Newell-Whitehead equation*, Procedia Computer Science, 3(2011), 1043–1048.

[2] J. H. He, *Variational iteration method for delay differential equations*, Communications in Nonlinear Science and Numerical Simulation, 2(1997), 235–236.

[3] J. H. He, *An approximate solution technique depending on an artificial parameter: A special example*, Communications in Nonlinear Science and Numerical Simulation, 3(1998), 92–97.

[4] J. H. He, *Variational iteration method a kind of non-linear analytical technique: some examples*, International Journal of Non-Linear Mechanics, 34(1999), 699–708.

[5] J. H. He, Y.Q. Wan and Q. Guo, *An iteration formulation for normalized diode characteristics*, International Journal of Circuit Theory and Applications, 32(2004), 629–632.

[6] J. H. He, *Some asymptotic methods for strongly nonlinear equations*, International Journal of Modern Physics B, 20(2006), 1141.

[7] J. H. He, *Non-perturbative Methods for Strongly Nonlinear Problems*, dissertation.de-Verlag im Internet GmbH, Berlin, 2006.

[8] J. H. He, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Computer Methods in Applied Mechanics and Engineering, 167(1998), 57–68.

[9] J. H. He, *Approximate solution of nonlinear differential equations with convolution product nonlinearities*, Computer Methods in Applied Mechanics and Engineering, 167(1998), 69–73.

[10] J. H. He, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering, 178(1999), 257–262.

[11] J. H. He, *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, International Journal of Non-Linear Mechanics, 35(2000), 37–43.

[12] J. H. He, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons & Fractals, 26(2005), 695–700.

[13] J. H. He, *Homotopy perturbation method for bifurcation of nonlinear problems*, International Journal of Nonlinear Sciences and Numerical Simulation, 6(2005), 207–208.

[14] J. H. He, *Nonlinear variants of the BBM equation with compact and noncompact physical structures*, Phys. Lett. A, 26(2005), 695.

[15] J. H. He, *Periodic solutions and bifurcations of delay-differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 347(2005), 228–230.

[16] J. H. He, *Construction of solitary solution and compaction-like solution by variational iteration method*, Chaos Solitons Fractals, 29(2006), 108–113.

[17] J. H. He, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett., A, 350(2006), 87–88.

[18] J. H. He, *Some asymptotic methods for strongly nonlinear equations*, Int. J. Mod. Phys. B, 20(2006), 41.

[19] H. Koçak, A. Yildirimm, D.H. Zhang, and S.T. Mohyud-Din, *The comparative Boubaker polynomials expansion scheme and homotopy perturbation method for solving a standard non- linear second-order boundary value problem*, Mathematical and Computer Modelling, 54(2011), 417–422.

[20] S. S. Nourazar, M. Soori and A. Nazari-Golshan, *On The exact solution of Newell-Whitehead- Segel equation using the Homotopy perturbation method*, Australian Journal of Basic and Applied Sciences & Fractals, 5(2011)(8), 1400–1411.

[21] A. M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2009.

[22] A. Yildirim, *Homotopy perturbation method for the mixed Volterra-Fredholm integral equations*, Chaos Solitons & Fractals, 42(2009), 2760–2764.