Variational Method to the Fractional Impulsive Equation with Neumann Boundary Conditions

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Abstract

We study the multiplicity of solutions for a class of fractional differential equations influenced by both instantaneous and non-instantaneous impulses, subject to Neumann boundary conditions. A key contribution of this paper is that we have established a new variational structure and successfully applied critical point theory to investigate the impulsive fractional Neumann boundary value problem. By using the critical point theorem, we give some new criteria to guarantee that the impulsive problem has at least three solutions. An example is also given to illustrate the main results.

Keywords: Fractional differential equation; Instantaneous impulses; Non-instantaneous impulses; Neumann boundary condition; Critical point theorem

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1. Introduction

The purpose of this paper is to establish the existence of solutions for the following impulsive fractional Neumann boundary value problem:

$$\begin{cases} {}_tD_T^\alpha({}_0^CD_t^\alpha u(t)) + q(t)u(t) = \lambda f_k(t, u(t)),\ t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \cdots, l, \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u))(t_k) = I_k(u(t_k)),\ k = 1, 2, \cdots, l, \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t_k^+),\ t \in (t_k, s_k],\ k = 1, 2, \cdots, l, \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(s_k^-) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(s_k^+),\ k = 1, 2, \cdots, l, \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(0) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(T) = 0, \end{cases} \tag{1.1}$$

where ${}_tD_T^\alpha, {}_0^CD_t^\alpha$ are the right Riemann-Liouville fractional derivative and the left Caputo fractional derivative, respectively, of order $\alpha \in (1/2, 1]$, $q(t) \in C([0, T])$ with $0 < q_0 = \min_{[0,T]}q(t) \le q(t) \le q^0 = \max_{[0,T]}q(t)$, λ is a positive parameter, $0 = s_0 < t_1 < s_1 < \cdots < s_l < t_{l+1} = T$, $I_k \in C(\mathbb{R}, \mathbb{R})(k{=}1, 2, \cdots, l)$ and $f_k \in C((s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})(k{=}0, 1, 2, \cdots, l)$,

$$\begin{aligned} &\Delta({}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u))(t_k) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t_k^+) - {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t_k^-), \\ &{}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t_k^\pm) = \lim_{t\to t_k^\pm} {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t), \\ &{}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(s_k^\pm) = \lim_{t\to s_k^\pm} {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t). \end{aligned}$$

Fractional calculus has a long history. Its origins can be traced back to more than 300 years ago [1], and it has been extensively studied and developed during the last few decades. One of the main reasons is that fractional differential equations(FDEs) have been successfully applied in many fields, such as: physics [2], economics [3], signal processing [4], control theory [5], viscoelasticity theory [6], rheology [7], etc. For more applications and references we refer the reader to [8, 9].

In the past two decades, a great deal of mathematical effort has been devoted to the study of fractional boundary value problems (BVPs) and achieved many profound results. However, these results are far from sufficient when compared to the research achievements of BVPs for integer-order differential equations. It is well known that the variational method

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is a very useful approach to studying BVPs of differential equations, and it has been widely used to investigate BVPs for integer-order differential equations with some classical boundary conditions (BCs), such as Dirichlet BCs [10], Neumann BCs [11-14], periodic BCs [15], anti-periodic BCs [16], Sturm-Liouville BCs [17] and multi-point BCs [18]. However, according to existing literature results, we find that the application of the variational method to fractional BVPs is currently limited to the study of fractional Dirichlet and Sturm-Liouville BVPs [19-21]. Therefore, a natural question is whether it is possible to establish the variational structure for FDEs under other boundary conditions, which is the main motivation for this study.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Such processes naturally occur in population dynamics, chemotherapy, and medicine[22, 23]. It follows from the existing literature that there are two prevalent types of impulses are recognized: instantaneous and non-instantaneous impulses[24]. In recent years, some scholars by using variational method studied on the impulsive FDEs with Dirichlet BCs and Sturm-Liouville BCs for the analysis of existence and multiplicity of solutions [25-28]. For example, Rodríguez-López and Tersian [26] studied the multiplicity of solutions for FDEs subject to Dirichlet BCs and instantaneous impulses with the help of three critical points theorem. Min and Chen [27] proved the infinitely many solutions for p-Laplacian FDEs subject to Sturm-Liouville BCs and instantaneous impulses by using symmetry mountain pass theorem. Zhao et al. [28] employed the least action principle and mountain pass theorem to investigate the existence and multiplicity of solutions for FDEs supplemented with Dirichlet BCs and non-instantaneous impulses.

More recently, there has been a growing interest among scholars in investigating the existence and multiplicity of solutions for fractional BVPs with both instantaneous and non-instantaneous impulses [29-34]. For instance, Wang et al. [30] investigated the existence of solutions for a fractional Dirichlet problem with both instantaneous and non-instantaneous impulses by using the least action principle. Li et al. [31] studied the multiplicity of solutions for p-Laplacian fractional Dirichlet problem with both instantaneous and non-instantaneous impulses by applying a three critical points theorem proved by Ricceri. Tian and Zhang [32] considered the existence of solutions for p-q-Laplacian fractional Dirichlet problem with instantaneous and non-instantaneous impulses by utilizing Ekeland's variational principle. Zhang and Ni [33] discussed the multiplicity of solutions for p-Laplacian FDEs subject to Sturm-Liouville BCs and involving both instantaneous and non-instantaneous impulses by the use of a three critical points theorem proposed by Bonanno and Marano.

To the best of our knowledge, no existing literature has utilized variational methods to study the Neumann BVPs of FDEs. Inspired by the above work, in this paper, we discuss a class of fractional Neumann BVP with both instantaneous and non-instantaneous impulses (1.1). Our analysis is based on a three critical points theorem (see Theorem 2.1 below) contained in [35]. The main contributions of this paper are highlighted as follows.

- The variational structure of the fractional Neumann BVPs has been established, and a class of fractional Neumann BVP (1.1) has been successfully studied via critical point theorem. This work expands the application scope of variational methods in fractional BVPs.
- Sufficient conditions for the existence of three solutions to problem (1.1) are given by using the three critical points theorem, and the validity of the main results is illustrated by an example.
- Observe that if $\alpha \to 1$, $t_k = s_k$, $k = 1, 2, \cdots, l$, the problem (1.1) reduces to the second order BVP. The existence of the existence and multiplicity of solutions for this problem is widely studied via variational methods among the papers, see for example ([11-14]). Therefore, the work in this paper extends the results for integer order BVP to the more general case of fractional order.

2. Preliminaries

In this section, we first recall some necessary definitions and properties of the fractional calculus, and then we introduce a fractional derivative space $\mathbb{E}^{\alpha}$ and some of related lemmas, and present a three critical points theorem which will be applied in the next section.

Definition 2.1. ([8]) Let $\nu > 0$, $u \in C[0,T]$. Then the left and right Riemann-Liouville fractional integrals ${}_0D_t^{-\nu}u(t)$

and ${}_tD_T^{-\nu}u(t)$ are respectively defined by

$$
{}_0D_t^{-\nu}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\theta)^{\nu-1}u(\theta)d\theta, \quad t\in[0,T],
$$
$$
{}_tD_T^{-\nu}u(t) = \frac{1}{\Gamma(\nu)}\int_t^T (\theta-t)^{\nu-1}u(\theta)d\theta, \quad t\in[0,T].
$$

Definition 2.2. ([8]) Let $\nu\in(0,1)$, $u\in C[0,T]$. Then the left and right Riemann-Liouville fractional derivatives ${}_0D_t^{\nu}u(t)$ and ${}_tD_T^{\nu}u(t)$ are respectively defined by

$$
{}_0D_t^{\nu}u(t) = \frac{d}{dt}{}_0D_t^{\nu-1}u(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t (t-\theta)^{-\nu}u(\theta)d\theta, \quad t\in[0,T],
$$
$$
{}_tD_T^{\nu}u(t) = -\frac{d}{dt}{}_tD_T^{\nu-1}u(t) = \frac{-1}{\Gamma(1-\nu)}\frac{d}{dt}\int_t^T (\theta-t)^{-\nu}u(\theta)d\theta, \quad t\in[0,T].
$$

Definition 2.3. ([8]) Let $\nu\in(0,1)$, $u\in AC[0,T]$. Then the left and right Caputo fractional derivatives ${}_0^CD_t^{\nu}u(t)$ and ${}_t^CD_T^{\nu}u(t)$ are respectively defined by

$$
{}_0^CD_t^{\nu}u(t) = {}_0D_t^{\nu-1}u'(t) = \frac{1}{\Gamma(1-\nu)}\int_0^t (t-\theta)^{-\nu}u'(\theta)d\theta, \quad t\in[0,T],
$$
$$
{}_t^CD_T^{\nu}u(t) = -{}_tD_T^{\nu-1}u'(t) = \frac{-1}{\Gamma(1-\nu)}\frac{d}{dt}\int_t^T (\theta-t)^{-\nu}u'(\theta)d\theta, \quad t\in[0,T].
$$

Definition 2.4. ([20]) Let $\alpha\in(1/2,1]$. The fractional derivative space

$$
\mathbb{E}^{\alpha} = \{u\in AC([0,T],\mathbb{R}^N) : {}_0^CD_t^{\alpha}u(t)\in L^2([0,T],\mathbb{R}^N)\},
$$

is defined by the closure of $C^{\infty}([0,T],\mathbb{R}^N)$ with the norm

$$
||u||_{\alpha,2} = \Big(\int_0^T |u(t)|^2dt + \int_0^T |{}_0^CD_t^{\alpha}u(t)|^2dt\Big)^{1/2}.
$$

Remark 2.1. For any $u\in\mathbb{E}^{\alpha}$, then $u\in L^2([0,T],\mathbb{R}^N)$ and ${}_0^CD_t^{\alpha}u(t)\in L^2([0,T],\mathbb{R}^N)$.

Lemma 2.1. ([21]) Let $q(t)\in C([0,T])$ is such that $0<q_0\le q(t)\le q^0$ and $u\in\mathbb{E}^{\alpha}$. The norm $||u||_{\alpha,2}$ is equivalent to

$$
||u||_{\alpha} = \Big(\int_0^T q(t)|u(t)|^2dt + \int_0^T |{}_0^CD_t^{\alpha}u(t)|^2dt\Big)^{1/2}. \tag{1.2}
$$

Lemma 2.2. ([20]) Let $\alpha\in(0,1]$, the space $\mathbb{E}^{\alpha}$ is a reflexive and separable Banach space.

Lemma 2.3. ([19]) Let $\alpha\in(0,1]$ and $p\in[1,+\infty)$. For any $u\in L^p([0,T],\mathbb{R}^N)$,

$$
||{}_0D_{\xi}^{-\alpha}u||_{L^p([0,t])} \le \frac{t^{\alpha}}{\Gamma(\alpha+1)}||u||_{L^p([0,t])}, \quad \xi\in[0,t], \quad t\in[0,T].
$$

Lemma 2.4. ([19]) Let $\eta>0$, $p,q\ge1$, $\frac{1}{p}+\frac{1}{q}\le 1+\eta$ or $p\ne1, q\ne1$, $\frac{1}{p}+\frac{1}{q}=1+\eta$, then the following property of fractional integration

$$
\int_a^b [{}_aD_t^{-\eta}u(t)]v(t)dt = \int_a^b [{}_tD_b^{-\eta}v(t)]u(t)dt,
$$

holds, provided that $u(t)\in L^p([a,b],\mathbb{R}^N)$, $v(t)\in L^q([a,b],\mathbb{R}^N)$.

Lemma 2.5. ([20]) Let $\alpha\in(1/2,1]$ and the sequence $\{u_n\}$ converges weakly to u in $\mathbb{E}^{\alpha}$, i.e., $u_n\rightharpoonup u$. Then $u_n\to u$ in $C([0,T],\mathbb{R}^N)$, i.e., $||u_n-u||_{\infty}\to0$ as $n\to\infty$.

Theorem 2.1. ([35]) Let X be a reflexive real Banach space, let $\Phi: X\to\mathbb{R}$ be a sequentially weakly lower semi-continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, and let $\Psi: X\to\mathbb{R}$ be a sequentially weakly upper semi-continuous and continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exist $r\in\mathbb{R}$ and $x_0,\bar{x}\in X$, with $\Phi(x_0)<r<\Phi(\bar{x})$ and $\Psi(x_0)=0$, such that

(i) $\sup\limits_{\Phi(x)\le r}\Psi(x) < (r-\Phi(x_0))\frac{\Psi(\bar{x})}{\Phi(\bar{x})-\Phi(x_0)}$,

(ii) for each $\lambda\in\Lambda_r := \Big(\frac{\Phi(\bar{x})-\Phi(x_0)}{\Psi(\bar{x})}, \frac{r-\Phi(x_0)}{\sup_{\Phi(x)\le r}\Psi(x)}\Big)$ the functional $\Phi-\lambda\Psi$ is coercive.

Then, for each $\lambda\in\Lambda_r$ the functional $\Phi-\lambda\Psi$ has at least three distinct critical points in X.

3. Main result

In this section, we introduce a new norm for space $\mathbb{E}^\alpha$ and demonstrate its equivalence to the standard norm (1.2). We then present the definition of weak solutions for problem (1.1) and define two functionals Φ and Ψ. To begin with, we define a new norm for fractional derivative space $\mathbb{E}^\alpha$ as follows:

$$||u|| = \left(\int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} q(t)|u(t)|^2 dt \right)^{1/2}. \tag{3.1}$$

Lemma 3.1. For $u \in \mathbb{E}^\alpha$, the norm $||u||_\alpha$ is equivalent to $||u||$, i.e., there exist two positive constants ρ_1 and ρ_2 such that

$$\rho_1 ||u||_\alpha \le ||u|| \le \rho_2 ||u||_\alpha, \quad \text{for all } u \in \mathbb{E}^\alpha.$$

Proof. Choosing ρ_2=1, it is easy to see that $||u|| \le \rho_2||u||_\alpha$. Let $\ell = \sum_{k=0}^{l} (t_{k+1} - s_k)$, by Lemma 2.3, we obtain

$$\begin{aligned}
|u(0)|^2 &= \frac{1}{\ell} \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} |u(0)|^2 dt = \frac{1}{\ell} \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} |u(t) - {}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))|^2 dt \\
&\le \frac{2}{\ell} \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} (|u(t)|^2 + |{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))|^2) dt \\
&\le \frac{2}{\ell} \left[\int_0^T |{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))|^2 dt + \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} |u(t)|^2 dt \right] \\
&\le \frac{2}{\ell} \left[\left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} |u(t)|^2 dt \right] \quad \text{(by using Lemma 2.3)} \\
&\le \frac{2}{\ell(\min\{1, q_0\})} \left[\left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} q(t)|u(t)|^2 dt \right] \\
&\le \frac{2}{\ell(\min\{1, q_0\})} \max\left\{ 1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \right\} ||u||^2.
\end{aligned}$$

This, together with Lemma 2.3, yields

$$\begin{aligned}
\int_0^T |u(t)|^2 dt &= \int_0^T |u(0) + {}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))|^2 dt \\
&\le 2 \int_0^T \left(|u(0)|^2 + |{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))|^2 \right) dt \\
&\le 2 \left[\left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + |u(0)|^2 T \right] \quad \text{(by using Lemma 2.3)} \\
&\le 2 \left[\left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt \right. \\
&\qquad \left. + \frac{2T||u||^2}{\ell(\min\{1, q_0\})} \max\left\{ 1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \right\} \right] \\
&\le 2 \left[\left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 + \frac{2T}{\ell(\min\{1, q_0\})} \max\left\{ 1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \right\} \right] ||u||^2 := \Xi ||u||^2.
\end{aligned}$$

It then follows that

$$\begin{aligned}
||u||_\alpha^2 &= \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + \int_0^T q(t)|u(t)|^2 dt \\
&\le \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + q^0 \int_0^T |u(t)|^2 dt \le (1 + q^0 \Xi)||u||^2.
\end{aligned}$$

Take $\rho_1 = (1 + q^0\Xi)^{-1/2}$, we get $\rho_1||u||_\alpha \le ||u||$. The proof is therefore complete. □

Lemma 3.2. If $\alpha \in (1/2, 1]$, then $||u||_\infty \le M||u||$, where

$$M := \frac{T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} + \left[\frac{2}{T_1(\min\{1, q_0\})} \max\left\{ 1, \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \right)^2 \right\} \right]^{1/2}.$$

Proof. For any $u \in \mathbb{E}^\alpha$, by using the Hölder's inequality, we have

$$\begin{aligned}
|u(t)| &\le |{}_0D_t^{-\alpha}({}_0^CD_t^\alpha u(t))| + |u(0)| \\
&= \frac{1}{\Gamma(\alpha)}\Big|\int_0^t (t-s)^{\alpha-1}{}_0^CD_s^\alpha u(s)ds\Big| + |u(0)| \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{2(\alpha-1)}ds\right)^{1/2}\left(\int_0^T |{}_0^CD_s^\alpha u(s)|^2 ds\right)^{1/2} + |u(0)| \\
&\le \frac{T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\left(\int_0^T |{}_0^CD_t^\alpha u(t)|^2 dt\right)^{1/2} + |u(0)| \le M||u||,
\end{aligned}$$

which implies that $||u||_\infty \le M||u||$. The proof is complete. □

Lemma 3.3. A function $u \in \mathbb{E}^\alpha$ is a solution of problem (1.1), then we have the following identity

$$\int_0^T ({}_0^CD_t^\alpha u(t))({}_0^CD_t^\alpha v(t))dt + \sum_{k=1}^n I_i(u(t_k))v(t_k) + \sum_{k=0}^n \int_{s_k}^{t_{k+1}} q(t)u(t)v(t)dt = \lambda\sum_{k=0}^n \int_{s_k}^{t_{k+1}} f_k(t,u(t))v(t)dt. \tag{3.2}$$

holds for any $v \in \mathbb{E}^\alpha$.

Proof. For $u, v \in \mathbb{E}^\alpha$, by Lemma 2.4, we have

$$\begin{aligned}
&\int_0^T ({}_0^CD_t^\alpha u(t))({}_0^CD_t^\alpha v(t))dt = \int_0^T {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t))v'(t)dt \\
&= \sum_{k=0}^l \int_{s_k}^{t_{k+1}} {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t))v'(t)dt + \sum_{k=1}^l \int_{t_k}^{s_k} {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t))v'(t)dt \\
&= \sum_{k=0}^l {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t))v(t)\Big|_{s_k^+}^{t_{k+1}^-} + \sum_{k=0}^l \int_{s_k}^{t_{k+1}} {}_tD_T^{\alpha}({}_0^CD_t^\alpha u(t))v(t)dt \\
&\quad + \sum_{k=1}^l {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t))v(t)\Big|_{t_k^+}^{s_k^-} - \sum_{k=1}^l \int_{t_k}^{s_k} \frac{d}{dt}({}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t)))v(t)dt \\
&= \sum_{k=1}^l \left[{}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t_k^-))v(t_k) - {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(t_k^+))v(t_k)\right] \\
&\quad + \sum_{k=1}^l \left[{}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(s_k^-))v(s_k) - {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(s_k^+))v(s_k)\right] \\
&\quad + {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(T))v(T) - {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(0))v(0) \\
&\quad + \lambda\sum_{k=0}^l \int_{s_k}^{t_{k+1}} f_k(t,u(t))v(t)dt - \sum_{k=0}^l \int_{s_k}^{t_{k+1}} q(t)u(t)v(t)dt \\
&= -\sum_{k=1}^n I_k(u(t_k))v(t_k) + \lambda\sum_{k=0}^l \int_{s_k}^{t_{k+1}} f_k(t,u(t))v(t)dt - \sum_{k=0}^l \int_{s_k}^{t_{k+1}} q(t)u(t)v(t)dt.
\end{aligned}$$

This lemma is proved. □

Definition 3.1. A function $u \in \mathbb{E}^\alpha$ is called a weak solution of problem (1.1), if (3.2) holds for any $v \in \mathbb{E}^\alpha$.

In order to apply Theorem 2.1 to our problem, we define the functionals $\Phi, \Psi : \mathbb{E}^\alpha \to \mathbb{R}$ as follows:

$$\Phi(u) = \frac{1}{2}||u||^2 + \sum_{k=1}^l \int_0^{u(t_k)} I_k(s)ds, \quad \Psi(u) = \sum_{k=0}^l \int_{s_k}^{t_{k+1}} F_k(t,u(t))dt, \tag{3.3}$$

for $u \in \mathbb{E}^\alpha$. Standard arguments show that $\Phi \in C^1(\mathbb{E}^\alpha, \mathbb{R})$, Ψ has continuous Gâteaux derivatives. For any $v(t) \in \mathbb{E}^\alpha$, their Gâteaux derivatives at the point $u(t) \in \mathbb{E}^\alpha$ are the functional $\Phi'(u), \Psi'(u) \in (\mathbb{E}^\alpha)^*$, respectively, given by

$$\Phi'(u)(v) = \int_0^T ({}_0^CD_t^\alpha u(t))({}_0^CD_t^\alpha v(t))dt + \sum_{k=0}^l \int_{s_k}^{t_{k+1}} q(t)u(t)v(t)dt + \sum_{k=1}^l I_k(u(t_k))v(t_k), \tag{3.4}$$

$$\Psi'(u)(v) = \sum_{k=0}^l \int_{s_k}^{t_{k+1}} f_k(t,u(t))v(t)dt. \tag{3.5}$$

Obviously, one way we look for the weak solutions of problem (1.1) is to establish the critical points of functional $\Phi-\lambda\Psi$. Moreover, similar to Lemma 3.3 in [34], the weak solution of FDEs (1.1) is also a classical one.

Theorem 3.1. Assume that the following conditions hold:

(C1) There exist positive constants $a, b>0$, such that $F_k(t,u)\le au^2+b$, where $F_k(t,u)=\int_0^u f_k(t,s)ds,\ k=0,1,\cdots,l.$

(C2) $I_k(u)$ is nondecreasing and $I_k(u)u\ge 0$, for all $u\in\mathbb{R}$, $k=1,\cdots,l$.

(C3) There exist positive constants $c, d>0$, such that

$$\frac{c^2}{2M^2}<\frac{1}{2}d^2\sum_{k=0}^{l}\int_{s_k}^{t_{k+1}}q(t)dt+\sum_{k=1}^{l}\int_0^d I_k(s)ds,$$

and

$$2aTM^2<\Lambda_1:=\frac{2M^2}{c^2}\sum_{k=0}^{l}\int_{s_k}^{t_{k+1}}\max_{u\in[-c,c]}F_k(t,u)dt<\Lambda_2:=\frac{\sum_{k=0}^{l}\int_{s_k}^{t_{k+1}}F_k(t,d)dt}{(d^2/2)\sum_{k=0}^{l}\int_{s_k}^{t_{k+1}}q(t)dt+\sum_{k=1}^{l}\int_0^d I_k(s)ds}.$$

Then, for any $\lambda\in[1/\Lambda_2, 1/\Lambda_1]$, the problem (1.1) has at least three distinct solutions in $\mathbb{E}^\alpha$.

In order to prove Theorem 3.1, we first prove three auxiliary lemmas.

Lemma 3.3. Assume that (C2) holds. Then the functional $\Phi:\mathbb{E}^\alpha\to\mathbb{R}$ is weakly lower semi-continuous and coercive.

Proof. Let $\{u_n\}\subset\mathbb{E}^\alpha$ satisfies $u_n\rightharpoonup u$ as $n\to\infty$, then we have $||u||\le\liminf_{n\to\infty}||u_n||$. Moreover, by Lemma 2.5, we also get $\{u_n\}$ converges uniformly to u on $C([0,T])$. As a consequence, we obtain

$$\liminf_{n\to\infty}\Phi(u_n)=\liminf_{n\to\infty}\left[\frac{1}{2}||u_n||^2+\sum_{k=1}^{l}\int_0^{u_n(t_k)}I_k(s)ds\right]\ge\frac{1}{2}||u||^2+\sum_{k=1}^{l}\int_0^{u(t_k)}I_k(s)ds=\Phi(u).$$

which means that Φ is weakly lower semi-continuous. Moreover, by using the condition (C2), it can easily be seen that Φ is coercive. The lemma is proved. □

Lemma 3.4. Assume that (C2) holds. Then the functional $\Phi':\mathbb{E}^\alpha\to(\mathbb{E}^\alpha)^*$ admits a continuous inverse on $(\mathbb{E}^\alpha)^*$.

Proof. We first show that Φ' is coercive. In fact, for every $u\in\mathbb{E}^\alpha\backslash\{0\}$, by (3.4) and condition (C2), we have

$$\Phi'(u)(u)=\int_0^T|{}_0^CD_t^\alpha u(t)|^2dt+\sum_{k=0}^{l}\int_{s_k}^{t_{k+1}}q(t)|u(t)|^2dt+\sum_{k=1}^{l}I_k(u(t_k))u(t_k)\ge||u||^2.$$

which implies that Φ' is coercive. Moreover, for given $u,v\in\mathbb{E}^\alpha$, it follows from condition (C2) that

$$(\Phi'(u)-\Phi'(v))(u-v)=||u-v||^2+\sum_{k=1}^{l}[I_k(u(t_k))-I_k(v(t_k))](u(t_k)-v(t_k))\ge||u-v||^2.$$

This implies, Φ' is uniformly monotone. In view of [36, Theorem 26.A (d)], we see that $(\Phi')^{-1}$ exists and is continuous on $(\mathbb{E}^\alpha)^*$. The proof is complete. □

Lemma 3.5. $\Psi:\mathbb{E}^\alpha\to\mathbb{R}$ is weakly upper semi-continuous and $\Psi':\mathbb{E}^\alpha\to(\mathbb{E}^\alpha)^*$ is a continuous and compact functional.

Proof. Easily, we can obtain that Ψ is weakly upper semi-continuous. In order to prove Ψ' is continuous and compact, we first show that Ψ' is strongly continuous on $\mathbb{E}^\alpha$. In fact, let $\{u_n\}\subset\mathbb{E}^\alpha$ satisfies $u_n\rightharpoonup u$ as $n\to\infty$, then by Lemma 2.5, we know that $\{u_n\}$ converges uniformly to u on $C([0,T])$. Since the functions $f_k\in C((s_k,t_{k+1}]\times\mathbb{R},\mathbb{R})$, it follows that $f_k(t,u_n)\to f_k(t,u)$, as $n\to\infty$. Then we obtain $\Psi'(u_n)\to\Psi'(u)$ as $n\to\infty$, that is, Ψ' is strongly continuous on $\mathbb{E}^\alpha$. Furthermore, by [36, Proposition 26.2] we can infer that Ψ' is a compact operator. This completes the proof. □

By using Theorem 2.1, we are now turning to the proof of Theorem 3.1.

Proof. (**Theorem 3.1.**) From Lemma 3.3 and Lemma 3.4 that Φ is a weakly lower semi-continuous, coercive and continuously Gâteaux differentiable functional, and its Gâteaux derivative admits a continuous inverse on $(\mathbb{E}^\alpha)^*$. By Lemma 3.5, we also obtain that Ψ is a weakly upper semi-continuous and continuously Gâteaux differentiable functional, and its Gâteaux derivative is compact. Let $r = c^2(2M^2)^{-1}$ and choose $u_0 = 0$, $u_1 = d$. Obviously, $u_0, u_1 \in \mathbb{E}^\alpha$. Furthermore,

$$\Phi(u_0) = 0, \quad \Psi(u_1) = \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} F_k(t, d)dt,$$

$$\Phi(u_1) = \frac{1}{2}d^2 \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} q(t)dt + \sum_{k=1}^{l} \int_0^d I_k(s)ds.$$

By condition (C3), we can obtain $\Phi(u_0) < r < \Phi(u_1)$. On the other hand, for any $u \in \mathbb{E}^\alpha$ such that $\Phi(u) \le r$, then $||u||^2 \le 2r$. It follows from Lemma 3.2 that $||u||_\infty \le c$. Therefore,

$$\sup_{\Phi(u)\le r} \Psi(u) \le \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} \max_{u\in[-c,c]} F_k(t, u)dt.$$

This combining with condition (C3), we can derive

$$\begin{aligned}(r - \Phi(u_0))\frac{\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= \frac{c^2}{2M^2} \cdot \frac{\sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} F_k(t, d)dt}{(d^2/2)\sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} q(t)dt + \sum_{k=1}^{l} \int_0^d I_k(s)ds} \\ &> \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} \max_{u\in[-c,c]} F_k(t, u)dt \ge \sup_{\Phi(u)\le r} \Psi(u),\end{aligned}$$

that is, the condition (i) of Theorem 2.1 holds. For any $u \in \mathbb{E}^\alpha$, in view of the conditions (C1), (C2), Lemma 3.2 and Eq. (3.3), we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &= \frac{1}{2}||u||^2 + \sum_{k=1}^{l} \int_0^{u(t_k)} I_k(s)ds - \lambda \sum_{k=0}^{l} \int_{s_k}^{t_{k+1}} F_k(t, u(t))dt \\ &\ge \left(\frac{1}{2} - \lambda aTM^2\right) ||u||^2 - \lambda bT. \end{aligned} \tag{3.6}$$

By condition (C3), one has

$$\frac{1}{2} - \lambda aTM^2 > 0, \quad \lambda \in [1/\Lambda_2, 1/\Lambda_1]. \tag{3.7}$$

It follows from the inequalities (3.6) and (3.7) that $\lim_{||u||\to\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty$. Thus, the condition (ii) of Theorem 2.1 holds. According to Theorem 2.1, for any $\lambda \in [1/\Lambda_2, 1/\Lambda_1]$, the functional $\Phi - \lambda\Psi$ possesses at least three distinct critical points in $\mathbb{E}^\alpha$, i.e., the impulsive problem (1.1) has at least three solutions. The proof is completed. □

Remark 3.1. Assume that the conditions (C1)-(C3) hold. If there exists a $k \in \{0, 1, 2, \cdots, l\}$, such that $f_k(t, 0) \ne 0$, then problem (1.1) has at least three nonzero solutions.

4. Example

Example 4.1. Consider the following fractional Neumann impulsive problem

$$\begin{cases} {}_tD_1^{3/4}({}_0^CD_t^{3/4}u(t)) + \frac{1}{45}u(t) = \lambda f_k(t, u(t)), \ t \in (s_i, t_{i+1}], \ i = 0, 1, \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u))(t_i) = I_1(u(t_1)), \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(t_1^+), \ t \in (t_1, s_1], \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(s_1^-) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(s_1^+), \\ {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(0) = {}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u)(1) = 0, \end{cases} \tag{4.1}$$

Corresponding to problem (1.1), here

$$l = 1,\ 0 = s_0 < t_1 = \frac{1}{2} < s_1 = \frac{3}{5} < t_2 = 1,\ T = 1,\ \alpha = \frac{3}{4},$$
$$f_k(t,u) = \cos u + \frac{1}{2}u,\ k = 0,1,\ I_1(u) = \frac{1}{100}u,\ q(t) = \frac{1}{45},$$

Obviously, the condition (C2) holds. Choose,

$$a = \frac{1}{4},\ b = c = 1,\ d = \frac{\pi}{2}.$$

Through direct calculation, we can obtain

$$\begin{aligned}
&T_1 = 0.9,\ M \approx 3.785,\ F_k(t,u) = \sin u + \frac{1}{4}u^2 \le \frac{1}{4}u^2 + 1,\ k = 0,1,\\
&0.035 \approx \frac{c^2}{2M^2} < \frac{1}{2}d^2 \sum_{k=0}^{1} \int_{s_k}^{t_{k+1}} q(t)dt + \int_0^d I_1(s)ds = \frac{\pi^2}{8}\left(\frac{9}{10}\cdot\frac{1}{45} + \frac{1}{100}\right) \approx 0.037,\\
&2aTM^2 = \frac{1}{2}M^2 < \Lambda_1 = 2M^2 \sum_{k=0}^{1} \int_{s_k}^{t_{k+1}} \max_{u\in[-1,1]} F_k(t,u)dt = \frac{9M^2}{5}\left(\sin 1 + \frac{1}{4}\right) \approx 28.15\\
&\qquad < \Lambda_2 = \frac{\sum_{k=0}^{1} \int_{s_i}^{t_{k+1}} F_k(t,\pi/2)dt}{(\pi^2/8)\sum_{k=0}^{1} \int_{s_k}^{t_{k+1}} q(t)dt + \int_0^d I_1(s)ds} \approx \frac{1}{0.037}\cdot\frac{9}{10}\left(1 + \frac{\pi^2}{16}\right) \approx 39.33.
\end{aligned}$$

Therefore, the assumptions (C1) and (C3) in Theorem 3.1 hold. Now all the assumptions in Theorem 3.1 are satisfied and, consequently, its conclusion implies that for any $\lambda \in [1/\Lambda_2, 1/\Lambda_1]$ the BVP (4.1) has at least three solutions.

Remark 4.1. For impulsive fractional BVP (4.1), note that $f_k(t,0) = 1 \neq 0,\ k = 0,1,$ thus the solutions of problem (4.1) are nonzero solutions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have made equal contributions to each part of this paper. All the authors read and approved the final manuscript.

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