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THEORETICAL STUDY OF A CLASS OF ζ-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract A study of an important class of nonlinear fractional differential equations driven by ζ-Caputo type derivative in a Banach space framework is presented. The classical Banach contraction principle associated with the Bielecki-type norm and a fixed-point theorem with respect to convex-power condensing operators are used to achieve some existence results. Two illustrative examples are provided to justify the theoretical results.

Keywords ζ-Caputo derivative, fixed point theorem, Hausdorff measure of noncompactness.

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1. Introduction

The present paper is devoted to analyzing the following problem with a constant coefficient $\rho > 0$ of the form:

$$\begin{cases} \left({}^c\mathcal{D}_{a^+}^{\vartheta;\zeta} + \rho\, {}^c\mathcal{D}_{a^+}^{\vartheta-1;\zeta}\right) y(t) = g(t, y(t)),\ t \in \mathcal{J} := [a, b], \\ y(a) = y'(a) = 0, \end{cases} \tag{1.1}$$

where $1 < \vartheta < 2$, ${}^c\mathcal{D}_{a^+}^{\theta;\zeta}$ is the Caputo fractional derivative with respect to ζ of order $\theta \in \{\vartheta, \vartheta - 1\}$, $g : \mathcal{J} \times \mathbb{F} \to \mathbb{F}$ is a given function verifying some assumptions that will be precised later and $(\mathbb{F}, \|\cdot\|)$ is a real Banach space.

The theory of differential equations involving non-integer order derivatives have become an indispensable tool as they arise in the modeling of various phenomena in numerous scientific and engineering disciplines. Numerous authors have investigated different aspects of the theory, see [1, 11, 15].

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Nowadays, differential equations of non-integer order with respect to another function, recently introduced in [4], occur in various concrete models. For instance, they appear in several anomalous diffusions, including ultra-slow processes [18], Heston model [6], random walks [13] financial crisis [19] and Verhulst model [7]. Therefore, a considerable attention has been given to the quantitative and qualitative proprieties of solutions of some kind of differential problems governed by φ-Caputo type [5,8,12].

Problem (1.1) has been considered in a finite-dimensional Banach space by Mahrouz et al. [22], they obtained some existence results under Lipschizianity and growth conditions (among other extra assumptions). By imposing reasonable conditions, we extend the previous results in general setting, namely, when the nonlinear function g acts on an infinite dimensional Banach space. The proof of our main results consists, firstly, to combine the classical Banach contraction principle with the Bielecki type norm which allows us to obtain a global existence and uniqueness result and dropping the extra assumption appearing in [22, Theorem 7]. Secondly, based on measure of noncompactness (MNC) technique and convex-power condensing (CPC) operator fixed point theorem, a new existence theorem is proved which improve considerably the result proved in [22, Theorem 6].

The current paper is divided into four sections: In Section 2, we collect a basic background needed in the sequel. In Section 3, Banach's fixed point theorem and fixed point theorem with respect to CPC operator is used to obtain a new existence criterion. Finally, two illustrative example are presented in Section 4.

2. Preliminaries

We endow the space $C(\mathcal{J},\mathbb{F})$ of continuous functions $z:\mathcal{J}\to\mathbb{F}$ by the norm

$$\|z\|_\infty=\sup_{t\in\mathcal{J}}\|z(t)\|,\quad \forall z\in C(\mathcal{J},\mathbb{F}).$$

$L^1(\mathcal{J},\mathbb{F})$ denotes the space of Bochner integrable functions $z:\mathcal{J}\to\mathbb{F}$ normed by

$$\|z\|_{L^1(\mathcal{J},\mathbb{F})}=\int_a^b\|z(t)\|dt,\quad \forall z\in L^1(\mathcal{J},\mathbb{F}).$$

We also define

$$\mathbb{T}^1_+(\mathcal{J},\mathbb{R})=\{\zeta\ :\ \zeta\in C^1(\mathcal{J},\mathbb{R})\ \text{ and }\ \zeta'(t)>0\ \text{ for all }\ t\in\mathcal{J}\}.$$

For $\zeta\in\mathbb{T}^1_+(\mathcal{J},\mathbb{R})$ and $t,s\in\mathcal{J}$, $(t>s)$, we pose

$$\zeta(t,s)=\zeta(t)-\zeta(s)\ \text{ and }\ \zeta(t,s)^\vartheta=(\zeta(t)-\zeta(s))^\vartheta.$$

Definition 2.1. [4,17] Let $\zeta\in\mathbb{T}^1_+(\mathcal{J},\mathbb{R})$ and $\vartheta>0$. The ζ-fractional integral (FI) of a function f of order ϑ is defined as

$$\mathcal{I}^{\vartheta,\zeta}_{a^+}f(t)=\frac{1}{\Gamma(\vartheta)}\int_a^t\zeta(t,s)^{\vartheta-1}\zeta'(s)f(s)ds,\quad t>a,$$

with $\Gamma(\cdot)$ the gamma function .

Lemma 2.1. [4,17] *Let $\vartheta, \gamma > 0$, then*

$$\mathcal{I}_{a^+}^{\vartheta;\zeta}\ \zeta(t,a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\vartheta+\gamma)}\zeta(t,a)^{\vartheta+\gamma-1}.$$

Definition 2.2. [4] Let $n-1 < \vartheta \le n$ with $n \in \mathbb{N}$, $\zeta \in \mathbb{T}_+^1(\mathcal{J}, \mathbb{R})$. The ζ-Caputo FD of a function f of order ϑ is defined as

$$\left({}^C\mathcal{D}_{a^+}^{\vartheta;\zeta} f\right)(t) = \mathcal{I}_{a^+}^{n-\vartheta;\zeta}\left(\frac{1}{\zeta'(t)}\frac{d}{dt}\right)^n f(t).$$

Definition 2.3. [10] Let $\mathbb{G} \subset \mathbb{F}$ be a bounded set. The Hausdorff MNC of $\mathbb{G}$ is given by

$$\Lambda(\mathbb{G}) = \inf\{\varepsilon > 0 : \mathbb{G} \ \text{ has a finite } \ \varepsilon-\text{net in } \mathbb{F}\}.$$

Recall that a set $\mathbb{S} \subset \mathbb{F}$ is called an $\varepsilon-$net of $\mathbb{G}$ if $\mathbb{G} \subset \mathbb{S}+\varepsilon\overline{\mathbb{B}} \equiv \{s+\varepsilon b,\ s \in \mathbb{S}, b \in \overline{\mathbb{B}}\}$, where $\overline{\mathbb{B}}$ is the closed unit ball in $\mathbb{F}$.

Lemma 2.2. [10] *Let $\mathbb{G}, \mathbb{V} \subset \mathbb{F}$ be bounded. Then $\Lambda(\cdot)$ satisfies.*

1. $\Lambda(\mathbb{G}) = 0 \Longleftrightarrow \mathbb{G}$ *is relatively compact,*
2. $\mathbb{G} \subset \mathbb{V} \Longrightarrow \Lambda(\mathbb{G}) \le \Lambda(\mathbb{V})$,
3. $\Lambda(\mathbb{G} \cup \mathbb{V}) = \max\{\Lambda(\mathbb{G}), \Lambda(\mathbb{V})\}$,
4. $\Lambda(\mathbb{G}) = \Lambda(\overline{\mathbb{G}}) = \Lambda(\mathrm{co}(\mathbb{G}))$, *where* $\mathrm{co}\,\mathbb{G}$ *and* $\overline{\mathbb{G}}$ *represent the convex hull and the closure of* $\mathbb{G}$*, respectively,*
5. $\Lambda(\mathbb{G} + \mathbb{V}) \le \Lambda(\mathbb{G}) + \Lambda(\mathbb{V})$,
6. $\Lambda(\lambda\mathbb{G}) \le |\lambda|\Lambda(\mathbb{G})$, *for any* $\lambda \in \mathbb{R}$.

Lemma 2.3. [2] *Let $\mathbb{G}$ be a bounded set of $\mathbb{F}$. Then, fix $\epsilon > 0$, there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{G}$, such that*

$$\Lambda(\mathbb{G}) \le 2\Lambda\left(\{x_n\}_{n=1}^{\infty}\right) + \epsilon.$$

The set $\mathbb{G} \subset L^1(\mathcal{J}, \mathbb{F})$ is called uniformly integrable if, for all $x \in \mathbb{G}$, we have

$$\|x(t)\| \le \delta(t), \quad \text{for a.e. } t \in \mathcal{J},$$

with $\delta \in L^1\left(J, \mathbb{R}^+\right)$.

Lemma 2.4. [16] *Assume that $\{x_n\}_{n=1}^{\infty} \subset L^1(\mathcal{J}, \mathbb{F})$ is uniformly integrable, the map $t \longmapsto \Lambda\left(\{x_n(t)\}_{n=1}^{\infty}\right)$ is measurable, and*

$$\Lambda\left(\left\{\int_a^t x_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right) \le 2\int_a^t \Lambda\left(\{x_n(s)\}_{n=1}^{\infty}\right)\mathrm{d}s.$$

Now, for $\varsigma > 0$, we endow the space $C(\mathcal{J}, \mathbb{F})$ by the Bielecky norm

$$\|z\|_B = \sup_{t\in\mathcal{J}} e^{-\varsigma\zeta(t,a)}\|z(t)\|. \tag{2.1}$$

Lemma 2.5. [20,24] *The norms $\|\cdot\|_B$ defined by (2.1) and $\|\cdot\|_\infty$ are equivalent, i.e; there exists $\ell \in (0,\infty)$ such that*

$$\|\cdot\|_B \le \|\cdot\|_\infty \le \ell\|\cdot\|_B.$$

Lemma 2.6. [9] *Let $\vartheta > 1$ and $\varsigma > 0$. Then for all $t \in \mathcal{J}$, one has*

$$\mathcal{I}_{a^+}^{\vartheta-1;\zeta} e^{\varsigma\zeta(t,a)} \leq \frac{1}{\varsigma^{\vartheta-1}} e^{\varsigma\zeta(t,a)}.$$

Definition 2.4. Let $\mathbb{K}$ be a real Banach space, $\mathbb{J} \subset \mathbb{K}$ is a closed and convex set. The operator $\mathcal{N} : \mathbb{J} \to \mathbb{J}$ is called convex-power condensing (CPC) operator about v_0 and m_0 if $\mathcal{N}$ is bounded and continuous, and there exist $v_0 \in \mathbb{J}$ and $m_0 \in \mathbb{N}^*$ such that for any bounded and not relatively compact $\mathbb{V} \subset \mathbb{J}$, with

$$\Lambda\left(\mathcal{N}^{(m_0,v_0)}(\mathbb{V})\right) < \Lambda(\mathbb{V}),$$

where

$$\mathcal{N}^{(1,v_0)}(\mathbb{V}) \equiv \mathcal{N}(\mathbb{V}), \quad \mathcal{N}^{(m,v_0)}(\mathbb{V}) = \mathcal{N}\left(\overline{co}\left\{\mathcal{N}^{(m-1,v_0)}(\mathbb{V})\right\}\right), \quad m = 2, 3, \cdots$$

Theorem 2.1. [21] *Let $\mathbb{K}$ be a real Banach space, and let $\mathbb{V} \subset \mathbb{K}$ be a bounded, closed and convex set. If $\mathcal{N} : \mathbb{V} \to \mathbb{V}$ is a CPC operator, then $\mathcal{N}$ has at least one fixed point in $\mathbb{V}$.*

3. Main Results

We present our first result dealing with the existence and uniqueness of solutions for (1.1) by using Banach's fixed point theorem.

Theorem 3.1. *Assume that*

(C1) The function $g : \mathcal{J} \times \mathbb{F} \to \mathbb{F}$ is continuous.

(C2) There exists $G \in L^\infty(\mathcal{J}, \mathbb{R}_+)$ such that

$$\|g(t,v) - g(t,u)\| \leq G(t)\|v - u\|, \qquad \textit{for all } v, u \in \mathbb{F} \textit{ and for a.e. } t \in \mathcal{J}.$$

Then, problem (1.1) admits a unique solution defined on $\mathcal{J}$.

Proof. According to [22, Theorem 1], let us introduce $\mathcal{U} : C(\mathcal{J}, \mathbb{F}) \to C(\mathcal{J}, \mathbb{F})$ given by:

$$\mathcal{U}y(t) = (\vartheta - 1)\int_a^t \zeta'(s)e^{-\rho\zeta(t,s)}\left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} g(\tau, y(\tau))d\tau\right)ds, \quad t \in \mathcal{J}. \tag{3.1}$$

Evidently, the solution of problem (1.1) can be regarded as the fixed point of $\mathcal{U}$.

We need to show that the operator $\mathcal{U}$ is a contraction mapping on $C(\mathcal{J}, \mathbb{F})$ via the Bielecki's norm. For each $y, x \in C(\mathcal{J}, \mathbb{F})$ and all $t \in \mathcal{J}$, using (C2), we can get

$$\begin{aligned}
&\|(\mathcal{U}y)(t) - (\mathcal{U}x)(t)\| \\
&\quad \leq (\vartheta - 1)\int_a^t e^{-\rho\zeta(t,s)} \int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \|g(\tau, y(\tau)) - g(\tau, x(\tau))\| d\tau \zeta'(s)ds \\
&\quad \leq (\vartheta - 1)\int_a^t e^{-\rho\zeta(t,s)} \int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} G(\tau)\|y(\tau) - x(\tau)\| d\tau \zeta'(s)ds,
\end{aligned}$$

which, by (2.1), can be written as

$$\begin{aligned}
&\|(\mathcal{U}y)(t)-(\mathcal{U}x)(t)\| \\
&\leq (\vartheta-1)\int_a^t e^{-\rho\zeta(t,s)}\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\frac{G(\tau)\|y(\tau)-x(\tau)\|}{e^{\varsigma\zeta(\tau,a)}e^{-\varsigma\zeta(\tau,a)}}d\tau\zeta'(s)ds \\
&\leq (\vartheta-1)\|G\|_{L^\infty}\|y-x\|_B\int_a^t e^{-\rho\zeta(t,s)}\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}e^{\varsigma\zeta(\tau,a)}d\tau\zeta'(s)ds.
\end{aligned}$$

By Lemma 2.6, one obtains

$$\begin{aligned}
&\|(\mathcal{U}y)(t)-(\mathcal{U}x)(t)\| \\
&\quad\leq (\vartheta-1)\|G\|_{L^\infty}\|y-x\|_B\int_a^t \frac{e^{-\rho\zeta(t,s)}e^{\varsigma\zeta(s,a)}}{\varsigma^{\vartheta-1}}\zeta'(s)ds \\
&\quad\leq (\vartheta-1)\|G\|_{L^\infty}\|y-x\|_B\frac{e^{-\rho\zeta(t)-\varsigma\zeta(a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}\int_a^t(\rho+\varsigma)e^{(\rho+\varsigma)\zeta(s)}\zeta'(s)ds \\
&\quad\leq \frac{(\vartheta-1)e^{-\rho\zeta(t)-\varsigma\zeta(a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}\left(e^{(\rho+\varsigma)\zeta(t)}-e^{(\rho+\varsigma)\zeta(a)}\right)\|G\|_{L^\infty}\|y-x\|_B.
\end{aligned}$$

By $e^{-\rho\zeta(t)-\varsigma\zeta(a)}\leq e^{-(\rho+\varsigma)\zeta(a)}$ and $e^{(\rho+\varsigma)\zeta(t)}-e^{(\rho+\varsigma)\zeta(a)}\leq e^{(\rho+\varsigma)\zeta(t)}$, we get

$$\begin{aligned}
\|(\mathcal{U}y)(t)-(\mathcal{U}x)(t)\| &\leq \frac{(\vartheta-1)e^{-(\rho+\varsigma)\zeta(a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}e^{(\rho+\varsigma)\zeta(t)}\|G\|_{L^\infty}\|y-x\|_B \\
&\leq \frac{(\vartheta-1)e^{(\rho+\varsigma)\zeta(b,a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}\|G\|_{L^\infty}\|y-x\|_B.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{U}y-\mathcal{U}x\|_B &\leq \frac{(\vartheta-1)e^{\rho\zeta(b,a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}\|G\|_{L^\infty}\|y-x\|_B \\
&\leq \mathfrak{M}_\varsigma\|y-x\|_B,
\end{aligned}$$

where $\mathfrak{M}_\varsigma=\dfrac{(\vartheta-1)e^{\rho\zeta(b,a)}}{(\rho+\varsigma)\varsigma^{\vartheta-1}}\|G\|_{L^\infty}$.

Choosing $\varsigma>0$ large enough, the quantity $\mathfrak{M}_\varsigma$ is less than 1. This produces that

$$\|\mathcal{U}y-\mathcal{U}x\|_B\leq \mathfrak{M}_\varsigma\|y-x\|_B.$$

Therefore, by applying Banach's contraction principle (see [14]), problem (1.1) admits a unique solution in $C(\mathcal{J},\mathbb{F})$.

□

Next, we present our second result, where Theorem 2.1 is applied.

Theorem 3.2. *Assume that*

(H1) $g:\mathcal{J}\times\mathbb{F}\to\mathbb{F}$ *is Carathéodory type function i.e.*

1. *for all* $x\in\mathbb{F}$, $g(\cdot,x)$ *is measurable,*
2. *for a.e.* $t\in\mathcal{J}$, $g(t,\cdot)$ *is continuous.*

(H2) There exists $\phi \in L^{\infty}(\mathcal{J}, \mathbb{R}_+)$ and a continous nondecreasing function $\kappa : [0, \infty) \to [0, \infty)$ such that

$$\|g(t, v)\| \leq \phi(t)\kappa(\|v\|), \qquad \textit{for a.e. } t \in \mathcal{J} \textit{ and } v \in \mathbb{F}.$$

(H3) There exists constant $\xi > 0$, such that for each $t \in \mathcal{J}$,

$$\Lambda(g(t, \mathbb{U})) \leq \xi \Lambda(\mathbb{U}),$$

where $\mathbb{U}$ is a bounded and countable set in $\mathbb{F}$.

(H4) There exists a constant $K > 0$ such that

$$(\vartheta - 1)\|\phi\|_{L^{\infty}}\kappa(K)\frac{\zeta(b, a)^{\vartheta}}{\Gamma(\vartheta + 1)} \leq K. \tag{3.2}$$

Then, problem (1.1) admits a solution on $\mathcal{J}$.

Proof. Introduce again the operator $\mathcal{U}$ represented by (3.1) and define a closed bounded convex set

$$\mathbb{B}_K = \{y \in C(\mathcal{J}, \mathbb{F}) \ : \ \|y\|_{\infty} \leq K\}.$$

To verify the conditions of Theorem 2.1, we split the proof into four steps:

Step 1. $\mathcal{U}$ maps the set $\mathbb{B}_K$ into itself.

For each $y \in \mathbb{B}_K$ and $t \in \mathcal{J}$, by the hypothesis (H2) and the fact that $0 < e^{-\rho\zeta(t,s)} < 1$ for $a < s < t < b$, we have

$$\begin{aligned}
\|\mathcal{U}y(t)\| &\leq (\vartheta - 1)\int_a^t \zeta'(s)\left(\int_a^s \frac{\zeta'(\tau)\zeta(s, \tau)^{\vartheta-2}}{\Gamma(\vartheta - 1)}\|g(\tau, y(\tau))\|d\tau\right)ds \\
&\leq (\vartheta - 1)\int_a^t \zeta'(s)\left(\int_a^s \frac{\zeta'(\tau)\zeta(s, \tau)^{\vartheta-2}}{\Gamma(\vartheta - 1)}\phi(\tau)\kappa(\|y(\tau)\|)d\tau\right)ds \\
&\leq (\vartheta - 1)\|\phi\|_{L^{\infty}}\kappa(K)\int_a^t \zeta'(s)\left(\int_a^s \frac{\zeta'(\tau)\zeta(s, \tau)^{\vartheta-2}}{\Gamma(\vartheta - 1)}d\tau\right)ds.
\end{aligned}$$

Using Lemma 2.1 with $\gamma = 1$, we get

$$\begin{aligned}
\int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s, \tau)^{\vartheta-2}}{\Gamma(\vartheta - 1)}d\tau\right)\zeta'(s)ds &= \frac{1}{\Gamma(\vartheta)}\int_a^t \zeta'(s)\zeta(s, a)^{\vartheta-1}ds \\
&= \frac{1}{\Gamma(\vartheta + 1)}\zeta(t, a)^{\vartheta}.
\end{aligned}$$

Using the above estimates and hypothesis (H4), we obtain

$$\begin{aligned}
\|\mathcal{U}y\| &\leq (\vartheta - 1)\|\phi\|_{L^{\infty}}\kappa(K)\frac{\zeta(t, a)^{\vartheta}}{\Gamma(\vartheta + 1)} \\
&\leq (\vartheta - 1)\|\phi\|_{L^{\infty}}\kappa(K)\frac{\zeta(b, a)^{\vartheta}}{\Gamma(\vartheta + 1)} \\
&\leq K.
\end{aligned}$$

This proves that $\mathcal{U}$ maps $\mathbb{B}_K$ into itself.

Step 2. The continuity of $\mathcal{U}$.

Assume that $\{y_n\}$ is a sequence such that $y_n \to y$ in $\mathbb{B}_K$ as $n \to \infty$. From (H1) we can see that $g(s, y_n(s)) \to g(s, y(s))$, as $n \to +\infty$.

Recalling (H2), we deduce that

$$\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y(\tau)) - g(\tau,y_n(\tau))\| \le 2\phi(\tau)\kappa(K)\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}.$$

From Lebesgue's dominated convergence theorem and the fact that the function

$$\tau \to 2\phi(\tau)\kappa(K)\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}$$

is Lebesgue integrable on $\mathcal{J}$, one gets

$$\begin{aligned}
&\|(\mathcal{U}y_n)(t) - (\mathcal{U}y)(t)\| \\
&\le (\vartheta-1)e^{-\rho\zeta(t,a)}\int_a^t \zeta'(s)e^{\rho\zeta(s,a)}\left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y_n(\tau)) - g(\tau,y(\tau))\|d\tau\right)ds \\
&\le (\vartheta-1)\int_a^t \zeta'(s)e^{\rho\zeta(s,a)}\left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y_n(\tau)) - g(\tau,y(\tau))\|d\tau\right)ds \\
&\le (\vartheta-1)e^{\rho\zeta(b,a)}\int_a^t \zeta'(s)\left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y_n(\tau)) - g(\tau,y(\tau))\|d\tau\right)ds,
\end{aligned}$$

where we have made use of the fact that $0 < e^{-\rho\zeta(t,a)} < 1$, for each $t \in \mathcal{J}$. Therefore

$$\|(\mathcal{U}y_n)(t) - (\mathcal{U}y)(t)\| \to 0 \text{ as } n \to \infty, \quad \forall t \in \mathcal{J}.$$

Hence,

$$\|\mathcal{U}y_n - \mathcal{U}y\|_\infty \to 0 \text{ when } n \to \infty. \tag{3.3}$$

This implies that $\mathcal{U}$ is continuous.

Step 3. $\mathcal{U}(\mathbb{B}_K)$ is equicontinuous.

Let $a < t_1 < t_2 < b$ and $y \in \mathbb{B}_K$, we have

$$\|(\mathcal{U}y)(t_2) - (\mathcal{U}y)(t_1)\| \le M_1 + M_2$$

where

$$M_1 = (\vartheta-1)e^{-\rho\zeta(t_2,t_1)}\int_{t_1}^{t_2} \zeta'(s)e^{\rho\zeta(s,t_1)}\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y(\tau))\|d\tau ds,$$

and

$$M_2 = (\vartheta-1)\int_a^{t_1} \zeta'(s)\left|e^{-\rho\zeta(t_2,s)} - e^{-\rho\zeta(t_1,s)}\right|\left\|\left(\mathcal{I}_{a^+}^{\vartheta-1;\zeta}g(\tau,y(\tau))\right)(s)\right\|ds.$$

From (H2) and the fact that $e^{-\rho\zeta(t_2,t_1)} < 1$, we get

$$\begin{aligned}
M_1 &\leq (\vartheta-1)\int_{t_1}^{t_2}\zeta'(s)e^{\rho\zeta(s,t_1)}\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y(\tau))\|d\tau ds \\
&\leq (\vartheta-1)e^{\rho\zeta(b,t_1)}\int_{t_1}^{t_2}\zeta'(s)\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|g(\tau,y(\tau))\|d\tau ds \\
&\leq (\vartheta-1)e^{\rho\zeta(b,t_1)}\|\phi\|_{L^\infty}\kappa(K)\int_{t_1}^{t_2}\zeta'(s)\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}d\tau ds \\
&\leq (\vartheta-1)e^{\rho\zeta(b,t_1)}\|\phi\|_{L^\infty}\kappa(K)\int_{t_1}^{t_2}\zeta'(s)\frac{\zeta(s,a)^{\vartheta-1}}{\Gamma(\vartheta)}ds \\
&\leq \frac{(\vartheta-1)e^{\rho\zeta(b,t_1)}\|\phi\|_{L^\infty}\kappa(K)}{\Gamma(\vartheta+1)}\left(\zeta(t_2,a)^{\vartheta}-\zeta(t_1,a)^{\vartheta}\right).
\end{aligned}$$

This produces that,

$$M_1 \longrightarrow 0 \quad \text{as} \quad t_2 \longrightarrow t_1. \tag{3.4}$$

On the other side,

$$M_2 = (\vartheta-1)\Big(e^{-\rho\zeta(t_1)} - e^{-\rho\zeta(t_2)}\Big)\int_a^{t_1} e^{\rho\zeta(s)}\Big\|\Big(\mathcal{I}_{a^+}^{\vartheta-1;\zeta}g(\tau,y(\tau))\Big)(s)\Big\|\zeta'(s)ds$$

Thus,

$$M_2 \longrightarrow 0 \quad \text{when} \quad t_2 \longrightarrow t_1. \tag{3.5}$$

From (3.4) and (3.5), the equicontinuity of $\mathcal{U}(\mathbb{B}_K)$ is deduced immediately.

Step 4. $\mathcal{U} : \mathbb{O} \to \mathbb{O}$ is a CPC operator, where $\mathbb{O} = \overline{\text{co}}\mathcal{U}(\mathbb{B}_K)$.

Let $y_0 \in \mathbb{O}$. In the following, we need to show that $\mathcal{U}$ satisfies Definition 2.4. To do this, for every bounded subset $\mathbb{A} \subset C(\mathcal{J},\mathbb{F})$ we define the MNC as

$$\Lambda_C\left(\mathcal{U}^{(n,y_0)}(\mathbb{A})\right) = \sup_{t\in\mathcal{J}}\Lambda\left(\mathcal{U}^{(n,y_0)}(\mathbb{A})(t)\right), \quad n \in \mathbb{N}^*. \tag{3.6}$$

Next, fix $\varepsilon > 0$. Lemma 2.3 yields the existence of $\{y_k\}_{k=1}^{\infty} \subset \mathbb{A}$ such that

$$\begin{aligned}
&\Lambda\left(\mathcal{U}^{(1,y_0)}(\mathbb{A})(t)\right) \\
&\quad= \Lambda\left(\mathcal{U}(\mathbb{A})(t)\right) \\
&\quad\leq 2\Lambda\left\{(\vartheta-1)\int_a^t\left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}g(\tau,\{y_k(\tau)\}_{k=1}^{\infty})d\tau\right)\zeta'(s)ds\right\}+\varepsilon.
\end{aligned}$$

Lemma 2.4 and the hypothesis (H3) imply that

$$\begin{aligned}
&\Lambda\left(\mathcal{U}^{(1,y_0)}(\mathbb{A})(t)\right)\\
&\quad\leq 8(\vartheta-1)\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Lambda(g(\tau,\{y_k(\tau)\}_{k=1}^{\infty}))d\tau\right)\zeta'(s)ds+\varepsilon\\
&\quad\leq 8(\vartheta-1)\xi\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Lambda(\{y_k(\tau)\}_{k=1}^{\infty})d\tau\right)\zeta'(s)ds+\varepsilon\\
&\quad\leq 8(\vartheta-1)\xi\Lambda(\mathbb{A})\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}d\tau\right)\zeta'(s)ds+\varepsilon.
\end{aligned}$$

Using Lemma 2.1, we get

$$\begin{aligned}
\Lambda\left(\mathcal{U}^{(1,y_0)}(\mathbb{A})(t)\right)&\leq 8(\vartheta-1)\xi\Lambda(\mathbb{A})\int_a^t\frac{\zeta(s,a)^{\vartheta-1}}{\Gamma(\vartheta)}\zeta'(s)ds+\varepsilon\\
&\leq 8(\vartheta-1)\xi\Lambda(\mathbb{A})\frac{\zeta(t,a)^{\vartheta}}{\Gamma(\vartheta+1)}+\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we obtain

$$\Lambda\left(\mathcal{U}^{(1,y_0)}(\mathbb{A})(t)\right)\leq 8\xi(\vartheta-1)\frac{\zeta(t,a)^{\vartheta}}{\Gamma(\vartheta+1)}\Lambda(\mathbb{A}). \tag{3.7}$$

Now, using Lemma 2.3 again, fix $\varepsilon>0$, there is a sequence $\{x_k\}_{k=1}^{\infty}\subset\overline{\mathrm{co}}\left\{\mathcal{U}^{(1,y_0)}(\mathbb{A}),y_0\right\}$ such that

$$\begin{aligned}
&\Lambda\left(\mathcal{U}^{(2,y_0)}(\mathbb{A})(t)\right)\\
&\quad=\Lambda\left(\mathcal{U}\left(\overline{\mathrm{co}}\left\{\mathcal{U}^{(1,y_0)}(\mathbb{A}),y_0\right\}\right)(t)\right)\\
&\quad\leq 2\Lambda\left\{(\vartheta-1)\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}g(\tau,\{x_k(\tau)\}_{k=1}^{\infty})d\tau\right)\zeta'(s)ds\right\}+\varepsilon.
\end{aligned}$$

Another recalling of Lemma 2.4 and (H3), it yields

$$\begin{aligned}
&\Lambda\left(\mathcal{U}^{(2,y_0)}(\mathbb{A})(t)\right)\\
&\leq 8\xi(\vartheta-1)\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Lambda(\{x_k(\tau)\}_{k=1}^{\infty})d\tau\right)\zeta'(s)ds+\varepsilon\\
&\leq 8\xi(\vartheta-1)\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Lambda\left(\overline{\mathrm{co}}\left\{\mathcal{U}^{(1,y_0)}(\mathbb{A}),y_0\right\}(\tau)\right)d\tau\right)\zeta'(s)ds+\varepsilon\\
&\leq 8\xi(\vartheta-1)\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\Lambda\left(\mathcal{U}^{(1,y_0)}(\mathbb{A})(\tau)\right)d\tau\right)\zeta'(s)ds+\varepsilon\\
&\leq\frac{(8\xi(\vartheta-1))^2}{\Gamma(\vartheta+1)}\Lambda(\mathbb{A})\int_a^t\left(\int_a^s\frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)}\zeta(\tau,a)^{\vartheta}d\tau\right)\zeta'(s)ds+\varepsilon.
\end{aligned}$$

Using Lemma 2.1, we have

$$\int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \zeta(\tau,a)^{\vartheta} d\tau \right) \zeta'(s)ds = \frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta)} \int_a^t \zeta'(s)\zeta(s,a)^{2\vartheta-1} ds$$
$$= \frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta+1)} \zeta(t,a)^{2\vartheta}.$$

By the above arguments, we get

$$\Lambda \left(\mathcal{U}^{(2,y_0)}(\mathbb{A})(t) \right) \leq \frac{(8\xi(\vartheta-1))^2}{\Gamma(2\vartheta+1)} \zeta(t,a)^{2\vartheta} \Lambda(\mathbb{A}) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\Lambda \left(\mathcal{U}^{(2,y_0)}(\mathbb{A})(t) \right) \leq \frac{(8\xi(\vartheta-1))^2}{\Gamma(2\vartheta+1)} \zeta(t,a)^{2\vartheta} \Lambda(\mathbb{A}).$$

Repeating the process for $n = 3, 4, \cdots$, for each $t \in \mathcal{J}$, we can show by mathematical induction, that

$$\Lambda \left(\mathcal{U}^{(n,y_0)}(\mathbb{A})(t) \right) \leq \frac{(8\xi(\vartheta-1))^n}{\Gamma(n\vartheta+1)} \zeta(t,a)^{n\vartheta} \Lambda(\mathbb{A}). \tag{3.8}$$

For this, we assume that (3.8) holds for some n and check that it is true for $n+1$.

Fix $\varepsilon > 0$. Lemma 2.3 yields the existence of $\{z_k\}_{k=1}^{\infty} \subset \overline{\text{co}} \left\{ \mathcal{U}^{(n,y_0)}(\mathbb{A}), y_0 \right\}$ such that

$$\begin{aligned}
&\Lambda \left(\mathcal{U}^{(n+1,y_0)}(\mathbb{A})(t) \right) \\
&\quad = \Lambda \left(\mathcal{U} \left(\overline{\text{co}} \left\{ \mathcal{U}^{(n,y_0)}(\mathbb{A}), y_0 \right\} \right)(t) \right) \\
&\quad \leq 2\Lambda \left\{ (\vartheta-1) \int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} g(\tau, \{z_k(\tau)\}_{k=1}^{\infty}) d\tau \right) \zeta'(s) ds \right\} + \varepsilon.
\end{aligned}$$

From (H3) and Lemma 2.4, one has

$$\begin{aligned}
&\Lambda \left(\mathcal{U}^{(n+1,y_0)}(\mathbb{A})(t) \right) \\
&\leq 8\xi(\vartheta-1) \int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \Lambda(\{z_k(\tau)\}_{k=1}^{\infty}) d\tau \right) \zeta'(s)ds + \varepsilon \\
&\leq 8\xi(\vartheta-1) \int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \Lambda \left(\overline{\text{co}} \left\{ \mathcal{U}^{(n,y_0)}(\mathbb{A}), y_0 \right\}(\tau) \right) d\tau \right) \zeta'(s)ds + \varepsilon \\
&\leq 8\xi(\vartheta-1) \int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \Lambda \left(\mathcal{U}^{(n,y_0)}(\mathbb{A})(\tau) \right) d\tau \right) \zeta'(s)ds + \varepsilon \\
&\leq \frac{(8\xi(\vartheta-1))^{n+1}}{\Gamma(n\vartheta+1)} \Lambda(\mathbb{A}) \int_a^t \left(\int_a^s \frac{\zeta'(\tau)\zeta(s,\tau)^{\vartheta-2}}{\Gamma(\vartheta-1)} \zeta(\tau,a)^{n\vartheta} d\tau \right) \zeta'(s)ds + \varepsilon \\
&\leq \frac{(8\xi(\vartheta-1))^{n+1}}{\Gamma((n+1)\vartheta+1)} \zeta(t,a)^{(n+1)\vartheta} \Lambda(\mathbb{A}) + \varepsilon.
\end{aligned}$$

Hence

$$\Lambda \left(\mathcal{U}^{(n+1,y_0)}(\mathbb{A})(t) \right) \leq \frac{(8\xi(\vartheta-1))^{n+1}}{\Gamma((n+1)\vartheta+1)} \zeta(t,a)^{(n+1)\vartheta} \Lambda(\mathbb{A}).$$

From (3.6) and (3.8), we get that

$$\Lambda_C\left(\mathcal{U}^{(n,y_0)}(\mathbb{A})\right) \leq \frac{(8\xi(\vartheta-1))^n}{\Gamma(n\vartheta+1)}\zeta(t,a)^{n\vartheta}\Lambda(\mathbb{A}). \tag{3.9}$$

Now, we prove that the series

$$\sum_{n=0}^{\infty}\frac{(8\xi(\vartheta-1))^n\zeta(t,a)^{n\vartheta}}{\Gamma(n\vartheta+1)}$$

is convergent. Applying the ratio test, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{(8\xi(\vartheta-1))^{(n+1)}\zeta(t,a)^{(n+1)\vartheta}}{\Gamma((n+1)\vartheta+1)}&\frac{\Gamma(n\vartheta+1)}{(8\xi(\vartheta-1))^n\zeta(t,a)^{n\vartheta}}\\ &=\lim_{n\to\infty}8\xi(\vartheta-1)\zeta(t,a)^{\vartheta}\frac{\Gamma(n\vartheta+1)}{\Gamma(n\vartheta+1+\vartheta)}\\ &=0.\end{aligned}$$

(Notice that (see eq. (1) in [23])

$$\frac{\Gamma(n\vartheta+1)}{\Gamma(n\vartheta+1+\vartheta)}=\frac{1}{((n+1)\vartheta+1)^{\vartheta}}\left(1-\frac{\vartheta(\vartheta-1)}{2((n+1)\vartheta+1)}+O(((n+1)\vartheta+1)^{-2})\right),$$

where O is the Landau symbol).

Hence, there exists a positive integer n_0, such that

$$\frac{(8\xi(\vartheta-1))^{n_0}}{\Gamma(n_0\vartheta+1)}\zeta(t,a)^{n_0\vartheta}<1. \tag{3.10}$$

Therefore, Definition 2.4 is verified, it follows that $\mathcal{U}:\mathbb{O}\to\mathbb{O}$ is a CPC operator.

Then, Theorem 2.1 entails that $\mathcal{U}$ admits a fixed point $y\in\mathbb{O}$ and it is the solution of (1.1). □

4. Examples

This section provides two examples illustrating our main results.

Example 1. Consider the following problem :

$$\begin{cases}\left({}^{c}\mathcal{D}_{1^+}^{\vartheta;\zeta}+\rho\,{}^{c}\mathcal{D}_{1^+}^{\vartheta-1;\zeta}\right)y(t)=g(t,y(t)),\ t\in\mathcal{J}\\ y(1)=y'(1)=(0,0,\cdots,0,\cdots).\end{cases} \tag{4.1}$$

Take

$$\vartheta=\frac{3}{2},\quad \rho=\frac{1}{5},\quad \zeta(t)=\ln(t),\quad \mathcal{J}=[1,e],$$

and $g:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ given by,

$$g(t,y)=\frac{5}{e^{t-1}+14}\left(13+\arctan(y)\right),\qquad \text{for } t\in\mathcal{J},\quad y\in\mathbb{R}.$$

The function g is clearly continuous. Next, for all $t \in \mathcal{J}$, $x, y \in \mathbb{R}$ one has

$$|g(t,y) - g(t,x)| \leq \frac{5}{e^{t-1} + 14}|y - x|.$$

Hence, hypothesis (C2) holds with $G(t) = \dfrac{5}{e^{t-1} + 14}$ for $t \in \mathcal{J}$, $\|G\|_{L^\infty} = \dfrac{1}{3}$. Moreover, if we choose $\varsigma \geq \dfrac{1}{2}$, the contraction of the corresponding solution operator yields immediately, i.e.

$$\mathfrak{M}_\varsigma = \frac{\frac{1}{2}e^{1/5}}{(1/5 + \varsigma)\varsigma^{1/2}} \frac{1}{3} < 1$$

Therefore, by Theorem 3.1, problem (4.1) admits a unique solution on $\mathcal{J}$.

Example 2. Let

$$\mathbb{F} = c_0 := \{u = (u_1, u_2, \cdots, u_n, \cdots) \ : \ u_n \to 0 \text{ as } n \to \infty\}$$

be the Banach space of real sequences converging to zero, equipped by

$$\|u\| = \sup_{n \geq 1} |u_n|.$$

Consider the following problem posed on c_0:

$$\begin{cases} \left({}^c\mathcal{D}_{0^+}^{\vartheta;\zeta} + \rho\, {}^c\mathcal{D}_{0^+}^{\vartheta-1;\zeta}\right) y(t) = g(t, y(t)),\, t \in \mathcal{J}, 0 < b < \left(\dfrac{91}{15}\right)^{1/\vartheta}, \\ \\ y(0) = y'(0) = (0, 0, \cdots, 0, \cdots). \end{cases} \tag{4.2}$$

Take $[0, b] := \mathcal{J}$, $\zeta(t) = t$ and $g : \mathcal{J} \times c_0 \to c_0$ given by

$$g(t, y) = \left\{ \frac{5}{13t + 91} \left(\frac{3}{n^2} + \sin(|y_n|) + \ln(1 + |y_n|) + \arctan(|y_n|) \right) \right\}_{n \geq 1}, \tag{4.3}$$

for $t \in \mathcal{J}$, $y = \{y_n\}_{n \geq 1} \in c_0$.
Evidently, g satisfies (H1). Next, for all $y \in c_0$ and $t \in \mathcal{J}$, one has

$$\begin{aligned} \|g(t, y)\| &\leq \frac{5}{13t + 91}(3 + 3\|y\|) \\ &\leq \phi(t)\kappa(\|y\|). \end{aligned}$$

Thus, condition (H2) holds with :

$$\phi(t) = \frac{15}{13t + 91}, \quad t \in \mathcal{J} \quad \text{ and } \quad \kappa(u) = 1 + u, \quad u \in [0, \infty).$$

Now, the Hausdorff MNC Λ in $(c_0, \|\cdot\|_{c_0})$ is defined as follows (see [10])

$$\Lambda(\mathbb{L}) = \lim_{n \to \infty} \sup_{y \in \mathbb{L}} \|(I - P_n)y\|_\infty,$$

where P_n is the projection onto the linear span of the first n vectors in the standard basis.

For a bounded set $\mathbb{L} \subset c_0$, we obtain

$$\Lambda(g(t,\mathbb{L})) \leq \frac{15}{91}\Lambda(\mathbb{L}), \text{ a.e. } t \in \mathcal{J}.$$

Thus (H3) is satisfied.

Next, we will show that (H4) is verified. $\kappa(u) = 1 + u$, we have to find $K > 0$ such that

$$\frac{15(\vartheta-1)}{91}\frac{(1+K)b^{\vartheta}}{\Gamma(\vartheta+1)} \leq K.$$

Since $\Gamma(\vartheta - 1) > 1$ for $1 < \vartheta < 2$, then we have to choose $K > 0$ such that

$$\frac{15(1+K)b^{\vartheta}}{91\vartheta} \leq K.$$

Thus

$$K \geq \frac{15b^{\vartheta}}{91\vartheta - 15b^{\vartheta}}.$$

Accordingly, all conditions of Theorem 3.2 are verified. Hence, the existence of at least one solution $y \in C(\mathcal{J}, c_0)$ of problem (4.2) follows from Theorem 3.2.

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