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EXPONENTIAL STABILITY AND APPLICATIONS OF SWITCHED POSITIVE LINEAR IMPULSIVE SYSTEMS WITH TIME-VARYING DELAYS AND ALL UNSTABLE SUBSYSTEMS

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Abstract The global uniform exponential stability of switched positive linear impulsive systems with time-varying delays and all unstable subsystems is studied in this paper, which includes two types of distributed time-varying delays and discrete time-varying delays. Switching behaviors dominating the switched systems can be either stabilizing and destabilizing in the new designed switching sequence. We design new linear programming algorithm process to find the feasible ratio of stabilizing switching behaviors, which can be compensated by unstable subsystems, destabilizing switching behaviors, and impulses. Specifically, we add a kind of nonnegative impulses which is consistent with the switching behaviors for the systems. Employing a multiple co-positive Lyapunov−Krasovskii functional, we present several new sufficient stability criteria and design new switching sequence. Then, we apply the obtained stability criteria to the exponential consensus of linear delayed multi-agent systems, and obtain the new exponential consensus criteria. Three simulations are provided to demonstrate the proposed stability criteria.

Keywords Exponential stability, switched systems, impulses, distributed time-varying delays.

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1. Introduction

Switched systems are the special dynamical systems, which consist of a number of subsystems and a switching rule among them [1–4]. As a special class of hybrid systems, when the subsystems of switched systems are all positive linear systems, they are called switched positive linear systems (SPLS). SPLS receive a lot of attentions since they can appropriately model some complex systems, such as automotive dynamic controls [5], multi-agent systems [6], aerospace engineering [7], traffic controls [8], and so forth. On the other hand, time-varying delays are frequently encountered in complex systems. Moreover, it is well known that even small time-varying delays may affect or even destroy the stability of the systems, which make difficulties for the stability analysis. Therefore, great interests and efforts have

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been focused on SPLS containing time-varying delays parameters [9,10], which have been an essential and appealing study topic.

For the stability of studying SPLS with delays, there mainly exist the following methods, such as co-positive Lyapunov function [11, 12], multiple co-positive Lyapunov−Krasovskii functional [13, 14], or diagonal Lyapunov−Krasovskii functional [15]. The exponential stability means that the systems state converges to zero over an infinite time interval, which has a better application because of its strong stability property. Therefore, many scholars devoted to study the exponential stability of switched systems with delays. For instance, Liu et al. [10] studied the exponential stability for SPLS with delays and impulses by the multiple linear co-positive Lyapunov functional approach and average dwell time method, which presented a delay-dependent exponential stability criterion for SPLS. Li and Xiang [16] addressed the exponential stability and L_1-again control for a class of SPLS with mixed time-varying delays and impulses by using the average dwell time approach and the co-positive Lyapunov−Krasovskii function technique, which developed an interactive convex optimization approach to verify the results. With the help of constructing an appropriate co-positive type Lyapunov−Krasovskii functional approach, Li et al. [18] investigated the exponential stability and l_1-again controller design for SPLS with mixed time-varying delays. According to a co-positive Lyapunov−Krasovskii functionals and average dwell time technique, Liu et al. [19] considered the exponential stability of SPLS with impulses and mixed time-varying delays.

Up to now, many scholars are interested in the fields that the subsystems of switching systems are all unstable, but the main switching systems are stable. Some significant results are presented in [20–23]. For instance, based on the theory of spherical covering and crystal point groups, Zhang et al. [20] obtained some sufficient algebraic conditions for stabilizing switched linear systems with all unstable subsystems. Furthermore, authors in [20] designed a switching law to stabilize the unstable switched behaviors. Liu et al. [21] investigated the stabilization problem of SPLS with discrete time-varying delays by utilizing multiple co-positive Lyapunov−Krasovskii functional, where all the subsystems of the main switching systems are unstable while all switching behaviors in the new designed switching sequence are stabilizing. However, the switching behaviors may make some increment of the energy functions, i.e., be destabilizing to the switched systems [22,23]. Specifically, Zhou et al. [22] considered the exponential stability of SPLS with all unstable subsystems and destabilizing switching behaviors in new switching sequence, which evaluates the ratio of stabilizing switching behaviors to compensate the state divergence caused by either unstable subsystems or destabilizing switching behaviors. After that, Zhou et al. [23] investigated the global uniform exponential stability of SPLS with all unstable subsystems and time-varying delays, which includes two types of distributed time-varying delays and discrete time-varying delays, by adjusting the ratio of the stabilizing switching behaviors, the state divergence caused by unstable subsystems and destabilizing switching behaviors can be compensated.

In some practical applications, impulsive interference is inevitable. Impulsive behaviors are regarded as a dynamical process, which expresses a state converts abruptly at some instants [16, 17, 24, 25, 29, 30]. Impulsive systems have been triumphantly applied to problems in physics, mechanics, and some fields of engineering. Recently, Ju et al. [26] took into account the impulsive effects of the switched linear time-varying systems, where the impulsive jumping is limited by a

linear matrix form. The results in [26] can be applied to the exponential consensus of linear multi-agent systems. Zhou et al. [23] investigated the influences of time-varying delays and the ratio of stabilizing switching behaviors for switched systems, but they did not consider the impulsive effects for switched systems. In fact, there may exist impulsive behaviors at switching instants for the switched systems in some practical situations, and the linear multi-agent systems may encounter the delays effects. Therefore, motivated by [23, 26], we will further study the effects of impulsive behaviors for SPLS with time-varying delays and all unstable subsystems, and consider the time-varying delays effects of the linear multi-agent systems with impulses.

In this paper, we add the impulses to the model in [23]. We first investigate the global uniform exponential stability for switched positive linear impulsive systems (SPLIS) with distributed time-varying delays and all unstable subsystems as follows

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}\int_{t-d(t)}^{t} x(s)ds,\ t \geq 0,\ t \neq t_m,\ m = 1, 2, 3, \cdots, \\ x(t^+) = I_{\sigma(t^+)\sigma(t^-)}(x(t^-)),\ t = t_m,\ m = 1, 2, 3, \cdots, \\ x(t_0 + \varphi) = \varpi(\varphi), \varphi \in [-\hat{d}, 0], \end{cases} \tag{1.1}$$

where $x(t) \in R^n$ stands for the state vector, the switching rule $\sigma(t) : [0, \infty) \to \langle n \rangle$, $\langle n \rangle = \{1, 2, \cdots, n\}$, $n > 1$ is an integer, $\sigma(t)$ denotes a piecewise right-continuous function and satisfies $\lim_{t \to t_m^+} \sigma(t) = \sigma(t_m)$, the matrices $A_i(t), B_i(t) \in R^{n \times n}$, $i \in \langle n \rangle$ are continuous time-varying matrix functions. The continuous switching instants fulfill $0 \leq t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} \cdots$, and $\lim_{m \to \infty} t_m = \infty$. The distributed time-varying delays function $d(t)$ fulfills $0 \leq d(t) \leq \hat{d}$ and $\dot{d}(t) \leq d < 1$, in other words, $d(t)$ is upper bounded and slowly varying. $I_{\sigma(t^+)\sigma(t^-)}(x) : R^n \to R^n$ are impulses, t_m, $m = 1, 2, \cdots$, those are not just the switching instants but the impulsive instants, and satisfying $0 < t_m < t_{m+1}$ and $\lim_{m \to \infty} t_m = \infty$. As $t \in [t_m, t_{m+1})$, the $\sigma(t_m)$th subsystem is actuated, $m = 0, 1, 2, \cdots$. At the switching and impulsive instants, define $x(t) = x(t^+)$ at $t = t_m$.

Next, we consider the SPLIS with discrete time-varying delays and all unstable subsystems as following

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - d(t)), t \geq 0, t \neq t_m, m = 1, 2, 3, \cdots, \\ x(t^+) = I_{\sigma(t^+)\sigma(t^-)}(x(t^-)),\ t = t_m,\ m = 1, 2, 3, \cdots, \\ x(t_0 + \varphi) = \varpi(\varphi), \varphi \in [-\hat{d}, 0], \end{cases} \tag{1.2}$$

where the definitions of all the variables and parameters are the same as system (1.1). Then, we apply the obtained stability criteria to the exponential consensus of linear delayed multi-agent systems with impulses.

By discretizing the dwell time interval, redefining the original switching sequence, and employing a multiple co-positive Lyapunov−Krasovskill functional. The global uniform exponential stability of the SPLIS is established as a new attempt. The global uniform exponential stability criteria imply that the ratio of stabilizing switching behaviors are crucial to the global uniform exponential stability of SPLIS with time-varying delays and unstable subsystems. Consequently, the primary contributions are highlighted as following: (1) Compared with [22, 23], we add the nonnegative impulses to the SPLS with time-varying delays, which includes two types of distributed time-varying delays and discrete time-varying delays. We first study the global uniform exponential stability of SPLIS with distributed time-varying delays, then we extend the distributed time-varying delays case to discrete

time-varying delays case. New sufficient global uniform exponential stability criteria of SPLIS are obtained by multiple co-positive Lyapunov−Krasovskii functional method. So the obtained stability criteria further improve the stability theory of SPLIS. (2) Compared with [26], both discrete time-varying delays and all unstable subsystems effects are contained for the SPLIS. Moreover, both stabilizing and destabilizing switching behaviors in new switching behaviors are included for SPLIS. By finding the feasible ratio of stabilizing switching behaviors, the system state divergence aroused through unstable subsystems, destabilizing switching behaviors, impulsive effects are compensated. (3) Different from the existing stability criteria in [16], our model is more comprehensive because of the unstable subsystems and destabilizing switching behaviors. We design new linear programming algorithm process to better verify the main stability criteria (Algorithm 3.3). Furthermore, our main stability criteria are used as the exponential consensus of linear impulsive multi-agent systems with switching communication topologies and discrete time-varying delays (Example 5.3).

The framework of this paper is arranged as following. Model formation and definitions are given in Section 2. Section 3 is dedicated to proving the main criteria. In Section 4, the obtained criteria are used for the exponential consensus of linear delayed multi-agent systems. Section 5 provides three simulation examples to support our theoretical results. A conclusion and future directions are discussed in Section 6.

2. Problem formation and preliminaries

We give the notations as follows. R^n represents the group of n-dimensional real vectors. $R^{n\times n}$ indicates $n \times n$-dimensional real matrices. $\langle n\rangle$ implies the group $\{1, 2, \cdots, n\}$, for n represents a positive integer. $im(t)$ means the impulsive signal. A vector $Z \in R^n$ is positive described $Z \succ 0$ provided that all its elements are positive. $Z^\top$ shows the transpose of the vector Z. E_n symbolizes the $n \times n$ unit matrix with appropriate dimension. A Metzler matrix denotes a real square matrix whose off-diagonal elements are all non-negative. A vector $y = (y_1, y_2, \cdots, y_n) \in R^n$, $\|y\|_1 = \sum_{i=1}^n \mid y_i \mid$. For a matrix $D = \{d_{ij}\}$, $\|D\|_1 = \max_j \sum_{i=1}^n \mid d_{ij} \mid$. $D \succeq 0$ denotes that all elements of matrix D are non-negative. Two vectors $p, q \in R^n$, $p \succ q$ signifies $p - q \succ 0$. Set $\zeta(t^+) = \lim_{d\to 0^+} \zeta(t + d)$ and $\zeta(t^-) = \lim_{d\to 0^-} \zeta(t + d)$. A function $\psi(t)$ described on $[0, \infty)$, $\psi(t^+)$ and $\psi(t^-)$ imply the right limitation and the left limitation of $\psi(t)$, separately. $\max[Z]$ is the maximum component of vector Z. Analogously, $\min[Z]$ is the minimum component of vector Z.

In this study, the switching sequence is redefined in a group of segments, and each switching interval is divided into several subintervals. Moreover, the increasing and decreasing behaviors of multiple co-positive Lyapunov−Krasovskill functional at the time of switches and impulses are analyzed, which will be evaluated to describe the stabilizing and destabilizing switching behaviors.

The switching sequence is redefined as $t_{l_k}, l_{k+1} = l_k + m, k = 0, 1, 2, \cdots$, set $t_{l_0} = t_0$, where m represents a predetermined positive integer. Under the recombination, a range of fragments are isolated from the initial switching sequence. In the new switching sequence, time interval $[t_{l_k}, t_{l_{k+1}})$ is defined as the $(k + 1)^{th}$ segment, and each segment includes m switching moments called as t_{l_k+g}, $\forall g \in M := \{0, 1, \cdots, m - 1\}$. So the new switching sequence still fulfills $0 < \bar{a}_1 \leq t_{l_k+g+1} - t_{l_k+g} \leq \bar{a}_2 < \infty$, $\forall g \in M$, $k \in N^+$.

On k^{th} part, the group of stabilizing and destabilizing switching behaviors are represented by $M_k^{\downarrow}$ and $M_k^{\uparrow}$, separately. Therefore, $|M_k^{\downarrow}|$ and $|M_k^{\uparrow}|$ are the number of stabilizing and destabilizing switching behaviors on the k^{th} part, separately. Accordingly, $|M_k^{\downarrow}|/m$ and $|M_k^{\uparrow}|/m$ represent the ratios of the stabilizing and destabilizing switching behaviors, separately. Apparently, there often exist a pair of positive integers m_1 and m_2 with $m_1 + m_2 = m$ in order that the number of the stabilizing and destabilizing switching behaviors satisfy $|M_k^{\uparrow}| = m_1$ and $|M_k^{\downarrow}| = m_2$.

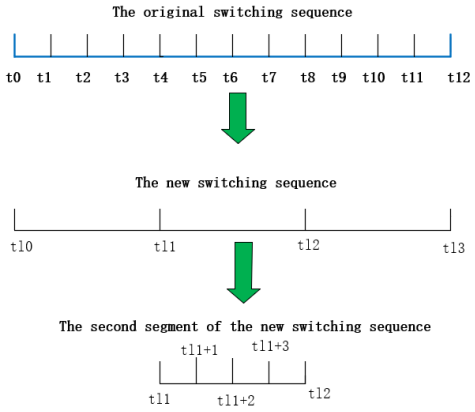


Figure 1. The sketch map process of adjusting the original switching sequence with $m = 4$. Then, the number of the stabilizing switching and destabilizing switching behaviors satisfy $|M_k^{\uparrow}| = m_1$ and $|M_k^{\downarrow}| = m_2$, and $m_1 + m_2 = 4$

The global uniform exponential stability criteria will be constructed for SPLIS (1.1) and (1.2) by multiple co-positive Lyapunov−Krasovskii functional in Section 3. Before beginning to construct stability conditions of SPLIS (1.1) and (1.2), describe a group of subsystem couples which cause stabilizing switching behaviors as $\tilde{N} := \{(j,i);\ \exists k \in N,\ \sigma(t) = j,\ t \in [t_{l_k+g}, t_{l_k+g+1}),\ \sigma(t) = i,\ t \in [t_{l_k+g+1}, t_{l_k+g+2}),\ g + 1 \in M_{k+1}^{\downarrow}\}$. Give a definition to matrix B as $B = (b_{uv}) \in R^{n\times n}$, where $(b_{uv}) = \max\{B_i{}^{(uv)}\}$, and $B_i{}^{(uv)}$ characterizes the $u-th$ row and $v-th$ column component of B_i. Obviously, $B \succeq B_i$.

Remark 2.1. As discussed above, it can be seen that the global uniform exponential stability of the SPLIS (1.1) and (1.2) in this paper is influenced by the following four aspects: (1) the property of subsystems, (2) the proportion of stabilizing switching behaviors, (3) the time-varying delays, (4) the impulsive effects.

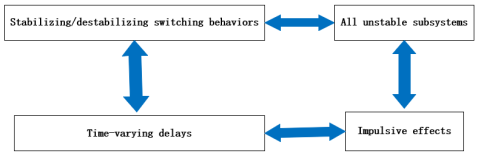


Figure 2. The influence factors of GUES for SPLIS (1.1) and (1.2).

In the next, several essential definitions, assumptions, and lemmas are proposed.

Definition 2.1. (see [27]) The SPLIS (1.1) or (1.2) is called global uniform exponential stability provided that $\forall\ x(t_0+\varphi)=\varpi(\varphi)$, $\varphi\in[-\hat{d},0]$, there are two parameters $\mu>0$ and $\nu>0$ in order that every solution $x(t)$ of system (1.1) or (1.2) fulfils $\|x(t)\|_1\leq\mu e^{-\nu(t-t_0)}\|x(t)\|_{1c}$, $\forall t\geq t_0$, and $\|x(t)\|_{1c}=\sup\limits_{-\hat{d}\leq\varphi\leq 0}\|x(t_0+\varphi)\|_1$.

Definition 2.2. (see [28]) The SPLIS (1.1) or (1.2) is called positive provided that $\forall x(t_0+\varphi)\geq 0$, $\varphi\in[-\hat{d},0]$ and for any switching rule $\sigma(t)$, the relevant trajectory $x(t)$ fulfils $x(t)\geq 0$, $\forall t\geq t_0$.

We present the following assumption in this paper.

(H_1) $I_{ji}\geq 0$ for $x\geq 0$, and there exist a class of positive matrices $C_i\in R^{n\times n}$, satisfying $I_{ji}(x)\leq C_i x$ for $x\in R^n$, $i,j\in\langle n\rangle$, $\sigma(t_m{}^+)=j$, $\sigma(t_m{}^-)=i$, $i\neq j$, $m=1,2,3,\cdots$.

Lemma 2.1. *(see [28]) The SPLIS* (1.1) *or* (1.2) *is positive when and only when A_i is a Metzler matrix and $B_i\succeq 0$, $\forall i\in\langle n\rangle$.*

3. Main results

In this Section, we establish the new global uniform exponential stability criteria for SPLIS (1.1) and (1.2), and design the new linear programming algorithm process to better verify the obtained stability criteria.

3.1. Global uniform exponential stability of SPLIS (1.1) with distributed time-varying delays case

Theorem 3.1. *Suppose that (H_1) is held. Given that $\bar{a}_1\leq\bar{a}_2$, positive integer T, three positive integers m_1, m_2, m satisfying $m_1+m_2=m$. If there are three constants ζ, ε, η, fulfilling $0<\zeta<1$, $\varepsilon\geq 1$, $\eta>0$, and a class of positive vectors $Z_{i,s}$, $i\in\langle n\rangle$, $s\in\{0,1,\cdots,T-1\}$, in order that the following inequalities hold,*

$$\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}(1-d)\,E_n+Z_{i,s+1}^T\Big((1-d)A_i+2\hat{d}B-\eta(1-d)E_n\Big)\prec 0, \tag{3.1}$$

$$\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T}{\bar{a}_1}(1-d)\,E_n+Z_{i,s}^T\Big((1-d)A_i+2\hat{d}B-\eta(1-d)E_n\Big)\prec 0, \tag{3.2}$$

$$Z_{i,T}^T\Big((1-d)A_i+2\hat{d}B-\eta(1-d)E_n\Big)\prec 0, \tag{3.3}$$

$$\left(\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T\hat{d}}{\bar{a}_1}-Z_{i,s+1}^T\right)B\prec 0, \tag{3.4}$$

$$\left(\frac{(Z_{i,s+1}^T-Z_{i,s}^T)T\hat{d}}{\bar{a}_1}-Z_{i,s}^T\right)B\prec 0, \tag{3.5}$$

$$Z_{i,0}\preceq\begin{cases}\zeta C_jZ_{j,T},\ (j,i)\in\tilde{N},\\ \varepsilon C_jZ_{j,T},otherwise,\end{cases} \tag{3.6}$$

$$\frac{m_2}{m}\ln\zeta+\frac{m_1}{m}\ln\varepsilon+\bar{a}_2\eta<0, \tag{3.7}$$

then the SPLIS (1.1) *is global uniform exponential stability.*

Remark 3.1. Conditions (3.1)−(3.5) assess the evolution of each subsystem between every two switching behaviors. Condition (3.6) estimates the changes of energy at every switching and impulsive instant when stabilizing switching behaviors (by ζ) and destabilizing switching behaviors (by ε) are dynamic. The ratios of all stabilizing and destabilizing switching behaviors are confined by condition (3.7).

Remark 3.2. Compared with [23], we add the impulses to the model in [23], and analyze the impulsive effects for the stability of the model. Then, we get the new global uniform exponential stability condition (3.6) in Theorem 3.1.

Proof. Firstly, each switching interval $[t_{l_k},t_{l_k+1})$ is divided into $T+1$ segments via the next two processes: (i) The time interval $[t_{l_k},t_{l_k+1})$ is parted into $[t_{l_k},t_{l_k}+\bar{a}_1)$ and $[t_{l_k}+\bar{a}_1,t_{l_k+1})$. (ii) The time interval $[t_{l_k},t_{l_k}+\bar{a}_1)$ is parted into T segments called as $G_{l_k,s}=[t_{l_k}+se,t_{l_k}+(s+1)e)$, $s=0,1,\cdots,T-1$, with same length $e=\frac{\bar{a}_1}{T}$. Then, in terms of the switching rule $\sigma(t)$, the following vector function $Z_i(t)$ is constructed.

$$Z_i(t)=\begin{cases}\varrho(t)Z_{i,s}+\tilde{\varrho}(t)Z_{i,s+1}, t\in G_{l_k,s}, & s=0,1,\cdots,T-1,\\ Z_{i,T}, & t\in[t_{l_k}+\bar{a}_1,t_{l_{k+1}}),\end{cases} \tag{3.8}$$

where $\varrho(t)=\dfrac{t_{l_k}+(s+1)e-t}{e}$, $t\in G_{k,s}$, $\tilde{\varrho}(t)=1-\varrho(t)=\dfrac{t-t_{l_k}-se}{e}$.

For $t\in[t_{l_k+g},t_{l_k+g+1})$, $Z_{\sigma(t)}(t)=Z_i(t)$ as $\sigma(t)=i$, $i\in\langle n\rangle$. From (3.8), $Z_{\sigma(t)}(t)$ is a piecewise right-continuous vector function, $\forall t\in[t_{l_k+g},t_{l_k+g+1})$. Choose the following multiple co-positive Lyapunov−Krasovskill functional for SPLIS (1.1):

$$V_{\sigma(t)}(t)=V^1_{\sigma(t)}(t)+V^2_{\sigma(t)}(t)+V^3_{\sigma(t)}(t), \tag{3.9}$$

where

$$V^1_{\sigma(t)}(t)=(1-d)Z^T_{\sigma(t)}(t)x(t), \tag{3.10}$$

$$V^2_{\sigma(t)}(t)=\int_{t-d(t)}^{t}(s-(t-d(t)))e^{\eta(t-s)}Z^T_{\sigma(t)}(t)Bx(s)ds, \tag{3.11}$$

$$V^3_{\sigma(t)}(t)=\int_{-\hat{d}}^{0}\int_{t+\varphi}^{t}e^{\eta(t-s)}Z^T_{\sigma(t)}(t)Bx(s)dsd\varphi. \tag{3.12}$$

For $t\in[t_{l_k+g},t_{l_k+g+1})$ as $\sigma(t)=i$, $i\in\langle n\rangle$, by (3.10)−(3.12), we obtain the time derivative of $V_i(x)$ along the trajectory of the SPLIS (1.1) as follows

$$\begin{aligned}\dot{V}^1_i(t)&=V_{11}+V_{12}+V_{13}\\&=(1-d)\dot{Z}^\top_i(t)x(t)+(1-d)Z^T_i(t)A_ix(t)\\&\quad+(1-d)Z^T_i(t)B_i\int_{t-d(t)}^{t}x(s)ds,\end{aligned} \tag{3.13}$$

$$\dot{V}^2_i(t)=V_{21}+V_{22}+V_{23}-V_{24}$$

$$= \eta V_i^2 + \int_{t-d(t)}^{t} (s-(s-d(t)))e^{\eta(t-s)}\dot{Z}_i^\top(t)Bx(s)ds$$
$$+d(t)Z_i^T(t)Bx(t) - (1-\dot{d}(t))\int_{t-d(t)}^{t} e^{\eta(t-s)}Z_i^T(t)Bx(s)ds, \quad (3.14)$$

$$\dot{V}_i^3(t) = V_{31}+V_{32}-V_{33} = \eta V_i^3 + \hat{d}Z_i^T(t)Bx(t) - \int_{t-\hat{d}}^{t} e^{\eta(t-s)}Z_i^T(s)Bx(s)ds. \quad (3.15)$$

Due to $\dot{d}(t) \le d < 1$, $\eta > 0$, it is acquired that $1-d < 1-\dot{d}(t)$ and $e^{\eta(t-s)} \ge 1$. Hence, $V_{13} - V_{24} \le 0$. According to the condition $0 \le d(t) \le \hat{d}$, we have

$$V_{23} + V_{32} \le 2\hat{d}Z_i^T(t)Bx(t). \quad (3.16)$$

Then, in terms of the integral formula, we obtain

$$V_{22} - V_{33} \le \int_{t-d(t)}^{t} e^{\eta(t-s)}\left((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\right)Bx(s)ds. \quad (3.17)$$

Afterwards, we gain

$$\dot{V}_i(t) \le V_{11}+V_{12}+\eta V_i^2+\eta V_i^3+2\hat{d}Z_i^T(t)Bx(t)$$
$$+\int_{t-d(t)}^{t} e^{\eta(t-s)}\left((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\right)Bx(s)ds. \quad (3.18)$$

According to (3.18), we acquire that

$$\begin{aligned}
&\dot{V}_i(t) - \eta V_i(t)\\
&\le (1-d)\dot{Z}_i^T(t)x(t) + (1-d)Z_i^T(t)A_ix(t)\\
&+\int_{t-d(t)}^{t} e^{\eta(t-s)}\left((s-(t-d(t)))\dot{Z}_i^\top(t) - Z_i^\top(s)\right)Bx(s)ds\\
&+2\hat{d}Z_i^T(t)Bx(t) - \eta(1-d)Z_i^\top(t)x(t)\\
&= (1-d)\dot{Z}_i^T(t)x(t) + Z_i^T(t)\left((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\right)x(t)\\
&+\int_{t-d(t)}^{t} e^{\eta(t-s)}\left((s-(t-d(t)))\dot{Z}_i^T(t) - Z_i^T(s)\right)Bx(s)ds. \quad (3.19)
\end{aligned}$$

Then, when $t \in [t_{l_k+g}, t_{l_k+g}+\bar{a}_1)$,

$$\begin{aligned}
&\dot{V}_i(t) - \eta V_i(t)\\
&\le \varrho(t)\left((1-d)\dot{Z}_i^T(t) + Z_{i,s}^T\left((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\right)\right)x(t)\\
&+\tilde{\varrho}(t)\left((1-d)\dot{Z}_i^T(t) + Z_{i,s+1}^T\left((1-d)A_i + 2\hat{d}B - \eta(1-d)E_n\right)\right)x(t)\\
&+\int_{t-d(t)}^{t} e^{\eta(t-s)}\left(\varrho(s)\left(\hat{d}\dot{Z}_i^T(t) - Z_{i,s}^T\right) + \tilde{\varrho}(s)\left(\hat{d}\dot{Z}_i^T(t) - Z_{i,s+1}^T\right)\right)Bx(s)ds.
\end{aligned} \quad (3.20)$$

By the definition of $Z_i(t)$, we gain that $\dot{Z}_i^T(t) = \frac{(Z_{i,s+1}^T - Z_{i,s}^T)T}{\bar{a}_1}$ for $t \in [t_{l_k}+g+se, t_{l_k}+g+(s+1)e)$, $s = 0, 1, \cdots, T-1$. Consequently, by (3.20) and

conditions (3.1), (3.2), (3.4), (3.5), we get that $\dot{V}_i(t) < \eta V_i(t)$.
For $t \in [t_{l_k+g} + \bar{a}_1, t_{l_k+g+1})$, it is apparent that $Z_i^\top(t) = Z_{i,T}^\top$. So it concludes from (3.19) and condition (3.3) that
$\dot{V}_i(t) < \eta V_i(t)$, $t \in [t_{l_k+g} + \bar{a}_1, t_{l_k+g+1})$. Subsequently, we obtain that for $t \in [t_{l_k+g}, t_{l_k+g+1})$, $\forall k \in N$, $g \in M$,

$$V_{\sigma(t_{l_k+g})}(t) < e^{\eta(t-t_{l_k+g})} V_{\sigma(t_{l_k+g})}(t_{l_k+g}^+). \tag{3.21}$$

Since at every switching and impulsive instant t_{l_k+g}, denote $\sigma(t_{l_k+g}^-) = j$, $\sigma(t_{l_k+g}^+) = i$, $j, i \in \langle n \rangle$, $j \neq i$, $Z_{j,T}^\top$ is replaced by $Z_{j,T}^\top C_j^\top$, from condition (3.6) and the definition of multiple co-positive Lyapunov−Krasovskill functional. Then, we calculate the value of multiple co-positive Lyapunov−Krasovskill functional at t_{l_k+g},
(i) when $g \in M_k^\downarrow$,

$$\begin{aligned}
&V({t_{l_k+g}}^+, x({t_{l_k+g}}^+)) - \zeta V({t_{l_k+g}}^-, x({t_{l_k+g}}^-)) \\
&= (1-d)\left(Z_{i,0}^T - \zeta Z_{j,T}^T C_j^\top\right) x(t) \\
&+ \int_{t-d(t)}^{t} (s-(t-d(t)))\, e^{\eta(t-s)} \left(Z_{i,0}^T - \zeta Z_{j,T}^T C_j^\top\right) Bx(s) ds \\
&+ \int_{-\hat{d}}^{0} \int_{t+\varphi}^{t} e^{\eta(t-s)} \left(Z_{i,0}^T - \zeta Z_{j,T}^T C_j^\top\right) Bx(s) ds d\varphi \\
&\leq 0,
\end{aligned}$$

(ii) when $g \in M_k^\uparrow$,

$$\begin{aligned}
&V({t_{l_k+g}}^+, x({t_{l_k+g}}^+)) - \varepsilon V({t_{l_k+g}}^-, x({t_{l_k+g}}^-)) \\
&= (1-d)\left(Z_{i,0}^T - \varepsilon Z_{j,T}^T C_j^\top\right) x(t) \\
&+ \int_{t-d(t)}^{t} (s-(t-d(t)))\, e^{\eta(t-s)} \left(Z_{i,0}^T - \varepsilon Z_{j,T}^T C_j^\top\right) Bx(s) ds \\
&+ \int_{-\hat{d}}^{0} \int_{t+\varphi}^{t} e^{\eta(t-s)} \left(Z_{i,0}^T - \varepsilon Z_{j,T}^T C_j^\top\right) Bx(s) ds d\varphi \\
&\leq 0.
\end{aligned}$$

Therefore,

$$V_{\sigma(t_{l_k+g})}(t) \leq \begin{cases} \varepsilon V_{\sigma(t_{l_k+g-1})}(t_{l_k+g}^-), g \in M_k^\uparrow, \\ \zeta V_{\sigma(t_{l_k+g-1})}(t_{l_k+g}^-), g \in M_k^\downarrow. \end{cases} \tag{3.22}$$

By the mathematical induction, when $t \in [t_0, t_1)$, we get

$$V_{\sigma(t_0)}(t) < e^{\eta(t-t_0)} V_{\sigma(t_0)}(t_0).$$

In the first segment $[t_{l_0}, t_{l_1})$, according to (3.22), we obtain

$$\begin{aligned}
V_{\sigma(t_{l_0})}(t_{l_1}^-) &\leq \zeta^{|M_k^\downarrow|} \varepsilon^{|M_k^\uparrow|} e^{\eta(t_{l_1}-t_{l_0})} V_{\sigma(t_{l_0})}(t_{l_0}^+) \\
&\leq \zeta^{m_2} \varepsilon^{m_1} e^{\eta(t_{l_1}-t_{l_0})} V_{\sigma(t_0)}(t_0).
\end{aligned} \tag{3.23}$$

From induction, the following conclusion can be drawn: in $(k+1)^{th}$ part, when $t \in [t_{l_k+g}, t_{l_k+g+1})$,

$$V_{\sigma(t_{l_k+g})}(t) \leq \varepsilon^{m_1} (\zeta^{m_2} \varepsilon^{m_1})^k e^{\eta(t-t_0)} V_{\sigma(t_0)}(t_0). \tag{3.24}$$

Because when $t \in [t_{l_k+g}, t_{l_k+g+1})$, $(k+1)m\bar{a}_2+\bar{a}_2 \geq t-t_0$, thus, $k \geq \dfrac{t-t_0-\bar{a}_2}{m\bar{a}_2}-1$. Furthermore, let $\gamma = m_2 \ln\zeta + m_1 \ln\varepsilon$. From condition (3.7), we get $\gamma < -m\bar{a}_2\eta < 0$. According to (3.24), we acquire that when $t \in [t_{l_k+g}, t_{l_k+g+1})$,

$$\begin{aligned} V_{\sigma(t_{l_k+g})}(t) &\leq \varepsilon^{m_1} e^{\eta(t-t_0)} e^{\gamma k} V_{\sigma(t_0)}(t_0) \\ &= \varepsilon^{m_1} e^{\eta(t-t_0)+\gamma k} V_{\sigma(t_0)}(t_0) \\ &\leq \varepsilon^{m_1} e^{\eta(t-t_0)+(\frac{t-t_0-\bar{a}_2}{m\bar{a}_2}-1)\gamma} V_{\sigma_{(t_0)}}(t_0). \end{aligned} \tag{3.25}$$

Through condition (3.7), there exits an adequate minor positive constant ω in order that $m_2 \ln\zeta + m_1 \ln\varepsilon + m\bar{a}_2\eta + \omega < 0$, $\gamma + m\bar{a}_2\eta + \omega < 0$. Therefore,

$$V_{\sigma(t_{l_k+g})}(t) \leq \kappa e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)} V_{\sigma_{(t_0)}}(t_0), \tag{3.26}$$

and $\kappa = \frac{\varepsilon^{m_1}}{e^{\gamma(1+1/m)}}$. In terms of the definition of $V_{\sigma(t)}(t)$, we get that

$$V_{\sigma(t)}(t) \geq (1-d) Z^{\top}_{\sigma(t)}(t) x(t) \geq (1-d) \min_{i,T}\{\min[Z_{i,T}]\} \|x(t)\|_1, \tag{3.27}$$

and

$$\begin{aligned} V_{\sigma(t_0)}(t_0) \leq{}& (1-d) \max_{i,T}\{\max[Z_{i,T}]\} \|x(t_0)\|_1 \\ &+ \hat{d} e^{\eta\hat{d}} \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 ds \\ &+ \hat{d}^2 \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 ds, \end{aligned} \tag{3.28}$$

where $T \in \{0, 1, 2, \cdots, T\}$. Therefore, we have

$$\begin{aligned} \|x(t)\|_1 \leq{}& \frac{1}{(1-d)\min_{i,T}\{\min[Z_{i,T}]\}} \kappa e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)} \{(1-d) \max_{i,T}\{\max[Z_{i,T}]\} \|x(t_0)\|_1 \\ &+ \hat{d} e^{\eta\hat{d}} \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 ds \\ &+ \hat{d}^2 \max_{i,T}\{\max[Z_{i,T}]\} \|B\|_1 \int_{t_0-\hat{d}}^{t_0} \|x(s)\|_1 ds\} \\ &\leq \mu e^{-\frac{\omega}{m\bar{a}_2}(t-t_0)} \sup_{-\hat{d}\leq\varphi\leq 0} \|x(t_0+\varphi)\|_1, \end{aligned} \tag{3.29}$$

where $\mu = \frac{\max_{i,T}\{\max[Z_{i,T}]\}}{\min_{i,T}\{\min[Z_{i,T}]\}} \left(1 + \frac{\hat{d}e^{\eta\hat{d}}+\hat{d}^2}{1-d}\|B\|_1\right)\kappa$. Let $\nu = \frac{\omega}{m\bar{a}_2}$. We can conclude that $\|x(t)\|_1 \leq \mu e^{-\nu(t-t_0)} \|x(t_0)\|_{1c}$ from (3.29), $\forall t \geq t_0$, so SPLIS (1.1) is global uniform exponential stability. □

3.2. Global uniform exponential stability of SPLIS (1.2) with discrete time-varying delays case

Corollary 3.1. *Suppose that (H_1) is held. Given that $\bar{a}_1 \leq \bar{a}_2$, positive integer T, three positive integers m_1, m_2, m are satisfying $m_1 + m_2 = m$. If there are three*

constants ζ, ε, η, *fulfilling* $0 < \zeta < 1$, $\varepsilon \geq 1$, $\eta > 0$ *and a class of positive vectors* $Z_{i,s}$, $i \in \langle n \rangle$, $s \in \{0, 1, \cdots, T-1\}$, *in order that the following inequalities hold,*

$$\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T}{\bar{a}_1}(1-d)E_n + Z_{i,s+1}^T\Big((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\Big) \prec 0, \tag{3.30}$$

$$\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T}{\bar{a}_1}(1-d)E_n + Z_{i,s}^T\Big((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\Big) \prec 0, \tag{3.31}$$

$$Z_{i,T}^T\Big((1-d)A_i + (1+\hat{d})B - \eta(1-d)E_n\Big) \prec 0, \tag{3.32}$$

$$\left(\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T\hat{d}}{\bar{a}_1} - Z_{i,s+1}^T\right)B \prec 0, \tag{3.33}$$

$$\left(\frac{(Z_{i,s+1}^T - Z_{i,s}^T)T\hat{d}}{\bar{a}_1} - Z_{i,s}^T\right)B \prec 0, \tag{3.34}$$

$$Z_{i,0} \preceq \begin{cases} \zeta C_j Z_{j,T}, \ (j,i) \in \tilde{N}, \\ \varepsilon C_j Z_{j,T}, otherwise, \end{cases} \tag{3.35}$$

$$\frac{m_2}{m}\ln\zeta + \frac{m_1}{m}\ln\varepsilon + \bar{a}_2\eta < 0, \tag{3.36}$$

then the SPLIS (1.2) *is global uniform exponential stability.*

Remark 3.3. Compared to [23], we add the impulses to the model, and we get the new global uniform exponential stability condition (3.35) in Corollary 3.1. Condition (3.35) is also the constraint of vector at every switching and impulsive instant.

Proof. Choose the following multiple co-positive Lyapunov−Krasovskill functional for SPLIS (1.2):

$$V_{\sigma(t)}(t) = V_{\sigma(t)}^1(t) + V_{\sigma(t)}^2(t) + V_{\sigma(t)}^3(t), \tag{3.37}$$

and

$$V_{\sigma(t)}^1(t) = (1-d)Z_{\sigma(t)}^T(t)x(t), \tag{3.38}$$

$$V_{\sigma(t)}^2(t) = \int_{t-d(t)}^t e^{\eta(t-s)}Z_{\sigma(t)}^T(t)Bx(s)ds, \tag{3.39}$$

$$V_{\sigma(t)}^3(t) = \int_{-\hat{d}}^0\int_{t+\varphi}^t e^{\eta(t-s)}Z_{\sigma(t)}^T(t)Bx(s)ds d\varphi. \tag{3.40}$$

For $t \in [t_{l_k+g}, t_{l_k+g+1})$, define $\sigma(t) = i$, $i \in \langle n \rangle$, we obtain the time derivative of $V_i(x)$ along the trajectory of the SPLIS (1.2) as following

$$\begin{aligned} \dot{V}_i^1(t) &= V_{11} + V_{12} + V_{13} \\ &= (1-d)\dot{Z}_i^\top(t)x(t) + (1-d)Z_i^T(t)A_ix(t) \\ &\quad + (1-d)Z_i^T(t)B_ix(t-d(t)), \end{aligned} \tag{3.41}$$

$$\dot{V}_i^2(t) = V_{21} + V_{22} + V_{23} - V_{24} = \eta V_i^2 + \int_{t-d(t)}^{t} e^{\eta(t-s)} \dot{Z}_i^{\top}(t) B x(s) ds + Z_i^T(t) B x(t) - (1-\dot{d}(t)) e^{\eta d(t)} Z_i^T(t) B x(t-d(t)), \tag{3.42}$$

$$\dot{V}_i^3(t) = V_{31} + V_{32} - V_{33} = \eta V_i^3 + \hat{d} Z_i^T(t) B x(t) - \int_{t-\hat{d}}^{t} e^{\eta(t-s)} Z_i^T(s) B x(s) ds. \tag{3.43}$$

Then, we acquire that

$$\dot{V}_i(t) - \eta V_i(t) = V_{11} + V_{12} + V_{13} + V_{22} + V_{23} - V_{24} + V_{32} - V_{33} - \eta(1-d) Z_i^{\top}(t) x(t). \tag{3.44}$$

Due to $d(t) \in [0, \hat{d}]$, $d \in [\dot{d}(t), 1)$, and $\int_{t-\hat{d}}^{t-d(t)} e^{\eta(t-s)} Z_i^T(s) B x(s) ds \geq 0$, we have $V_{13} - V_{24} \leq 0$, $V_{22} - V_{33} \leq \int_{t-d(t)}^{t} e^{\eta(t-s)} \left(\dot{Z}_i^T(t) - Z_i^{\top}(s) \right) B x(s) ds$. Therefore, we gain the following

$$\begin{aligned}
&\dot{V}_i(t) - \eta V_i(t) \\
&= (1-d) \dot{Z}_i^T(t) x(t) \\
&+ Z_i^T(t) \left((1-d) A_i + (1+\hat{d}) B - \eta(1-d) E_n \right) x(t) \\
&+ \int_{t-d(t)}^{t} e^{\eta(t-s)} \left(\dot{Z}_i^T(t) - Z_i^T(s) \right) B x(s) ds \\
&\leq \varrho(t) \left((1-d) \dot{Z}_i^T(t) + Z_{i,s}^T \left((1-d) A_i + (1+\hat{d}) B - \eta(1-d) E_n \right) \right) x(t) \\
&+ \tilde{\varrho}(t) \left((1-d) \dot{Z}_i^T(t) + Z_{i,s+1}^T \left((1-d) A_i + (1+\hat{d}) B - \eta(1-d) E_n \right) \right) x(t) \\
&+ \int_{t-d(t)}^{t} e^{\eta(t-s)} \left(\varrho(s) \left(\dot{Z}_i^T(t) - Z_{i,s}^T \right) + \tilde{\varrho}(s) \left(\dot{Z}_i^T(t) - Z_{i,s+1}^T \right) \right) B x(s) ds.
\end{aligned} \tag{3.45}$$

From conditions (3.30)−(3.34), we give that

$$\dot{V}_{\sigma(t)} - \eta V_{\sigma(t)} \leq 0.$$

The following procedure is the same as Theorem 3.1. Then we get the conclusion that SPLIS (1.2) is global uniform exponential stability. □

Furthermore, for the circumstance as all switching behaviors are all stabilizing, Corollary 3.1 is reduced to Corollary 3.2:

Corollary 3.2. *Suppose that* (H_1) *is held. Given that* $\bar{a}_1 \leq \bar{a}_2$*, positive integer* T*, two positive integers* m_2*,* m *satisfying* $m_2 = m$*. If there are two constants* ζ*,* η*, fulfilling* $0 < \zeta < 1$*,* $\eta > 0$*, and a class of positive vectors* $Z_{i,s}$*,* $i \in \langle n \rangle$*,* $s \in \{0, 1, \cdots, T-1\}$*, in order that* (3.30)−(3.34) *and the next inequalities hold:*

$$Z_{i,0} \leq \zeta C_j Z_{j,T}, \ (j,i) \in \tilde{N}, \tag{3.46}$$

$$\ln \zeta + \bar{a}_2 \eta < 0, \tag{3.47}$$

then the SPLIS (1.2) *is global uniform exponential stability.*

Remark 3.4. Compared to [23], we add the impulses to the model, and we get the new global uniform exponential stability condition (3.47) in Corollary 3.2. Condition (3.47) is also the constraint of vector at every switching and impulsive instant.

3.3. The algorithm to find feasible solution

We present the following algorithm to verify Theorem 3.1. Then the similar algorithm can be easily obtained for Corollary 3.1, Corollary 3.2, and the Theorems 4.1-4.2 in Section 4.

Step 1). Choose a group of constants $0 < \zeta < 1$, $\varepsilon \geq 1$, $\bar{a}_2 > 0$, $\eta > 0$, m_1, m_2, m are three positive integers satisfying $m_1 + m_2 = m$, all parameters satisfying (3.7).
Step 2). Choose the lower bound of switching interval $\bar{a}_1$ satisfying $\bar{a}_1 \leq \bar{a}_2$, $T > 0$, $T \in N^+$, then $Z_{i,s}$ satisfying (3.1)-(3.6) can be obtained by linear programming; otherwise, go back to Step 1.
Step 3). With the feasible parameters ζ, ε, η, $\bar{a}_2 > 0$, find the feasible value of $\frac{m_2}{m}$ for designing the new switching sequence.

Remark 3.5. Compared with [23], we add the impulsive effects to the SPLS with distributed time-varying delays and unstable subsystems. The number of linear matrix inequalities in this paper are more than [23], so our results are better than [23]. Furthermore, compared with [16], our model is comprehensive because of the existences of unstable subsystems and destabilizing switching behaviors. We design new linear programming algorithm process to better verify the global uniform exponential stability criteria for SPLIS (1.1) and (1.2).

4. Applications in the consensus of linear delayed multi-agent systems

In this Section, we apply the obtained criteria to the exponential consensus of linear delayed multi-agent system. We consider the linear delayed multi-agent system with switching topologies and impulses

$$\begin{cases} \dot{u}_i(t) = \sum_{j=0}^{n} a_{ij}^{(\sigma(t))}(u_j(t) - u_i(t)) + \sum_{j=0}^{n} b_{ij}^{(\sigma(t))}(u_j(t-d(t)) - u_i(t-d(t))), \\ \qquad t \geq 0,\ t \neq t_m, m = 1, 2, 3, \cdots, \\ u(t^+) = \sum_{j=0}^{n} c_{ij}^{(\sigma(t^+))} u_j(t^-), t = t_m, m = 1, 2, 3, \cdots, \end{cases} \tag{4.1}$$

where $u_i(t) \in R$ denotes the state of the ith agent for $i \in \langle n \rangle$, $\sigma(t) : [0, \infty) \to \langle r \rangle$ represents the switching rule, $u_0 \in R$ is the state of leader with dynamic equation $\dot{u}_0 = 0$, $a_{ij}^{(s)}$ is nonnegative weights satisfying $a_{ii}^{(s)} = 0$, $i, j \in \langle n \rangle$, $s \in \langle r \rangle$, $b_{ij}^{(s)} \geq 0$, impulses coefficients $c_{ij}^{(s)}$, where $i \in \langle n \rangle$ and $s \in \langle r \rangle$.

Do the transformation $v_i(t) = u_i(t) - u_0(t)$. System (4.1) converts into the next matrix form

$$\begin{cases} \dot{v}(t) = A_{\sigma(t)} v(t) + B_{\sigma(t)} v(t - d(t)), t \geq 0,\ t \neq t_m, m = 1, 2, 3, \cdots, \\ v(t^+) = C_{\sigma(t^+)} v(t^-), t = t_m, m = 1, 2, 3, \cdots, \end{cases} \tag{4.2}$$

and $A_s = D_s - L_s$, $D_s = -\text{diag}\{a_{10}^{(s)}, a_{20}^{(s)}, \ldots, a_{n0}^{(s)}\}$, $B_s = [b_{ij}^{(s)}]_{n \times n}$ and $B_s \geq 0$, $C_s = [C_{ij}^{(s)}]_{n \times n}$ and $L_s = (l_{ij}^{(s)})$ denotes the Laplace matrix with $l_{ii}^{(s)} = \sum_{j=1}^{n} a_{ij}^{(s)}$ and $l_{ij}^{(s)} = -a_{ij}^{(s)}$ for $i \neq j$, $i, j \in \langle n \rangle$ and $s \in \langle r \rangle$. When $\lim_{t \to \infty} [u_i(t) - u_0(t)] = 0$,

and $i \in \langle n \rangle$, we call system (4.1) reaches exponential consensus, which is equal to the exponential stability of system (4.2). Thus, we obtain the following exponential consensus criteria according to Corollary 3.1 and Corollary 3.2.

Theorem 4.1. *Suppose that (H_1) is held. Given that $\bar{a}_1 \leq \bar{a}_2$, positive integer T, three positive integers m_1, m_2, m are satisfying $m_1 + m_2 = m$. If there are three constants ζ, ε, η, fulfilling $0 < \zeta < 1$, $\varepsilon \geq 1$, $\eta > 0$ and a class of positive vectors $Z_{i,s}$, $i \in \langle n \rangle$, $s \in \{0, 1, \cdots, T-1\}$, in order that (3.30)−(3.36) hold, then the linear delayed multi-agent system (4.1) achieves exponential consensus.*

Theorem 4.2. *Suppose that (H_1) is held. Given that $\bar{a}_1 \leq \bar{a}_2$, positive integer T, two positive integers m, m_2, satisfying $m_2 = m$. If there are two constants ζ, η, fulfilling $0 < \zeta < 1$, $\eta > 0$, and a class of positive vectors $Z_{i,s}$, $i \in \langle n \rangle$, $s \in \{0, 1, \cdots, T-1\}$, in order that (3.30)−(3.34), (3.46)−(3.47) hold, then the linear delayed multi-agent system (4.1) achieves exponential consensus.*

Remark 4.1. When there is no time delay, the consensus of the system (4.1) was investigated in [26] by utilizing discrete absolute value Lyapunov function. We consider the effects of discrete time-varying delays for system (4.1), so the obtained criteria enhance the relevant stability criteria in [26].

5. Simulation examples

The theoretical results are verified through three simulation examples.

Example 5.1. Consider continuous-time SPLIS (1.1) with the following parameters:

$$A_1 = \begin{bmatrix} 0.45 & 0.45 \\ 0.9 & -15.4 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0.015 \\ 0.015 & 0.015 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.49 & 0.98 \\ 0.78 & -9.9 \end{bmatrix}, B_2 = \begin{bmatrix} 0.015 & 0 \\ 0.015 & 0.015 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -9.2 & 0.46 \\ 0.09 & 0.08 \end{bmatrix}, B_3 = \begin{bmatrix} 0.015 & 0.015 \\ 0.015 & 0 \end{bmatrix}.$$

Impulsive matrices taking the form of

$$C_1 = \begin{bmatrix} 1.05 & 0 \\ 0 & 1.04 \end{bmatrix}, C_2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}, C_3 = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.6 \end{bmatrix}.$$

And $d(t) = 0.15 - 0.15\cos(t)$. According to above, we acquire that $\hat{d} = 0.3$, $0.15 \leq d < 1$, $B = [0.015, 0.015; 0.015, 0.015]$.

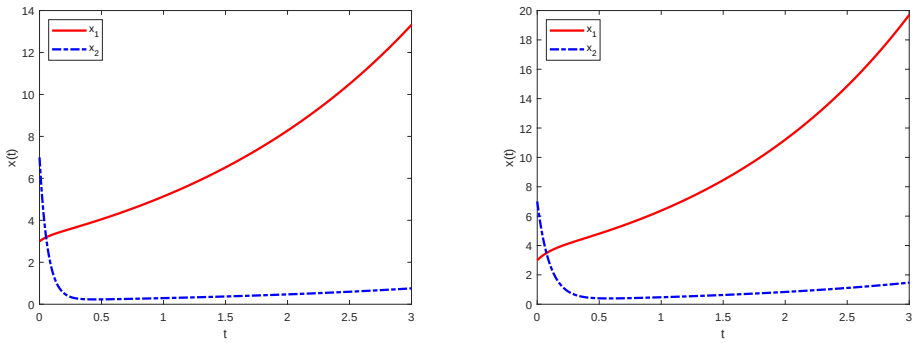
Choosing the initial state $x(0) = (3, 7)^T$, and the three subsystems solution trajectories of system (1.1) are displayed in Figure 3 and Figure 4, which indicate that the three subsystems are all unstable. Let $\eta = 1.15$, $\bar{a}_1 = 0.12$, $\bar{a}_2 = 0.15$, $\zeta = 0.7$, $\varepsilon = 1.2$, $m_1 = 1$, $m_2 = 8$, $m = 9$ and $T = 1$, it can be gained that

conditions in Theorem 3.1 hold with Z_i^j given as follows.

$$Z_{1,0} = \begin{bmatrix} 80.4001 \\ 169.1412 \end{bmatrix}, Z_{1,1} = \begin{bmatrix} 59.1225 \\ 236.9019 \end{bmatrix},$$
$$Z_{2,0} = \begin{bmatrix} 72.4943 \\ 152.5653 \end{bmatrix}, Z_{2,1} = \begin{bmatrix} 56.4351 \\ 205.3150 \end{bmatrix},$$
$$Z_{3,0} = \begin{bmatrix} 41.4550 \\ 170.4646 \end{bmatrix}, Z_{3,1} = \begin{bmatrix} 90.5496 \\ 152.8046 \end{bmatrix}.$$

It is observed that $\tilde{N} = \{(2,3),(3,2),(1,3),(3,1)\}$, i.e., the stabilizing switching behaviors occur not only subsystems couples of 2 and 3, but also subsystems couples of 1 and 3. Furthermore, the feasible value of $\frac{m_2}{m}$ is 0.88889. The new switching sequence is designed as follows,

$$\underbrace{2 \to 3 \to 2 \to 3 \to 1 \to 3 \to 1 \to 2 \to 3.}_{Loop\ this\ sequence}$$



(a) (b)

Figure 3. The state trajectories of subsystem 1 and subsystem 2 of SPLIS (1.1).

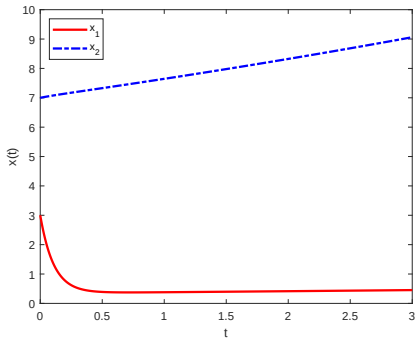
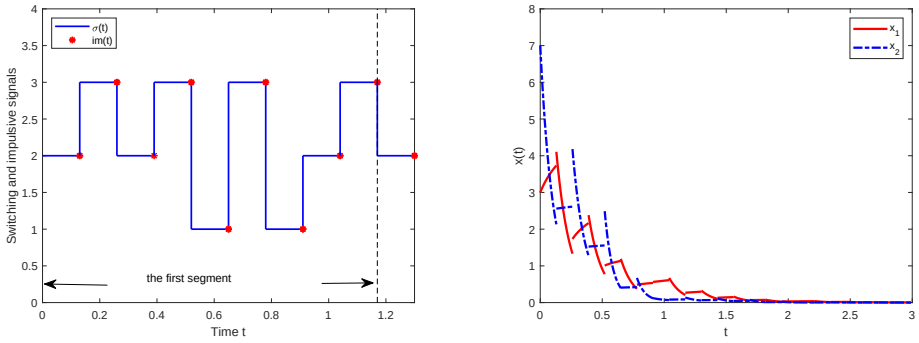


Figure 4. The state trajectories of subsystem 3 of SPLIS (1.1).

Choosing initial condition $x(0) = (3, 7)^T$, the switching and impulsive signals are in Figure 5(a) . The solution trajectory of SPLIS (1.1) is displayed in Figure 5(b), which implies that the solution trajectory of SPLIS (1.1) is global uniform exponential stability. Furthermore, SPLIS (1.1) contains the impulsive effects, so that the main results in [23] cannot be applied to the example.



(a) (b)

Figure 5. The signals and state trajectories of SPLIS (1.1).

Remark 5.1. In Example 5.1, there are nine switching parts on the first segment from Figure 5(a). According to the definitions of the designed sequence and stabilizing switching behaviors, only $1 \to 2$(the 8th part) indicates destabilizing switching behavior in the first segment. Therefore, the number of destabilizing and stabilizing switching behaviors is "1" and "8", respectively. Examples 5.2-5.3 have similar explanations as Example 5.1.

Example 5.2. Consider continuous-time SPLIS (1.2) with the following parame-

ters:

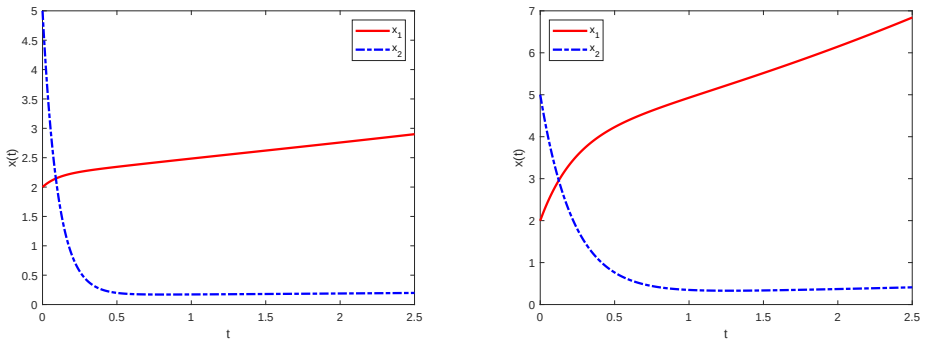
$$A_1 = \begin{bmatrix} 0.06 & 0.42 \\ 0.68 & -9.9 \end{bmatrix}, B_1 = \begin{bmatrix} 0.012 & 0.012 \\ 0 & 0.012 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.09 & 1.9 \\ 0.27 & -4.4 \end{bmatrix}, B_2 = \begin{bmatrix} 0.012 & 0.012 \\ 0.012 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -8.5 & 0.47 \\ 1.58 & 0.09 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0.012 \\ 0.012 & 0.012 \end{bmatrix}.$$

Impulsive matrices taking the form of

$$C_1 = \begin{bmatrix} 1.02 & 0 \\ 0 & 1.035 \end{bmatrix}, C_2 = \begin{bmatrix} 1.08 & 0 \\ 0 & 1.014 \end{bmatrix}, C_3 = \begin{bmatrix} 1.038 & 0 \\ 0 & 1.059 \end{bmatrix}.$$

And $d(t) = 0.18 - 0.18\cos(t)$. Therefore, we gain that $\hat{d} = 0.36$, $0.18 \leq d < 1$, $B = [0.012, 0.012; 0.012, 0.012]$.

Choosing the initial state $x(0) = (2, 5)^T$, and the three subsystems state trajectories are shown in Figure 6 and Figure 7, which show that the three subsystems are all unstable.



(a) (b)

Figure 6. The state trajectories of subsystem 1 and subsystem 2 of SPLIS (1.2).

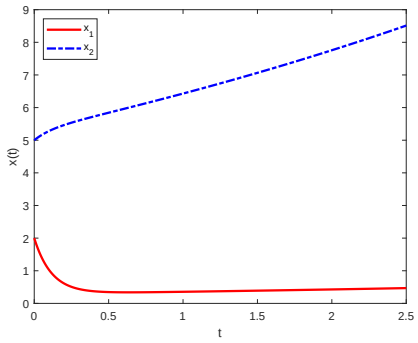


Figure 7. The state trajectories of subsystem 3 of SPLIS (1.2).

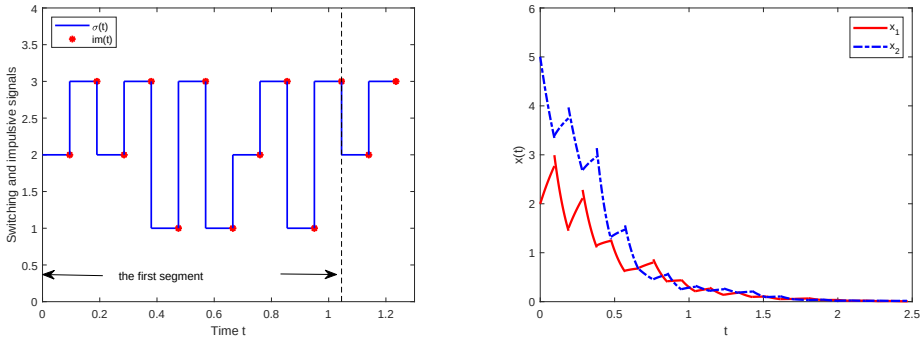
Let $\eta = 0.76$, $\bar{a}_1 = 0.09$, $\bar{a}_2 = 0.0955$, $\zeta = 0.86$, $\varepsilon = 1.02$, $m_1 = 1$, $m_2 = 10$, $m = 11$ and $T = 1$, it can be acquired that conditions in Corollary 3.1 hold with Z_i^j given as follows.

$$Z_{1,0} = \begin{bmatrix} 155.7359 \\ 197.1935 \end{bmatrix}, Z_{1,1} = \begin{bmatrix} 149.2218 \\ 250.2733 \end{bmatrix},$$
$$Z_{2,0} = \begin{bmatrix} 153.2504 \\ 197.3689 \end{bmatrix}, Z_{2,1} = \begin{bmatrix} 140.9317 \\ 255.4565 \end{bmatrix},$$
$$Z_{3,0} = \begin{bmatrix} 128.8974 \\ 220.7683 \end{bmatrix}, Z_{3,1} = \begin{bmatrix} 176.6993 \\ 218.7161 \end{bmatrix}.$$

It can be observed that group $\tilde{N} = \{(2,3),(3,2),(1,3),(3,1)\}$, i.e., the stabilizing switching behaviors occur between subsystems couples of 2 and 3 and subsystem couples of 1 and 3. Furthermore, the feasible limit of $\frac{m_2}{m}$ is counted as 0.90909. The new switching sequence is designed as follows,

$$\underbrace{2 \to 3 \to 2 \to 3 \to 1 \to 3 \to 1 \to 2 \to 3 \to 1 \to 3.}_{Loop\ this\ sequence}$$

Choosing the initial condition $x(0) = (2,5)^T$, the switching and impulsive signals are exhibited in Figure 8(a). The solution trajectory of SPLIS (1.2) is shown in Figure 8(b), which implies that the solution trajectory of SPLIS (1.2) is global uniform exponential stability. Moreover, SPLIS (1.2) contains the impulsive effects, the results in Zhou et al. [23] are invalid.



(a) (b)

Figure 8. The signals and state trajectories of SPLIS (1.2).

Example 5.3. Consider the linear delayed multi-agent system (4.1) with $r = 2$, $n = 3$, $d(t) = 0.12 - 0.12\cos(t)$, $D_1 = \text{diag}\{0, -1.02, 0\}$, $D_2 = \text{diag}\{-1.04, 0, 0\}$, and Laplace matrices fulfilling

$$L_1 = \begin{bmatrix} 0.08 & -0.08 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -0.07 & 0.07 \end{bmatrix},$$

and

$$B_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.002 \end{bmatrix},$$

Impulsive matrices taking the form of

$$C_1 = \begin{bmatrix} 1.03 & 0 & 0 \\ 0 & 1.02 & 0 \\ 0 & 0 & 1.04 \end{bmatrix}, C_2 = \begin{bmatrix} 1.003 & 0 & 0 \\ 0 & 1.002 & 0 \\ 0 & 0 & 1.004 \end{bmatrix},$$

which infers from $A_1 = D_1 - L_1, A_2 = D_2 - L_2$, we have

$$A_1 = \begin{bmatrix} -0.08 & 0.08 & 0 \\ 0 & -1.02 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -1.04 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.07 & -0.07 \end{bmatrix}, B = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.002 \end{bmatrix}.$$

Since there exist the discrete time-varying delays effects, Corollary 3 in [26] cannot be applied to this example. $\hat{d} = 0.24$, $0.12 \leq d < 1$. Choosing $\eta = 1.09$, $\bar{a}_1 = 0.42$,

$\bar{a}_2 = 0.47$, $\zeta = 0.58$, $T = 1$, $m_2 = m = 2$. It is obtained that conditions in Theorem 4.2 hold true for

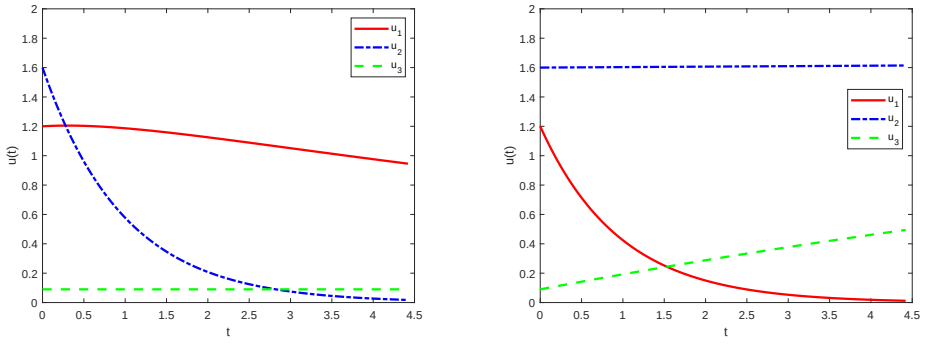
$$Z_{10} = \begin{bmatrix} 623.1397 \\ 626.1294 \\ 620.2074 \end{bmatrix}, Z_{11} = \begin{bmatrix} 838.6341 \\ 846.8560 \\ 830.5703 \end{bmatrix}, Z_{20} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}, Z_{21} = \begin{bmatrix} 500 \\ 500 \\ 500 \end{bmatrix}.$$

It can be observed that group $\tilde{N} = \{(1,2),(2,1)\}$, so all switching behaviors are stabilizing. The switching sequence is designed as follows

$$\underbrace{1 \to 2.}_{Loop\ this\ sequence}$$

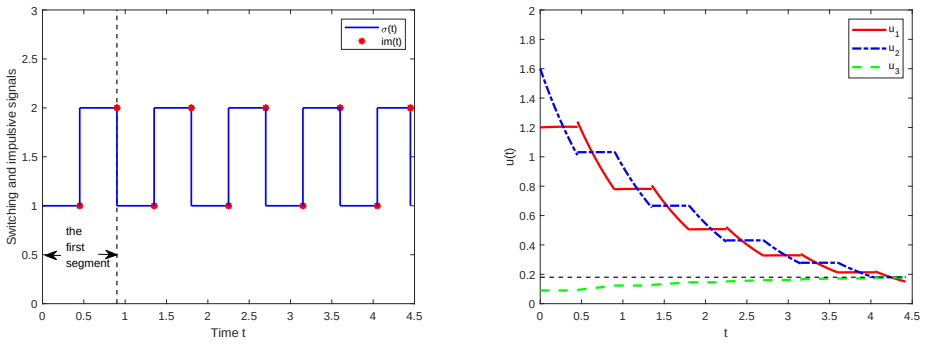
Choosing the initial state $u(0) = [1.2, 1.6, 0.09]^\top$, and two subsystems of system (4.1) are shown in Figure 9, which shows that the two subsystems are all unstable.

Therefore, the system (4.1) reaches exponential consensus. Choosing initial condition $u_0(0) = 0.18$, $u_1(0) = 1.2$, $u_2(0) = 1.6$, $u_3(0) = 0.09$ and the signals are shown in Figure 10(a). The solution trajectories of the system (4.1) is shown in Figure 10(b), which implies that the system (4.1) is exponential consensus.



(a) (b)

Figure 9. The state trajectories of two subsystems of system (4.1).



(a) (b)

Figure 10. The signals and state trajectories of system (4.1).

Remark 5.2. It should be noted that the recent results in [16, 22, 23, 26] cannot be applied to Examples 5.1, 5.2 and 5.3 due to the coexistence of distributed time-varying delays, discrete time-varying delays, unstable subsystems, stabilizing/destabilizing switching behaviors, and impulsive effects. To some extent, the proposed results in this paper are more effective than previous ones.

6. Discussions

The global uniform exponential stability of SPLIS with time-varying delays and all unstable subsystems is investigated in this paper. By utilizing multiple co-positive Lyapunov−Krasovskill functional, new specific global uniform exponential stability criteria for the SPLIS in the case of switching-impulse signals are obtained in the fields of linear matrix inequalities, which are verified via the linear programming algorithm. After that, the main stability criteria are applied to the exponential consensus of linear delayed multi-agent systems with switching communication topologies.

There exist some limitations on our work. It is worth noting that the impulsive instants and switching instants may may not be synchronous. We deal with the impulsive effects for the SPLIS in this paper, which are synchronous with the switching signals. Then, we will gradually explore the case of global uniform exponential stability for SPLIS (1.1) and SPLIS (1.2) with asynchronous switching impulsive signals in the future. Furthermore, the method in this paper aren't directly generalized to the case of two types of time-varying delays. In fact, there exists the corresponding results of two types of time-varying delays case in theory. The main difficulty is how to define Lyapunov-Krasovskii functional. This will be left for our future study.

Declaration of competing interest

There is no competing interest.

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