On Hermite-Hadamard-type characterizations of higher-order differential inequalities

Mohamed Jleli Bessem Samet*

Department of Mathematics, College of Science, King Saud University P.O. Box 2455, Riyadh, 11451, Saudi Arabia

E-mail addresses: jleli@ksu.edu.sa; bsamet@ksu.edu.sa

Abstract

Let I be an open interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$. It is well-known that f is convex in I if and only if, for all $x, y \in I$ with $x < y$, it holds that

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{f(x)+f(y)}{2}.$$

The above inequality is known in the literature as Hermite-Hadamard inequality. In the first part of this paper, we extend the above result to the set of functions $f \in C^{2n}(I)$ satisfying the higher-order differential inequality $(-1)^n f^{(2n)} \leq 0$ in I. In particular, when f satisfies the above inequality with $n = 2$, and f is convex, we obtain an interesting refinement of Hermite-Hadamard inequality. The second part of this paper is devoted to the study of sub-biharmonic functions, i.e., the set of functions $f \in C^4(\Omega)$, Ω is an open subset of $\mathbb{R}^N$ ($N \geq 2$), satisfying $\Delta^2 f \leq 0$ in Ω. Namely, a characterization of this set of functions is established. In particular, when f is subharmonic ($\Delta f \geq 0$ in Ω) and f is sub-biharmonic, an interesting refinement of Hermite-Hadamard inequality in higher dimension is obtained.

2010 Mathematics Subject Classification. 26A51; 26B25; 31B05; 26D15.

Key words and phrases. Convexity; higher-order differential inequalities; subharmonic functions; sub-biharmonic functions; Hermite-Hadamard inequality; characterization.

1 Introduction

The Hermite-Hadamard inequality can be stated as follows: Let $f : [a, b] \to \mathbb{R}$ be a convex function, then

$$\frac{1}{b-a}\int_a^b f(z)\,dz \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality dates back to an 1883 observation of Hermite [15] with an independent use by Hadamard [13] in 1893. The Hermite-Hadamard inequality has been frequently used in the study of the properties of convex functions and their applications

*Corresponding author

in optimization and approximation theory (see e.g. [3, 12, 18]). Many generalizations and extensions of (1) can be found in the literature. For instance, we refer to [1, 2, 10, 11, 20, 21, 22] (see also the references therein).

It is interesting to notice that (1) provides a characterization of convex functions. Namely, if f is a continuous function in an interval I, then the following statements are equivalent:

(i) f is convex in I;

(ii) For all $x, y \in I$ with $x < y$,

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{f(x)+f(y)}{2}. \tag{2}$$

The proof of the above characterization can be found in [25]. On the other hand, it is well-known that, if f is twice differentiable, then f is convex if and only if $f'' \geq 0$. So, if f is twice differentiable, then $f'' \geq 0$ in I if and only if, for all $x, y \in I$ with $x < y$, (2) holds. From this observation, it is natural to ask whether it is possible to extend this result to more general differential inequalities. The first part of this paper is concerned with the study of this question. Namely, we are concerned with the characterization of higher-order differential inequalities of the form

$$(-1)^n f^{(2n)} \leq 0 \text{ in } I, \tag{3}$$

where n is a positive natural number. A particular attention is devoted to the study of fourth-order differential inequalities, i.e., the case $n = 2$. Here, by $f^{(2n)}$, we mean the derivative of order $2n$ of f. Observe that in the special case $n = 1$, (3) means that f is convex in I. To the best of our knowledge, except for the case $n = 1$ (the convexity case), the study of characterization of higher-order differential inequalities of the form (3) has never been considered in the literature.

In the higher-dimensional case, one can expect that for a given convex function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is convex, it holds that

$$\frac{1}{|\Omega|}\int_\Omega f(z)\,dz \leq \frac{1}{|\partial\Omega|}\int_{\partial\Omega} f(z)\,dS_z. \tag{4}$$

In [8], Dragomir proved (4) in the two-dimensional case, where Ω is a ball in $\mathbb{R}^2$. The same author [9] proved that the same result holds true in the three-dimensional case. Later, (4) has been extended to the N-dimensional case by de la Cal & Carcamo [7]. On the other hand, it is well-known that any twice differentiable convex function f is subharmonic, that is, f verifies the elliptic inequality $\Delta f \geq 0$, where Δ is the Laplacian operator. So, it is natural to ask whether it is possible to obtain a Hermite-Hadamard inequality for subharmonic functions. This question has been studied by many authors. For instance, Niculescu & Persson [21] proved that, if $f \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is subharmonic, then

$$\int_\Omega f(z)\,dz \leq \int_{\partial\Omega} f(z)\frac{\partial\varphi}{\partial\nu}(z)\,dS_z,$$

where $\varphi \leq 0$ solves the Dirichlet boundary value problem

$$\begin{cases} \Delta\varphi = 1 \text{ in } \Omega, \\ \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

Here, by $\frac{\partial}{\partial \nu}$, we mean the normal derivative. Other results related to Hermite-Hadamard inequalities for subharmonic functions can be found in [4, 16, 17, 19, 26] (see also the references therein).

It is interesting to mention that inequality (4) characterizes subharmonic functions. In the two-dimensional case, this result already appears in [5, 23], and was also stated in an equivalent form in [24]. These two results can be stated as follows: $\Delta f \geq 0$ in Ω if and only if, for all $x \in \Omega$ and $\delta > 0$ with $\overline{B(x,\delta)} \subset \Omega$, it holds that

$$\frac{1}{|B(x,\delta)|}\int_{\Omega} f(z)\,dz \leq \frac{1}{|\partial B(x,\delta)|}\int_{\partial B(x,\delta)} f(z)\,dS_z. \tag{5}$$

Here, $B(x,\delta)$ is the open ball of center x and radius δ, and $\overline{B(x,\delta)}$ is the closure of $B(x,\delta)$. From this result, it is natural to ask whether it is possible to obtain a similar characterization for sub-biharmonic functions f, i.e., f satisfies $\Delta^2 f \leq 0$ in Ω, where Δ^2 is the biLaplacian operator. The second part of this paper is devoted to the study of this question. Namely a Hermite-Hadamard-type characterization of sub-biharmonic functions is obtained. To the best of our knowledge, the characterization of sub-biharmonic functions has not been previously considered in the literature.

We fix below some notations that will be used throughout this paper.

- I: an open interval of $\mathbb{R}$;
- $C^k(I)$: the space of k-times continuously differentiable functions in I;
- $f^{(k)}$: the derivative of order k of f;
- N: a natural number ≥ 2;
- Ω: an open subset of $\mathbb{R}^N$;
- $\|\cdot\|$: the Euclidean norm in $\mathbb{R}^N$;
- ∇: the gradient operator;
- Δ: the Laplacian operator;
- Δ^2: the biLaplacian operator;
- $\dfrac{\partial}{\partial \nu}$: the normal derivative.

Let $x \in \Omega$ and $\delta > 0$.

- $B(x,\delta) = \left\{z \in \mathbb{R}^N : \|z - x\| < \delta\right\}$;
- $\overline{B(x,\delta)}$ is the closure of $B(x,\delta)$;
- $\partial B(x,\delta) = \left\{z \in \mathbb{R}^N : \|z - x\| = \delta\right\}$;
- ν is the outward unit normal on $\partial B(x,\delta)$;
- $V_N(\delta) = |B(x,\delta)|$;

- $A_{N-1}(\delta) = |\partial B(x,\delta)|$

Finally, we recall that (see e.g. [6])

$$V_N(\delta) = \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)}\delta^N$$

and

$$A_{N-1}(\delta) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\delta^{N-1},$$

where $\Gamma(\cdot)$ is the Gamma function.

The rest of the paper is organized as follows.

In Section 2, we consider higher-order differential inequalities of the form (3), where n is a positive natural number. A characterization of the set fo functions $f \in C^{2n}(I)$ satisfying (3) is established (see Theorem 2.2). Next, a special attention is devoted to the special case $n = 2$ (fourth-order differential inequalities). In particular, when f is a convex function satisfying $f'''' \leq 0$, we obtain an interesting refinement of Hermite-Hadamard inequality (see Corollary 2.7).

Section 3 is devoted to the study of sub-biharmonic functions. Namely, we establish a characterization of the set of functions $f \in C^4(\Omega)$ satisfying $\Delta^2 f \leq 0$ in Ω (see Theorem 3.1). In particular, when f is both subharmonic and sub-biharmonic, we obtain an improvement of inequality (5) (see Corollary 3.2).

2 Differential inequalities of the form $(-1)^n f^{(2n)} \leq 0$

In this section, we are concerned with the characterization of the set of functions $f \in C^{2n}(I)$ satisfying the higher-order differential inequality

$$(-1)^n f^{(2n)}(z) \leq 0, \quad z \in I, \tag{6}$$

where n is a positive natural number.

We first need the following lemma that can be easily proved by means of the general Leibniz rule.

Lemma 2.1. *Let $x, y \in I$ with $x < y$. Let*

$$g_n(z) = -\frac{1}{(2n)!}(z-x)^n(z-y)^n, \quad x \leq z \leq y.$$

Then g_n is a solution to the boundary value problem

$$\begin{cases} g_n^{(2n)}(z) = -1, & x < z < y, \\ g_n^{(k)}(x) = g_n^{(k)}(y) = 0, & k = 0, 1, \cdots, n-1. \end{cases} \tag{7}$$

Moreover, for all $k \in \{n, n+1, \cdots, 2n-1\}$, we have

$$g_n^{(k)}(x) = \frac{(-1)^{k+1}}{(2n)!}\frac{k!n!}{(k-n)!(2n-k)!}(y-x)^{2n-k} \tag{8}$$

and

$$g_n^{(k)}(y) = -\frac{1}{(2n)!}\frac{k!n!}{(k-n)!(2n-k)!}(y-x)^{2n-k}. \tag{9}$$

2.1 Main result

The main result of this section is the following theorem.

Theorem 2.2. *Let $f \in C^{2n}(I)$. The following statements are equivalent:*

(i) *The function f verifies (6);*

(ii) *For all $x, y \in I$ with $x < y$, it holds that*

$$\frac{1}{y-x}\int_x^y f(z)\,dz \le \frac{n!}{(2n)!}\sum_{k=n}^{2n-1}\frac{k!(y-x)^{2n-k-1}}{(k-n)!(2n-k)!}\left(f^{(2n-k-1)}(x)+(-1)^{k+1}f^{(2n-k-1)}(y)\right). \tag{10}$$

Proof. (i) $\Longrightarrow$ (ii): Assume that f verifies (6), and let $x, y \in I$ with $x < y$. By (7), one has

$$\int_x^y f(z)\,dz = -\int_x^y g_n^{(2n)}(z)f(z)\,dz.$$

Integrating by parts ($2n$-times), we obtain

$$\int_x^y f(z)\,dz = \sum_{k=0}^{2n-1}(-1)^k\left[g_n^{(k)}(z)f^{(2n-k-1)}(z)\right]_{z=x}^{y} - \int_x^y g_n(z)f^{(2n)}(z)\,dz.$$

Moreover, we have

$$\left[g_n^{(k)}(z)f^{(2n-k-1)}(z)\right]_{z=x}^{y} = 0, \quad k = 0, 1, \cdots, n-1.$$

Hence, it holds that

$$\int_x^y f(z)\,dz = \sum_{k=n}^{2n-1}(-1)^k\left[g_n^{(k)}(z)f^{(2n-k-1)}(z)\right]_{z=x}^{y} - \int_x^y g_n(z)f^{(2n)}(z)\,dz.$$

Then, making use of (8) and (9), we obtain

$$\begin{aligned}
&\int_x^y f(z)\,dz \\
&= \sum_{k=n}^{2n-1}(-1)^k\left(g_n^{(k)}(y)f^{(2n-k-1)}(y) - g_n^{(k)}(x)f^{(2n-k-1)}(x)\right) - \int_x^y g_n(z)f^{(2n)}(z)\,dz \\
&= \frac{n!}{(2n)!}\sum_{k=n}^{2n-1}\frac{k!(y-x)^{2n-k}}{(k-n)!(2n-k)!}\left(f^{(2n-k-1)}(x)+(-1)^{k+1}f^{(2n-k-1)}(y)\right) - \int_x^y g_n(z)f^{(2n)}(z)\,dz \\
&= \frac{n!}{(2n)!}\sum_{k=n}^{2n-1}\frac{k!(y-x)^{2n-k}}{(k-n)!(2n-k)!}\left(f^{(2n-k-1)}(x)+(-1)^{k+1}f^{(2n-k-1)}(y)\right) \\
&\quad + \frac{1}{(2n)!}\int_x^y (z-x)^n(y-z)^n\left[(-1)^n f^{(2n)}(z)\right]\,dz.
\end{aligned} \tag{11}$$

On the other hand, in view of (6), one has

$$\int_x^y (z-x)^n(y-z)^n\left[(-1)^n f^{(2n)}(z)\right]\,dz \le 0. \tag{12}$$

Thus, (10) follows from (11) and (12). This proves that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i): Assume that for all $x, y \in I$ with $x < y$, (10) holds. Let $x \in I$ be fixed, and $\varepsilon > 0$ (small enough), so that $[x-\varepsilon, x+\varepsilon] \subset I$. Then, it holds that

$$\int_{x-\varepsilon}^{x+\varepsilon} f(z)\,dz \leq \frac{n!}{(2n)!}\sum_{k=n}^{2n-1}\frac{k!(2\varepsilon)^{2n-k}}{(k-n)!(2n-k)!}\left(f^{(2n-k-1)}(x-\varepsilon)+(-1)^{k+1}f^{(2n-k-1)}(x+\varepsilon)\right). \tag{13}$$

Let

$$g_{n,\varepsilon}(z) = -\frac{1}{(2n)!}(z-x+\varepsilon)^n(z-x-\varepsilon)^n, \quad x-\varepsilon \leq z \leq x+\varepsilon.$$

Taking $x-\varepsilon$ (resp. $x+\varepsilon$) instead of x (resp. y) in Lemma 2.1, we can see that $g_{n,\varepsilon}$ is a solution to the boundary value problem

$$\begin{cases} g_{n,\varepsilon}^{(2n)}(z) = -1, \quad x-\varepsilon < z < x+\varepsilon, \\ g_{n,\varepsilon}^{(k)}(x-\varepsilon) = g_{n,\varepsilon}^{(k)}(x+\varepsilon) = 0, \quad k = 0, 1, \cdots, n-1. \end{cases} \tag{14}$$

Moreover, for all $k \in \{n, n+1, \cdots, 2n-1\}$, we have

$$g_{n,\varepsilon}^{(k)}(x-\varepsilon) = \frac{(-1)^{k+1}}{(2n)!}\frac{k!n!}{(k-n)!(2n-k)!}(2\varepsilon)^{2n-k} \tag{15}$$

and

$$g_{n,\varepsilon}^{(k)}(x+\varepsilon) = \frac{-1}{(2n)!}\frac{k!n!}{(k-n)!(2n-k)!}(2\varepsilon)^{2n-k}. \tag{16}$$

By (14), (15), (16), and using integrations by parts, we obtain

$$\begin{aligned}
&\int_{x-\varepsilon}^{x+\varepsilon} f(z)\,dz \\
&= -\int_{x-\varepsilon}^{x+\varepsilon} f(z)g_{n,\varepsilon}^{(2n)}(z)\,dz \\
&= \frac{n!}{(2n)!}\sum_{k=n}^{2n-1}\frac{k!(2\varepsilon)^{2n-k}}{(k-n)!(2n-k)!}\left(f^{(2n-k-1)}(x-\varepsilon)+(-1)^{k+1}f^{(2n-k-1)}(x+\varepsilon)\right) \\
&\quad + \frac{1}{(2n)!}\int_{x-\varepsilon}^{x+\varepsilon}(z-x+\varepsilon)^n(x+\varepsilon-z)^n\left[(-1)^n f^{(2n)}(z)\right]\,dz,
\end{aligned}$$

which implies by (13) that

$$\int_{x-\varepsilon}^{x+\varepsilon}(z-x+\varepsilon)^n(x+\varepsilon-z)^n\left[(-1)^n f^{(2n)}(z)\right]\,dz \leq 0.$$

Since

$$(z-x+\varepsilon)^n(x+\varepsilon-z)^n \geq 0, \quad x-\varepsilon \leq z \leq x+\varepsilon,$$

then there exists

$$z_\varepsilon \in (x-\varepsilon, x+\varepsilon)$$

such that

$$(-1)^n f^{(2n)}(z_\varepsilon) \leq 0. \tag{17}$$

Since $f^{(2n)}$ is continuous in I, passing to the limit as $\varepsilon \to 0^+$ in (17), we obtain

$$(-1)^n f^{(2n)}(x) \leq 0.$$

This shows that (ii) $\implies$ (i). The proof of Theorem 2.2 is then completed. □

Remark 2.3. Taking $n = 1$ in Theorem 2.2, we obtain the following result: If $f \in C^2(I)$, then $f'' \geq 0$ in I if and only if, for all $x, y \in I$ with $x < y$, it holds that

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{f(x)+f(y)}{2}.$$

2.2 Fourth-order differential inequalities

Taking $n = 2$ in Theorem 2.2, we obtain the following result.

Corollary 2.4. *Let $f \in C^4(I)$. The following statements are equivalent:*

(i) *$f'''' \leq 0$ in I;*

(ii) *For all $x, y \in I$ with $x < y$, it holds that*

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{f(x)+f(y)}{2} - \frac{y-x}{12}\left(f'(y)-f'(x)\right). \tag{18}$$

We now consider the the set of functions $f \in C^4(I)$ satisfying the fourth-order differential inequality

$$f''''(z) \leq m, \quad z \in I, \tag{19}$$

where $m \in \mathbb{R}$ is a constant. From Corollary 2.4, we deduce the following result.

Corollary 2.5. *Let $f \in C^4(I)$. The following statements are equivalent:*

(i) *The function f verifies (19);*

(ii) *For all $x, y \in I$ with $x < y$, it holds that*

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{f(x)+f(y)}{2} - \frac{y-x}{12}\left(f'(y)-f'(x)\right) + \frac{m}{720}(y-x)^4. \tag{20}$$

Proof. Observe that (19) is equivalent to

$$h''''(z) \leq 0, \quad z \in I,$$

where

$$h(z) = f(z) - \frac{m}{24}z^4, \quad z \in I.$$

Hence, applying Corollary 2.4 with h instead of f, we obtain that f verifies (19) if and only if, for all $x, y \in I$ with $x < y$, it holds that

$$\frac{1}{y-x}\int_x^y h(z)\,dz \leq \frac{h(x)+h(y)}{2} - \frac{y-x}{12}\left(h'(y)-h'(x)\right),$$

that is,

$$\frac{1}{y-x}\int_x^y f(z)\,dz \le \frac{f(x)+f(y)}{2} - \frac{y-x}{12}\left(f'(y)-f'(x)\right) + mP(x,y),$$

where

$$P(x,y) = \frac{1}{120}\frac{y^5-x^5}{y-x} - \frac{1}{48}(x^4+y^4) + \frac{1}{72}(y-x)(y^3-x^3).$$

Elementary calculations show that

$$P(x,y) = \frac{1}{720}(y-x)^4.$$

This shows that f verifies (19) if and only if, for all $x, y \in I$ with $x < y$, (20) holds. □

Remark 2.6. Observe that, if $m < 0$, then (20) improves (18).

We next consider the set of functions $f \in C^4(I)$ satisfying

$$f''''(z) \le 0,\ f''(z) \ge 0, \quad z \in I. \tag{21}$$

Clearly, if $f \in C^4(I)$ verifies (21), then f is convex in I. From Corollary 2.4, we deduce the following interesting refinement of Hermite-Hadamard inequality.

Corollary 2.7 (Refinement of Hermite-Hadamard inequality)**.** *Let $f \in C^4(I)$ verifies (21). Then, for all $x, y \in I$ with $x < y$, it holds that*

$$\frac{1}{y-x}\int_x^y f(z)\,dz \le \frac{f(x)+f(y)}{2} - \frac{y-x}{12}\left(f'(y)-f'(x)\right) \le \frac{f(x)+f(y)}{2}. \tag{22}$$

Proof. We have just to observe that, if $f \in C^4(I)$ verifies (21), then f' is nondecreasing in I. So, for all $x, y \in I$ with $x < y$, one has

$$f'(y) - f'(x) \ge 0.$$

Hence, by Corollary 2.4, we obtain (22). □

From Corollary 2.4, we also deduce the following result.

Corollary 2.8. *Let $f \in C^4(I)$ verifies $f'''' \le 0$ in I. Let $k \ge 2$ be a natural number and $\{x_i\}_{i=1}^k \subset I$ with $x_1 < x_2 < \cdots < x_k$. Then, it holds that*

$$\int_{x_1}^{x_k} f(z)\,dz \le \sum_{i=1}^{k-1}(x_{i+1}-x_i)\left(\frac{f(x_i)+f(x_{i+1})}{2} - \frac{x_{i+1}-x_i}{12}\left(f'(x_{i+1})-f'(x_i)\right)\right). \tag{23}$$

Proof. We have

$$\int_{x_1}^{x_k} f(z)\,dz = \sum_{i=1}^{k-1}\int_{x_i}^{x_{i+1}} f(z)\,dz. \tag{24}$$

On the other hand, by Corollary 2.4, for all $i \in \{1, 2, \cdots, k-1\}$, we have

$$\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(z)\,dz \le \frac{f(x_i)+f(x_{i+1})}{2} - \frac{x_{i+1}-x_i}{12}\left(f'(x_{i+1})-f'(x_i)\right). \tag{25}$$

Combining (24) with (25), we obtain (23). □

From Corollary 2.8, we deduce the following result.

Corollary 2.9. *Let $f \in C^4(I)$ verifies $f'''' \leq 0$ in I. Then, for all $x, y \in I$ with $x < y$, it holds that*

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{1}{2}\left(\frac{f(x)+f(y)}{2} + f\left(\frac{x+y}{2}\right)\right) - \frac{y-x}{48}\left(f'(y)-f'(x)\right). \tag{26}$$

Proof. Applying Corollary 2.8 with

$$k = 3,\ x_1 = x,\ x_2 = \frac{x+y}{2},\ x_3 = y,$$

we immediately obtain (26). □

In particular, if $f \in C^4(I)$ verifies (21), we deduce from Corollary 2.9 the following result.

Corollary 2.10. *Let $f \in C^4(I)$ verifies (21). Then, for all $x, y \in I$ with $x < y$, it holds that*

$$\begin{aligned}\frac{1}{y-x}\int_x^y f(z)\,dz &\leq \frac{1}{2}\left(\frac{f(x)+f(y)}{2} + f\left(\frac{x+y}{2}\right)\right) - \frac{y-x}{48}\left(f'(y)-f'(x)\right)\\ &\leq \frac{1}{2}\left(\frac{f(x)+f(y)}{2} + f\left(\frac{x+y}{2}\right)\right).\end{aligned} \tag{27}$$

Remark 2.11. In [21], it was shown that, if f is a convex function in I, then for all $x, y \in I$ with $x < y$, it holds that

$$\frac{1}{y-x}\int_x^y f(z)\,dz \leq \frac{1}{2}\left(\frac{f(x)+f(y)}{2} + f\left(\frac{x+y}{2}\right)\right). \tag{28}$$

Therefore, if $f \in C^4(I)$ verifies (21), then (27) improves (28).

3 Sub-biharmonic functions

In this section, we are concerned with the characterization of the set of functions $f \in C^4(\Omega)$ satisfying

$$\Delta^2 f(z) \leq 0, \quad z \in \Omega. \tag{29}$$

A function f satisfying the above property is said to be sub-biharmonic in Ω (see e.g. [14]).

Our main result in this section is stated below.

Theorem 3.1 (A characterization of sub-biharmonic functions)**.** *Let $f \in C^4(\Omega)$. The following statements are equivalent:*

(i) *The function f verifies (29);*

(ii) *For all $x \in \Omega$ and $\delta > 0$ with $\overline{B(x,\delta)} \subset \Omega$, it holds that*

$$\frac{1}{V_N(\delta)}\int_{B(x,\delta)} f(z)\,dz \leq \frac{1}{A_{N-1}(\delta)}\int_{\partial B(x,\delta)} \left(f(z) - \frac{\delta}{N+2}\frac{\partial f}{\partial \nu}(z)\right) dS_z. \tag{30}$$

Proof. (i) $\Longrightarrow$ (ii): Assume that f verifies (29). Let $x \in \Omega$ and $\delta > 0$ with $\overline{B(x,\delta)} \subset \Omega$. Let us introduce the function

$$H(z) = -\frac{1}{8N(N+2)} \left(\|z-x\|^2 - \delta^2 \right)^2, \quad z \in B(x,\delta). \tag{31}$$

Elementary calculations show that H is a solution to the boundary value problem

$$\left\{ \begin{array}{l} \Delta^2 H(z) = -1, \quad z \in B(x,\delta), \\ \\ H(z) = \dfrac{\partial H}{\partial \nu}(z) = 0, \quad z \in \partial B(x,\delta). \end{array} \right. \tag{32}$$

By (32), one has

$$\int_{B(x,\delta)} f(z)\, dz = -\int_{B(x,\delta)} f(z) \Delta^2 H(z)\, dz.$$

Applying Green identity to the right side of the last equality, we obtain

$$\begin{aligned} \int_{B(x,\delta)} f(z)\, dz \quad &= \quad -\int_{\partial B(x,\delta)} f(z) \frac{\partial \Delta H}{\partial \nu}(z)\, dS_z + \int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z) \Delta H(z)\, dS_z \\ &\quad -\int_{\partial B(x,\delta)} \Delta f(z) \frac{\partial H}{\partial \nu}(z)\, dS_z + \int_{\partial B(x,\delta)} H(z) \frac{\partial \Delta f}{\partial \nu}(z)\, dS_z \\ &\quad -\int_{B(x,\delta)} H(z) \Delta^2 f(z)\, dz. \end{aligned}$$

Due to the boundary conditions in (32), we get

$$\begin{aligned} \int_{B(x,\delta)} f(z)\, dz \quad &= \quad -\int_{\partial B(x,\delta)} f(z) \frac{\partial \Delta H}{\partial \nu}(z)\, dS_z + \int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z) \Delta H(z)\, dS_z \\ &\quad -\int_{B(x,\delta)} H(z) \Delta^2 f(z)\, dz. \end{aligned} \tag{33}$$

Since $H \leq 0$ and $\Delta^2 f \leq 0$, it holds that

$$\int_{B(x,\delta)} f(z)\, dz \leq -\int_{\partial B(x,\delta)} f(z) \frac{\partial \Delta H}{\partial \nu}(z)\, dS_z + \int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z) \Delta H(z)\, dS_z. \tag{34}$$

On the other hand, by (31), we obtain

$$\Delta H(z) = \frac{\delta^2 - \|z-x\|^2}{2(N+2)} - \frac{\|z-x\|^2}{N(N+2)}, \quad z \in B(x,\delta)$$

and

$$\nabla \Delta H(z) = -\frac{z-x}{N+2} - \frac{2(z-x)}{N(N+2)}, \quad z \in B(x,\delta).$$

Thus, we get

$$\Delta H(z) = -\frac{\delta^2}{N(N+2)}, \quad z \in \partial B(x,\delta) \tag{35}$$

and

$$\begin{aligned}
\frac{\partial \Delta H}{\partial \nu}(z) &= \nabla \Delta H(z) \cdot \nu(z) \\
&= \nabla \Delta H(z) \cdot \frac{z-x}{\|z-x\|} \\
&= -\frac{\delta}{N+2} - \frac{2\delta}{N(N+2)} \\
&= -\frac{\delta}{N}, \quad z \in \partial B(x,\delta). \qquad (36)
\end{aligned}$$

Hence, in view of (34), (35) and (36), we obtain

$$\int_{B(x,\delta)} f(z)\,dz \le \frac{\delta}{N} \int_{\partial B(x,\delta)} \left(f(z) - \frac{\delta}{N+2}\frac{\partial f}{\partial \nu}(z) \right) dS_z.$$

Multiplying the above inequality by $\frac{1}{V_N(\delta)}$, (30) follows. This shows that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i): Assume now that for all $x \in \Omega$ and $\delta > 0$ with $\overline{B(x,\delta)} \subset \Omega$, (30) holds. Let $x \in \Omega$ be fixed, and $\delta > 0$ (small enough) so that $B(x,\delta) \subset \Omega$. Then, by (30) and the first part of the proof, (33) and (34) hold. Consequently, we get

$$\int_{B(x,\delta)} H(z)\Delta^2 f(z)\,dz \ge 0.$$

Since $H \le 0$, we deduce that there exists $z_\delta \in B(x,\delta)$ such that

$$\Delta^2 f(z_\delta) \le 0.$$

Since $f \in C^4(\Omega)$ (so $\Delta^2 f \in C(\Omega)$), passing to the limit as $\delta \to 0^+$ in the above inequality, we obtain

$$\Delta^2 f(x) \le 0.$$

This shows that (ii) $\Longrightarrow$ (i). The proof of Theorem 3.1 is then completed. $\square$

We now consider the set of functions $f \in C^4(\Omega)$ such that f is subharmonic and f is sub-biharmonic, that is,

$$\Delta f(z) \ge 0, \ \Delta^2 f(z) \le 0, \quad z \in \Omega. \qquad (37)$$

In this case, from Theorem 3.1, we deduce the following refinement of (5).

Corollary 3.2. *Let $f \in C^4(\Omega)$ verifies (37). Then, for all $x \in \Omega$ and $\delta > 0$ with $\overline{B(x,\delta)} \subset \Omega$, it holds that*

$$\begin{aligned}
\frac{1}{V_N(\delta)} \int_{B(x,\delta)} f(z)\,dz &\le \frac{1}{A_{N-1}(\delta)} \int_{\partial B(x,\delta)} \left(f(z) - \frac{\delta}{N+2}\frac{\partial f}{\partial \nu}(z) \right) dS_z \\
&\le \frac{1}{A_{N-1}(\delta)} \int_{\partial B(x,\delta)} f(z)\,dS_z. \qquad (38)
\end{aligned}$$

Proof. Let $f \in C^4(\Omega)$ verifies (37). Applying Green identity, we obtain

$$\begin{aligned}\int_{B(x,\delta)} \Delta f(z)\, dz &= \int_{B(x,\delta)} 1\, \Delta f(z)\, dz \\ &= -\int_{B(x,\delta)} \nabla 1 \cdot \nabla f(z)\, dz + \int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z)\, dS_z \\ &= \int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z)\, dS_z.\end{aligned}$$

Since $\Delta f \geq 0$, it holds that

$$\int_{\partial B(x,\delta)} \frac{\partial f}{\partial \nu}(z)\, dS_z \geq 0.$$

Thus, by Theorem 3.1, we obtain (38). □

Acknowledgement

The second author is supported by Researchers Supporting Project number (RSP2023R4), King Saud University, Riyadh, Saudi Arabia.

References

[1] S. Abramovich, L.E. Persson, Fejér and Hermite-Hadamard type inequalities for N-quasiconvex functions, Math Notes. 102 (2017) 599–609.

[2] B. Ahmad, A. Alsaedi, M. Kirane, B.T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals, J Comput Appl Math. 353 (2019) 120–129.

[3] J. Barić, L. Kvesić, J. Pecarić, M.R. Penava, Estimates on some quadrature rules via weighted Hermite-Hadamard inequality, Appl. Anal. Discret. Math. 16 (2022) 232–245.

[4] T. Beck, B. Brandolini, K. Burdzy, A. Henrot, J.J. Langford, S. Larson, R.G. Smits, S. Steinerberger, Improved bounds for Hermite-Hadamard inequalities in higher dimensions, J. Geom. Anal. 31 (2021) 801–816.

[5] E.F. Beckenbach, T. Radó, Subharmonic functions and surfaces of negative curvature, Trans. Am. Math. Soc. 35 (1933) 662–674

[6] L. Debnath, Nonlinear Partial Differential Equations for Scientists and Engineers, Birkhäuser Mass, USA, Boston (1997).

[7] J. de la Cal, J. Carcamo, Multidimensional Hermite-Hadamard inequalities and the convex order, J. Math. Anal. Appl. 324 (2006) 248–261.

[8] S.S. Dragomir, On Hadamard's inequality on a disk, J. Inequal. Pure Appl. Math. 1 (1) (2000), Article 2; http://jipam.vu.edu.au.

[9] S.S. Dragomir, On Hadamard's inequality for the convex mappings defined on a ball in the space and applications, Math. Ineq. & Appl. 3 (2) (2000) 177–187.

[10] S.S. Dragomir, C. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs (2000).

[11] S.S. Dragomir, B.T. Torebek, Some Hermite-Hadamard type inequalities in the class of hyperbolic p-convex functions, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 113(4) (2019) 3413–3423.

[12] A. Guessab, B. Semisalov, Optimal general Hermite-Hadamard-type inequalities in a ball and their applications in multidimensional numerical integration, Applied Numerical Mathematics. 170 (2021) 83–108.

[13] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl. 58 (1893) 171–215.

[14] H. Hedenmalm, S. Jakobsson, S. Shimorin, A maximum principle à la Hadamard for biharmonic operators with applications to the Bergman spaces, C. R. Acad. Sci. Paris Sér. I Math. 328 (11) (1999) 973–978.

[15] C. Hermite, Sur deux limites d'une intégrale défine, Mathesis. 3 (1883) 82.

[16] S. Larson, A sharp multidimensional Hermite-Hadamard inequality, Int. Math. Res. Not. IMRN. 2022 (2) (2022) 1297–312.

[17] J. Lu, S. Steinerberger, A dimension-free Hermite-Hadamard inequality via gradient estimates for the torsion function, Proc. Amer. Math. 148(2) (2020) 673–679.

[18] P.R. Mercer, Hadamard's inequality and trapezoid rules for the Riemann-Stieltjes integral, J. Math. Anal. Appl. 344 (2008) 921–926.

[19] M. Mihailescu, C. Niculescu, An extension of the Hermite-Hadamard inequality through subharmonic functions, Glasg. Math. J. 49 (3) (2007) 509–514.

[20] P.O. Mohammed, M.Z. Sarikaya, On generalized fractional integral inequalities for twice differentiable convex functions, J. Comput. Appl. Math. 372 (2020) Article Number 112740.

[21] C.P. Niculescu, L.-E. Persson, Old and new on the Hermite-Hadamard inequality, Real Anal. Exch. 29(2) (2003/04) 663–685.

[22] C.P. Niculescu, L.-E. Persson, Convex Functions and Their Applications. A Contemporary Approach, Springer Science+Business Media, Inc., 2006.

[23] T. Radó, Subharmonic functions. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5. Springer, Berlin (1937).

[24] M. Reade, Some remarks on subharmonic functions, Duke Math. J. 10 (1943) 531–536.

[25] A.W. Roberts, D.E. Varberg, Convex Functions, Academic Press (1973).

[26] S. Steinerberger, The Hermite-Hadamard inequality in higher dimension, Journal of Geometric Analysis, 30 (1) (2020) 466–483.