Monotone iterative technique for fractional measure differential equations in ordered Banach space *

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Abstract: In this paper, based on monotone iterative method in the presence of the lower and upper solutions, we investigate the existence and uniqueness of S-asymptotically ω-periodic mild solutions to a class of multi-term time-fractional measure differential equations with nonlocal conditions in an ordered Banach spaces. Firstly, we look for suitable the concept of S-asymptotically ω-periodic mild solution to our concern problem, by means of Laplace transform and (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$. Secondly, we construct monotone iterative method in the presence of the lower and upper solutions to the delayed fractional measure differential equations, and obtain the existence of maximal and minimal S-asymptotically ω-periodic mild solutions for the mentioned system under wide monotone conditions and noncompactness measure condition of nonlinear term. Finally, as the application of abstract results, an example is given to illustrate our main results.

Keywords: Regulated functions, Henstock-Lebesgue-Stieltjes integral, Measure differential equations, Monotone iterative technique

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1 Introduction

Fractional calculus has been used extensively in the study of linear and nonlinear fractional differential equations (FDEs) arising from real-world challenges. Many researchers in such areas make considerable use of FDEs to make some concerns more approachable, such as the modeling of nonlinear phenomena, optimal control of complex systems, and other scientific research (see, for example, [48, 47]). Furthermore, multi-term time-fractional differential systems can describe nonlinear phenomena seen in physics, mathematics, and engineering. These types of equations have sparked a lot of attentions in recent years, as demonstrated in [73, 69, 63] and the references included therein.

The theory of measure differential equations (MDEs, for short) covers some well known cases. When give an absolutely continuous function, a step function, or the sum of an absolutely continuous function with a step function, this kind of system corresponds to usual ordinary differential equations, difference equations or impulsive differential equations respectively. Another advantage of considering MDEs is that we can possibly model Zeno trajectories because gas a function of bounded variation may exhibit infinitely many discontinuities in a finite interval. This type of system arises in many areas of applied mathematics such as nonsmooth mechanics, game theory etc. see [7, 60, 74]. MDEs were investigated early by [71, 70, 62, 23]. One can refer to the review paper [21] for a complete introduction of measure differential systems. Recently, the theory of MDEs for $\mathbb{R}^n$ space has developed to some extent [31, 32, 59, 77].

On the other hand, it is well known that the periodic law of the development or movement of things is a common phenomenon in nature and human activities. However, in real life, many phenomena do not have strict periodicity. In order to better characterize these mathematical models, many scholars have introduced other definitions of generalized periodicity, such as almost periodicity, asymptotic periodicity, asymptotic almost periodicity, pseudo almost periodicity and S-asymptotic periodicity. Since S-asymptotically periodic functions were first studied in Banach space by Henríquez et al. [43], there are some papers about S-asymptotically periodic solutions for fractional evolution equations, one can refer to [14, 15, 64, 53, 67, 54]. It is worth noting that S-asymptotically period function, which was first proposed and established by Henríquez et al. [43], is a more general approximate period function between asymptotically periodic function and asymptotically almost periodic function.

The properties of periodic solutions to functional differential equations, integral equations and partial differential equations have been extensively studied. Specially, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems for fractional differential equations can not be extended periodically to time t in $\mathbb{R}^+$. Therefore, many scholars began to study various extended solutions of periodic solutions of fractional evolution equations(such as almost periodic solutions, asymptotically almost periodic solutions, pseudo almost periodic solutions, asymptotically periodic solutions, S-asymptotically periodic solutions and so on). For the related research on the S-asymptotically periodic solutions of fractional evolution equations, one can refer to [14, 15, 64, 53, 67, 54]. In [72], Shu et al. discussed the existence and uniqueness of positive S-asymptotically ω-periodic mild solutions for a class of semilinear neutral fractional evolution equations with delay by using the contraction mapping principle on positive cones. In [56], Li et al. discussed the positive S-asymptotically ω-periodic mild solutions for the abstract fractional evolution equation on infinite interval.

Due to the structures of such equations, investigating their solutions is challenging. To the best of the authors' knowledge, the existence of S-asymptotically ω-periodic mild solutions for abstract damped elastic systems with delay is a subject that has not been treated in the literature. This fact and the interesting relationship between S-asymptotically ω-periodic mild solutions and S-asymptotically ω-periodic functions are the main motivations of this work.

Based on previous work ideas and methods [31, 35, 37, 42, 25], in this work, we investigate the existence of S-asymptotically ω-periodic mild solution to multi-term fractional measure differential equations with nonlocal conditions and delay

$$\begin{cases} {}^cD_t^{1+\beta}u(t) + \sum\limits_{k=1}^{n} \alpha_k {}^cD_t^{\gamma_k}u(t) = Au(t) + F(t, u(t), u_t)dg(t), \quad t \geq 0, \\ u(t) = Q(u)(t) + \varphi(t), \quad t \in [-r, 0], \\ u'(0) = Q_0(u) + \psi, \end{cases} \tag{1.1}$$

where $u(\cdot)$ take values in a Banach space E; ${}^cD_t^{\eta}$ stand for the Caputo fractional derivative of order η, $\alpha_k > 0$ and all γ_k, $k = 1, 2, \cdots, n, n \in \mathbb{N}$, are positive real numbers such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. We assume that $A : \mathcal{D}(A) \subset E \to E$ is a κ-sectorial operator, and A generates a strongly continuous family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ of bounded and linear operators on E, $f : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is a suitable nonlinear function and $g : \mathbb{R}^+ \to \mathbb{R}$ is nondeacreasing and continuous from the left, dg denote the distributional derivative of g (see [77]), the functions $Q, Q_0 : G(\mathbb{R}^+, E) \to E$ will be specified later, where $G(\mathbb{R}^+, E)$ denotes the space of regulated functions on $\mathbb{R}^+$, $\mathcal{B} := G([-r, 0], E)$. For $t \geq 0$, $u_t \in \mathcal{B}$ is the history state defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$, $\varphi \in \mathcal{B}$ and $\varphi(0) \in \mathcal{D}(A)$, $\psi \in E$,

$r > 0$ is a constant.

The highlights and advantages of this paper are presented as follows:

(1) This paper is to construct the general principle for lower and upper solutions coupled with the monotone iterative technique for the delay evolution equation involving nonlocal in ordered Banach space, and obtain the existence of maximal and minimal S-asymptotically ω-periodic mild solutions, which will fill the research gap in this area.

(2) The main method used in this paper is the monotone iterative technique in the presence of the lower and upper solutions, which is an effective and widely used method to study the nonlinear differential equations as an application of the ordered fixed point theorem. This method can not only study the solvability of the equations, but also obtain the iterative sequence of the solutions, which provides a reasonable and effective theoretical basis for solving the approximate solutions by computer.

(3) Monotone iterative technique is utilized to derive the existence results of S-asymptotically ω-periodic mild solutions, which will fill the research gap in this area by using regulated functions, Henstock-Lebesgue-Stieltjes integral settings for measure driven equation involving multi-term time fractional derivatives.

(4) The topological method that some authors have chosen to study existence of S-asymptotically ω-periodic solutions is the theory of fixed points, which has been a very powerful and important tool to the study of nonlinear phenomena. Specifically, authors have used contraction mapping principle, Leray-Schauder alternative theorem, Schauder theorem and Krasnoselkii's theorem. However, monotone iterative method in the presence of the lower and upper solutions is the first time that it has been used to study our concerned problem in ordered Banach space. Therefore, our results are novel and meaningful.

This paper is organized as follows. The second part of the paper demonstrates preliminary details. The third part states the existence of S-asymptotically ω-periodic mild solution by monotone iterative method in the presence of the lower and upper solutions combined with (β, γ_k)-resolvent family. And the last section is provided an example to illustrate the applications of the obtained results. Concluding part close this article.

2 Preliminaries

In this section, we briefly recall some basic known results which will be used in the sequel. Throughout this paper, let $(E, \|\cdot\|)$ be an ordered Banach space with partial order "$\leq$" induced by the positive cone $K = \{u \in E | u \geq \theta\}$ (θ is the zero element of E), K is normal with normal constant N. Let $r > 0$ be constants, we denote by $C_b(\mathbb{R}^+, E)$ the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to E equipped with the norm

$$\|u\|_\infty = \sup_{t \in \mathbb{R}^+} \|u(t)\|$$

and $G(\mathbb{R}^+, E)$ denotes the Banach space of regulated functions on $\mathbb{R}^+$ equipped with a norm $\|u\|_\infty = \sup\limits_{t \in \mathbb{R}^+} \|u(t)\|$, $\mathcal{B} := G([-r, 0], E)$ the Banach space of regulated functions from $[-r, 0]$ to E with the norm

$$\|\phi\|_{\mathcal{B}} = \sup_{s \in [-r,0]} \|\phi(s)\|.$$

Let $SAP_\omega(E)$ represent the subspace of $C_b(\mathbb{R}^+, E)$ consisting all the E-value S-asymptotically ω-periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|$. Then $SAP_\omega(E)$ is a Banach space (see [43], Proposition 3.5]). If $u \in SAP_\omega(E)$, then it is not difficult to test and verify that the function $t \to u_t$ belongs to $SAP_\omega(\mathcal{B})$ (see [53, 54]).

For the rest of this paper, we define

$$\Omega := \{u \in G([-r, \infty), E) \cap C_b([-r, \infty), E)|\ u|_{[-r,0]} \in \mathcal{B} \text{ and } u|_{\mathbb{R}^+} \in SAP_\omega(E)\}.$$

It is easy to see that Ω is a Banach space equipped with the norm

$$\|u\|_\Omega = \sup_{t \in [-r, \infty)} \|u(t)\|.$$

Define a positive cone K_Ω by

$$K_\Omega = \{u \in \Omega | u(t) \in K, \quad t \in [-r, \infty)\},$$

then Ω is an ordered Banach space with the partial order relation "$\leq$" induced by the cone K_Ω, and K_Ω is normal with the normal constant N. Similarly, $\mathcal{B}$ is also an order Banach space whose partial ordering "$\leq$" induced by a positive cone

$$K_{\mathcal{B}} = \{\phi \in \mathcal{B} | \phi(s) \in K, s \in [-r, 0]\}$$

with the normal constant N. For $v, w \in \Omega$ with $v \leq w$, we denote the order interval $\{u | v \leq u \leq w\} \subset \Omega$ by $[v, w]$. Furthermore, we denote $\{u(t) | v(t) \leq u(t) \leq w(t), t \in [-r, \infty)\}$ in E by $[v(t), w(t)]$ and $\{u_t | v_t \leq u_t \leq w_t, t \in [0, \infty)\}$ in $\mathcal{B}$ by $[v_t, w_t]$, respectively.

A partition of $[a,b]$ is a finite collection of pairs $\{([t_{i-1},t_i],e_i), i=1,2,\cdots,n\}$, where $[t_{i-1},t_i]$ are nonoverlapping subintervals of $[a,b]$, $e_i \in [t_{i-1},t_i], i=1,\cdots,n$ and $\bigcup_{i=1}^{n}[t_{i-1},t_i]=[a,b]$. A gauge δ on $[a,b]$ is a positive function on $[a,b]$. For a given guage δ we say that a partition is δ-fine if $[t_{i-1},t_i]\subset(e_i-\delta(e_i),e_i+\delta(e_i))$, $i\in\{1,\cdots,n\}$. Let $u(t^-)$ and $u(t^+)$ denote the left limit and right limit of the function u at the point t, respectively.

We recall some basic definitions and properties of regulated function. For more details, we refer to [68, 25]

Definition 2.1. *[68] A function $u:[a,b]\to E$ is said to be regulated on $[a,b]$, if the limits*

$$\lim_{s\to t^-} u(s)=u(t^-),\ t\in(a,b] \quad \textit{and} \quad \lim_{s\to t^+} u(s)=u(t^+),\ t\in[a,b)$$

exist and are finite.

We denote by $G([a,b],E)$ the space of all regulated function from $[a,b]$ into E. It is well known that the space $G([a,b],E)$ is a Banach space endowed with the supremum norm.

Definition 2.2. *[68] A set $B\subset G([a,b],E)$ is called equiregulated, if for every $\epsilon>0$ and $\tau\in[a,b]$, there exists $\delta>0$ such that*

(i) If $u\in B, t\in[a,b]$ and $t\in(\tau-\delta,\tau)$, then $\|u(\tau^-)-u(t)\|_E<\epsilon$.

(ii) If $u\in B, t\in[a,b]$ and $t\in(\tau,\tau+\delta)$, then $\|u(t)-u(\tau^+)\|_E<\epsilon$.

Lemma 2.1. *[68] Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions from $[a,b]$ to E. If u_n converge pointwisely to u_0 as $n\to\infty$ and the sequence $\{u_n\}_{n=1}^{\infty}$ is equiregulated, then u_n converges uniformly to u_0.*

Lemma 2.2. *[12, 68] Let $B\subset G([a,b],E)$. If B is bounded and equiregulated, then the set $\overline{co}(B)$ is also bounded and equiregulated, where $\overline{co}(B)$ define the closed convex hull of B.*

The following result is a variant of the Arzelà-Ascoli theorem for regulated functions with Banach space values.

Lemma 2.3. *[57] Assume that $B\subset G([a,b],E)$ is equiregulated and, for every $t\in[a,b]$ the set $\{u(t):u\in B\}$ is relatively compact in E. Then the set B is relatively compact in $G([a,b],E)$.*

Next, we recall the definition of Henstock-Lebesgue-Stieltjes integral.

Definition 2.3. *[68] A function $\psi : [0,b] \to E$ is said to be Henstock-Lebesgue-Stieltjes integrable w.r.t. $g : [0,b] \to \mathbb{R}$, if there exists a function denoted by $(HLS)\int_a^{\cdot} : [0,b] \to E$ such that, for every $\epsilon > 0$, there is a gauge δ_ϵ on $[0,b]$ with*

$$\Big\| \sum_{i=1}^{n} \psi(e_i)(g(t_i) - g(t_{i-1})) - \Big((HLS) \int_0^{t_i} \psi(s)dg(s) - (HLS) \int_0^{t_{i-1}} \psi(s)dg(s) \Big) \Big\| < \epsilon,$$

for every δ_ϵ-fine partition $\{(e_i, [t_{i-1}, t_i]) : i = 1, 2, \ldots, n\}$ of $[0,b]$.

We denote by $\mathbb{HLS}_g^p([a,b],\mathbb{R})(p > 1)$ the space of all p-ordered Henstock-Lebesgue-Stieltjes integral regulated from $[a,b]$ to $\mathbb{R}$ with respect to g, with norm $\|\cdot\|_{\mathbb{HLS}_g^p}$ defined by

$$\|\psi\|_{\mathbb{HLS}_g^p} = \Big((HLS) \int_a^b \|\psi(s)\|^p dg(s) \Big)^{\frac{1}{p}}.$$

Lemma 2.4. *[35] Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\Psi \in \mathbb{HLS}_g^p([a,b],\mathbb{R}^+)$ and $g : [a,b] \to \mathbb{R}$ be regulated. Then the function $H(t) = \int_0^t (t-s)^\beta \Psi(s)dg(s)$ is regulated and*

$$H(t) - H(t^-) \leq \Big(\int_{t^-}^{t} (t-s)^{q\beta} dg(s) \Big)^{\frac{1}{q}} \Psi(t)(\Delta^- g(t))^{\frac{1}{p}}, \quad t \in (a,b],$$

$$H(t^+) - H(t) \leq \Big(\int_{t^+}^{t} (t^+ - s)^{q\beta} dg(s) \Big)^{\frac{1}{q}} \Psi(t)(\Delta^+ g(t))^{\frac{1}{p}}, \quad t \in [a,b),$$

where $\Delta^+ g(t) = g(t^+) - g(t)$ and $\Delta^- g(t) = g(t) - g(t^-)$.

Let $L_\mu^1([a,b],E)$ be the set of all μ-integrable functions, where μ is a measure.

Lemma 2.5. *[26] Let for $t \in [a,b]$, $Z(t)$ be weakly relatively compact in E. Suppose that $B \subset L_\mu^1([a,b],E)$ is a bounded set and there is a function $N(\cdot) \in L_\mu^1([a,b],\mathbb{R}^+)$ such that $\|b(t)\|_E \leq N(t)$ μ-a.e. $t \in [a,b]$ for all $b \in B$. If for every $b \in B, b(t) \in Z(t)$ for μ-a.e. $t \in [a,b]$, then B is weakly relatively compact in $L_\mu^1([a,b],E)$.*

Definition 2.4. [66] *An (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ on E is said to be positive, if $S_{\beta,\gamma_k}(t)x \geq \theta$ for each $x \geq \theta, x \in E$, and $t \geq 0$.*

Definition 2.5. [66] *An (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ on E is said to be equicontinuous, if the function $t \to S_{\beta,\gamma_k}(t)$ is continuous from $(0,\infty) \to \mathcal{L}(E)$ on the operator norm $\|\cdot\|_{\mathcal{L}(E)}$.*

In the following, we present the theory of fractional calculus about the Riemann-Liouville integral and Caputo fractional derivative.

Definition 2.6. *The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0,\infty), E)$ of order $\eta > 0$ with lower limit zero is defined as follows*

$$\mathbb{I}^\eta f(t) = \int_0^t \frac{(t-s)^{\eta-1}}{\Gamma(\eta)} f(s)ds, \quad t > 0$$

and $\mathbb{I}^0 f(t) = f(t)$, provided that side integral is point-wise defined in $[0,\infty)$.

Definition 2.7. *Let $\eta > 0$ be given and denote $m = [\eta]$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in C^m([0,\infty), E)$ with lower limit zero is given by*

$${}^cD^\eta f(t) = \mathbb{I}^{m-\eta} D^m f(t) = \int_0^t \frac{(t-s)^{m-\eta-1}}{\Gamma(m-\eta)} D^m f(s)ds,$$

and ${}^cD^0 f(t) = f(t)$, where $D^m = d^m/dt^m$ and $[\cdot]$ is ceiling function.

For more progress and important properties about fractional calculus and its applications, we refer the reader to [48, 65] and references therein.

Let A be a closed linear operator on the Banach space E with domain $\mathcal{D}(A)$ and denote by $\rho(A)$ the resolvent set of A.

Definition 2.8. [47]. *Let E be a Banach space and let $\beta > 0$, $\gamma_k, \alpha_k, k = 1, 2, \ldots n$ be real positive numbers. Then A is called the generator of (β, γ_k)-resolvent family if there exists $\kappa \geq 0$ and a strongly continuous function $S_{\beta,\gamma_k} : \mathbb{R}^+ \to \mathcal{L}(E)$ such that*

$$\Big\{\lambda^{\beta+1} + \sum_{k=1}^n \alpha_k \lambda^{\gamma_k} : Re(\lambda) > \kappa\Big\} \subset \rho(A)$$

and

$$\lambda^\beta \Big(\lambda^{\beta+1} + \sum_{k=1}^n \alpha_k \lambda^{\gamma_k} - A\Big)^{-1} u = \int_0^\infty S_{\beta,\gamma_k}(t) u dt,$$

where $Re(\lambda) > \kappa$ and $u \in E$.

A operator A is said to be κ-sectorial of angle θ if there exist $\theta \in [0, \frac{\pi}{2})$ and $\kappa \in \mathbb{R}$ such that its resolvent is in the sector

$$\kappa + S_\theta := \Big\{\kappa + \lambda : \lambda \in \mathbb{C}, |arg(\lambda)| < \frac{\pi}{2} + \theta\Big\} \backslash \{\omega\},$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in \omega + S_\theta.$$

Lemma 2.6. *[47] Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given, $\mu > 0$ and $\kappa < 0$. Assume that A is a κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}$. Then A generates a (β, γ_k)-resolvent family $S_{\beta,\gamma_k}(t)$ satisfying the estimate*

$$\|S_{\beta,\gamma_k}(t)\| \leq \frac{C}{1 + |\kappa|(t^{\beta+1} + \sum_{k=1}^n \alpha_k t^{\gamma_k})}, \quad t \geq 0, \tag{2.1}$$

for some constant $C > 0$ depending only on β, γ_k.

Definition 2.9. [43]. *A function $u \in C_b(\mathbb{R}^+, E)$ is called S-asymptotically ω-periodic if there exists ω such that*

$$\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0, \quad \forall t \geq 0.$$

In this case, we say that ω is an asymptotic of u. It is clear that if ω is an asymptotic period for u, then every $k\omega, k = 1, 2, \cdots$, is also an asymptotic period of u.

We now look for suitable the concept of mild solution to problem (1.1).

First of all, we consider the following linear multi-term time-fractional measure differential equations with nonlocal conditions of the form

$$\begin{cases} {}^cD_t^{1+\beta}u(t) + \sum\limits_{k=1}^{n} \alpha_k {}^cD_t^{\gamma_k}u(t) = Au(t) + h(t)dg(t), \quad t \geq 0, \\ u(t) = Q(u)(t) + \varphi(t), \qquad t \in [-r, 0], \\ u'(0) = Q_0(u) + \psi, \end{cases} \tag{2.2}$$

where $h \in G(\mathbb{R}^+, E)$.

Lemma 2.7. *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given and A be a generator of a bounded (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$. Then mild solution of the problem (2.2) is given by*

$$\begin{aligned} u(t) = {} & S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ & + \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0) + \varphi(0)] \\ & + \int_0^t T_{\beta,\gamma_k}(t-s)h(s)dg(s) \end{aligned}$$

for $t \geq 0$, where $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$.

Proof. Let $\mathcal{L}(\cdot)$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively. With the goal of constructing a representation of the solution of (2.2) in terms of the family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, and applying the Laplace transform to (2.2), we have

$$\begin{aligned} \lambda^{\beta+1}\mathcal{L}[u](\lambda) - & \sum_{j=0}^{[\beta+1]-1} u^{(j)}(0)\lambda^{\beta-j} \\ & + \sum_{k=1}^{n} \alpha_k \Big[\lambda^{\gamma_k}\mathcal{L}[u](\lambda) - \sum_{j=0}^{[\gamma_k]-1} u^{(j)}(0)\lambda^{\gamma_k-1-j}\Big] \\ & = A\mathcal{L}[u](\lambda) + \mathcal{L}[h](\lambda). \end{aligned}$$

Applying the given nonlocal conditions, we have

$$\begin{aligned}\lambda^{\beta+1}\mathcal{L}[u](\lambda) - \lambda^{\beta}[Q(u)(0)+\varphi(0)] - \lambda^{\beta-1}[\psi + Q_0(u)]\\
+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k}\mathcal{L}[u](\lambda) - \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}[Q(u)(0)+\varphi(0)]\\
= A\mathcal{L}[u](\lambda) + \mathcal{L}[h](\lambda).\end{aligned}$$

This is equivalent to

$$\begin{aligned}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)\mathcal{L}[u](\lambda) = \lambda^{\beta}[Q(u)(0)+\varphi(0)] + \lambda^{\beta-1}[\psi + Q_0(u)]\\
+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}[Q(u)(0)+\varphi(0)] + \mathcal{L}[h](\lambda), \quad Re(\lambda) > \omega.\end{aligned}$$

Hence, assuming the existence of the family $S_{\beta,\gamma_k}(t)$ we obtain

$$\begin{aligned}\mathcal{L}[u](\lambda) = \lambda^{\beta}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}[Q(u)(0)+\varphi(0)]\\
+\lambda^{\beta-1}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}[\psi + Q_0(u)]\\
+\sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k-1}\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}[Q(u)(0)+\varphi(0)]\\
+\Big(\lambda^{\beta+1} + \sum_{k=1}^{n}\alpha_k\lambda^{\gamma_k} - A\Big)^{-1}\mathcal{L}[h](\lambda)\end{aligned}$$

for all λ such that $Re(\lambda) > \omega$, $\lambda^{\beta+1} + \sum\limits_{k=1}^{n}\alpha_k\lambda^{\gamma_k} \in \rho(A)$, then

$$\begin{aligned}\mathcal{L}[u](\lambda) = \mathcal{L}[S_{\beta,\gamma_k}](\lambda)[Q(u)(0)+\varphi(0)] + \mathcal{L}[\varphi_1]\mathcal{L}[S_{\beta,\gamma_k}](\lambda)[\psi + Q_0(u)]\\
+\sum_{k=1}^{n}\alpha_k\mathcal{L}[\varphi_{1+\beta-\gamma_k}]\mathcal{L}[S_{\beta,\gamma_k}](\lambda)[Q(u)(0)+\varphi(0)]\\
+\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\mathcal{L}[\varphi_\beta](\lambda)\mathcal{L}[h](\lambda), \quad Re(\lambda) > \omega,\end{aligned}$$

where

$$\varphi_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0, \beta > 0.$$

Inversion of the Laplace transform shows that

$$\begin{aligned}u(t) &= S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\\ &+ \sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)]\\ &+ \int_0^t T_{\beta,\gamma_k}(t-s)h(s)dg(s).\end{aligned}$$

This completed the proof. ∎

The above representation formula allows us to give the following definition.

Definition 2.10 *Let $0<\beta\le\gamma_n\le\cdots\le\gamma_1\le 1$ and $\alpha_k\ge 0, k=1,2,\dots n$ be given and A be a generator of a bounded (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge0}$. Then a regulated function $u(\cdot):\mathbb{R}^+\to E$ is said to be mild solution of problem (1.1) if $u(t)=Q(u)(t)+\varphi(t), u'(0)=Q_0(u)+\psi$ and satifies the following integral equation*

$$\begin{aligned}u(t) &= S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\\ &+ \sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)]\\ &+ \int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s), \quad t\ge 0.\end{aligned}$$

Moreover, if u is S-asymptotically ω-periodic, then it is called S-asymptotically ω-periodic mild solution of problem (1.1).

Moreover, we noted that by the estimate (2.1), we have

$$\begin{aligned}\|T_{\beta,\gamma_k}(t)\|_{\mathcal{L}(E)} = \|(\varphi_\beta * S_{\beta,\gamma_k})(t)\|_{\mathcal{L}(E)} &\le \int_0^t \varphi_\beta(t-\tau)\|S_{\beta,\gamma_k}(\tau)\|d\tau\\ &\le \Gamma(1-\gamma_k)\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)\tau^{\gamma_k}\|S_{\beta,\gamma_k}(\tau)\|d\tau\\ &\le \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n\alpha_k}\int_0^t \varphi_\beta(t-\tau)\varphi_{1-\gamma_k}(\tau)d\tau\\ &= \frac{\Gamma(1-\gamma_k)}{|\kappa|\sum_{k=1}^n\alpha_k}\varphi_{\beta-\gamma_k+1}(t),\end{aligned}$$

for all $t>0$. Hence there exists a constant $C>0$ such that

$$\|T_{\beta,\gamma_k}(t)\|_{\mathcal{L}(E)} \le Ct^{\beta-\gamma_k}. \tag{2.3}$$

First of all, in view of (2.1), we denote $M:=\sup_{t\ge0}\|S_{\beta,\gamma_k}(t)\|<+\infty, M>0$. We estimate

$\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\|$ as follows. Let $0 < \varepsilon < \gamma_k - \beta$ be given, then

$$\begin{aligned}
&\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| \\
&= \|\Gamma(\gamma_k - \beta - \varepsilon) \int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1} S_{\beta,\gamma_k}(\tau)d\tau\| \\
&\leq \Gamma(\gamma_k - \beta - \varepsilon) \int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)\tau^{\beta-\gamma_k+\varepsilon+1} \|S_{\beta,\gamma_k}(\tau)\|d\tau,
\end{aligned}$$

where, thanks to (2.1), we have that

$$\begin{aligned}
\Gamma(\gamma_k - \beta - \varepsilon)\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| &\leq \frac{M\tau^{\beta-\gamma_k+\varepsilon-1}}{1+|\kappa|\tau^{\beta+1}} \\
&= \frac{M\tau^{-\gamma_k+\varepsilon}}{\frac{1}{\tau^{\beta+1}}+|\kappa|}, \qquad \varepsilon > 0.
\end{aligned}$$

Since $\varepsilon < \gamma_k$, there exists a constant $C > 0$ such that

$$\tau^{\beta-\gamma_k+\varepsilon+1}\|S_{\beta,\gamma_k}(\tau)\| \leq C.$$

Therefore,

$$\begin{aligned}
\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| &\leq C \int_0^t \varphi_{1+\beta-\gamma_k}(t-\tau)\varphi_{\gamma_k-\beta-\varepsilon}(\tau)d\tau \\
&= C\varphi_{1-\varepsilon}(t) = Ct^{-\varepsilon}. \qquad (2.4)
\end{aligned}$$

And then, in view of (2.1), we denote $M := \sup_{t\geq 0} \|S_{\beta,\gamma_k}(t)\| < +\infty, M > 0$. We note that

$$\int_0^\infty \frac{1}{1+|\kappa|t^{\beta+1}}dt = \frac{|\kappa|^{-\frac{1}{\beta+1}}\pi}{(\beta+1)\sin(\frac{\pi}{\beta+1})}$$

for $1 < \beta + 1 < 2$ and therefore $S_{\beta,\gamma_k}(t)$ is, in fact, integrable. Hence, we have

$$\begin{aligned}
\|(\varphi_1 * S_{\beta,\gamma_k})(t)\| = \left\|\int_0^t S_{\beta,\gamma_k}(s)ds\right\| &\leq \int_0^t \|S_{\beta,\gamma_k}(s)\|ds \\
&\leq \int_0^t \frac{C}{1+|\kappa|(s^{\beta+1}+\sum_{k=1}^n \alpha_k s^{\gamma_k})}ds \\
&< C\int_0^\infty \frac{1}{1+|\kappa|s^{\beta+1}}ds = \frac{C|\kappa|^{-\frac{1}{\beta+1}}\pi}{(\beta+1)\sin(\frac{\pi}{\beta+1})}.
\end{aligned}$$

Moreover, we denote

$$\widetilde{M} := \sup_{t\geq 0}\|(\varphi_1 * S_{\beta,\gamma_k})(t)\| = \frac{C|\kappa|^{-\frac{1}{\beta+1}}\pi}{(\beta+1)\sin(\frac{\pi}{\beta+1})}. \qquad (2.5)$$

In addition, we present the definitions of lower and upper solutions for the nonlocal problem (1.1).

Definition 2.11. *If a function $v \in \Omega$ with $v|_{\mathbb{R}^+} \in C(\mathbb{R}^+, E) \cap C^{1+\beta}(\mathbb{R}^+, E)$ satisfies $Av(0) \leq A[Q(u)(0) + \varphi(0)]$ and*

$$\begin{cases} {}^cD_t^{1+\beta} v(t) + \sum\limits_{k=1}^{n} \alpha_k {}^cD_t^{\gamma_k} v(t) \leq Av(t) + F(t, v(t), v_t)dg(t), & t \geq 0, \\ v(t) \leq Q(v)(t) + \varphi(t), & t \in [-r, 0], \\ v'(0) \leq Q_0(v) + \psi, \end{cases}$$

then $v(t)$ is named a lower solution of nonlocal problem (1.1). And if the inequalities in above are all reversed, then $v(t)$ is named an upper solution of nonlocal problem (1.1).

We give the definition of upper and lower mild solution.

Definition 2.12. *If a function $v \in \Omega$ satisfies*

$$\begin{aligned} v(t) \leq\ & S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(v)] \\ & + \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0) + \varphi(0)] \\ & + \int_0^t T_{\beta,\gamma_k}(t-s) F(s, v(s), v_s) dg(s), \end{aligned}$$

then $v(t)$ is named a lower mild solution of nonlocal problem (1.1). And if the inequality in above is reversed, then $v(t)$ is named an upper mild solution of nonlocal problem (1.1).

Next, we recall some properties of noncompactness which will be used in the sequel. The Hausdorff measure of noncompactness of a bounded subset S of E is defined to be the infimum of the set of all real numbers $\epsilon > 0$ such that S can be covered by a finite number of balls radius smaller than ϵ, that is,

$$\alpha(S) = \inf\{\epsilon > 0 : S \subset \cup_{i=1}^{n} B(\xi_i, r_i), \xi_i \in E, r_i < \epsilon (i = 1, \dots, n), n \in \mathbb{N}\},$$

where $B(\xi_i, r_i)$ denotes the open ball centered at ξ_i and of radius r_i.

Now, we give the following useful lemmas.

Lemma 2.8. [8, 49] *Let S, T be bounded subsets of E and $\lambda \in \mathbb{R}$. Then*

(1) $\alpha(S) = 0$ if and only if S is relatively compact;

(2) $S \subseteq T$ implies $\alpha(S) \leq \alpha(T)$;

(3) $\alpha(\overline{S}) = \alpha(S)$;

(4) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;

(5) $\alpha(\lambda S) = |\lambda|\alpha(S)$, *where* $\lambda S = \{x = \lambda z : z \in S\}$;

(6) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$, *where* $S + T = \{x = y + z : y \in S, z \in Z\}$;

(7) $\alpha(co(S)) = \alpha(S)$;

(8) *If the map* $Q : D(Q) \subseteq E \to Z$ *is Lipschitz continuous with a constant* k*, then* $\alpha_z(Q\Omega) \leq k\alpha(\Omega)$ *for any bounded subset* $\Omega \subseteq D(\Omega)$*, where* Z *is a Banach space.*

Let W be a subset of $G([a, b], E)$. For each fixed $t \in [a, b]$, we denote $W(t) = \{x(t) : x \in W\}$. Next we will present some results of the Hausdorff measure of noncompactness in the space of regulated functions $G([a, b], E)$.

Lemma 2.9. [12] *Let* $W \subset G([a, b], E)$ *be bounded and equiregulated on* $[a, b]$*. Then* $\alpha(W(t))$ *is regulated on* $[a, b]$*.*

Lemma 2.10. [12] *Let* $W \subset G([a, b], E)$ *be bounded and equiregulated on* $[a, b]$*. Then* $\alpha(W) = \sup\{\alpha(W(t)) : t \in [a, b]\}$*.*

Lemma 2.11. [22] *Let* E *be a Banach space and* $B \subset E$ *be bounded. Then there exists a countable subset* $B_0 \subset B$*, such that* $\alpha(B) \leq \alpha(B_0)$*.*

Denote by $\mathcal{LS}_g([a, b], E)$ the space of all functions $f : [a, b] \to E$ that are Lebesgue-Stieltjes integrable with respect to g. Let μ_g be the Lebesgue-stieltjes measure on $[a, b]$ induced by g. Since the Lebesgue-stieltjes measure is a regular Borel measure, then the following results holds by Theorem 3.1 in [46].

Lemma 2.12. [46] *Let* $W_0 \subset \mathcal{LS}_g([a, b], E)$ *be a countable set. Assume that there exists a positive function* $k \in \mathcal{LS}_g([a, b], \mathbb{R}^+)$ *such that* $\|w(t)\| \leq k(t)$ μ_g*-a.e. holds for all* $w \in W_0$*. Then we have*

$$\alpha\Big(\int_a^b W_0(t)dg(t)\Big) \leq 2\int_a^b \alpha(W_0(t))dg(t).$$

We will make use of a version of Bellman integral inequality for Henstock-Lebesgue-Stieltjes integrals, which can be found in [50](Theorm 4.4).

Lemma 2.13. [50] Let $T > 0$. Assume that $a, m \in G([0, T], \mathbb{R}^+)$. If the function $y \in G([0, T], \mathbb{R}^+)$ satisfies the inequality

$$y(t) \leq m(t) + \int_0^t a(s)y(s)dg(s)$$

for every $t \in [0, T]$, then

$$y(t) \le m(t) + \int_0^t a(s)m(s)e^{\int_s^t a(\tau)dg(\tau)}dg(s).$$

3 Main Result

In this section, we discuss the existence of S-asymptotically ω-periodic mild solutions and for the system (1.1).

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal. Let $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\alpha_k \ge 0, k = 1, 2, \dots n$ be given. A is an κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}, k = 1, 2, \cdots, n$ with $\kappa < 0$, and A generates a positive and compact (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$ on E. Assume that $\omega > 0$ is a constant and the nonlocal problem (1.1) has a lower mild solution $v^{(0)}$ and an upper mild solution $w^{(0)}$ with $v^{(0)} \le w^{(0)}$. If $\varphi \in K_{\mathcal{B}}$, $Q(u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is continuous as well as the following conditions are established:*

$(H1)$ For each constant $R > 0$, there exists $P(\cdot) \in \mathbb{HLS}_g^p(\mathbb{R}^+, \mathbb{R}^+)$ for some $p > 1$ such that

$$\sup_{\|u\| \le R} \|F(t, u(t), u_t)\| \le P(t)W(R), \ t \ge 0,$$

where $W : [0, +\infty) \to \mathbb{R}^+$ is a continuous nondecreasing function and

$$\lim_{R\to+\infty} \inf \frac{W(R)}{R} = w_0 < +\infty.$$

$(H2)$ (1) There exists $\omega > 0$ such that for all $x \in E, \phi \in \mathcal{B}$

$$\lim_{t\to\infty} \|F(t + \omega, x, \phi) - F(t, x, \phi)\| = 0.$$

(2) $F(t, u, u_t)$ is measurable for all $u \in G(\mathbb{R}^+, E)$.

$(H3)$ For any $t \in \mathbb{R}^+$, $x_1, x_2 \in E$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $v^{(0)}(t) \le x_1 \le x_2 \le w^{(0)}(t)$ and $v_t^{(0)} \le \phi_1 \le \phi_2 \le w_t^{(0)}$,

$$F(t, x_2, \phi_2) - F(t, x_1, \phi_1) \ge \theta;$$

$(H4)$ (1) The nonlocal functions $Q(u), Q_0(u)$ is increasing in order interval $[v^{(0)}, w^{(0)}]$;

(2) $Q, Q_0 : G([-r, +\infty), E) \to E$ are continuous and compact mapping, and there are two positive constants c_0, c_1, d_0, d_1 such that

$$\|Q_0(u)\| \le c_0\|u\| + d_0, \quad \|Q(u)\| \le c_1\|u\| + d_1.$$

Then the nonlocal problem (1.1) has minimal and maximal S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedures starting from $v^{(0)}$ and $w^{(0)}$, respectively.

Proof. For each $u \in [v^{(0)}, w^{(0)}]$, we have $u_t \in [v^{(0)}_t, w^{(0)}_t] = [v^{(0)}(t+s), w^{(0)}(t+s)] \subset SAP_\omega(\mathcal{B})$ for $t \in \mathbb{R}^+, s \in [-r, 0]$. Now, we define an operator $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to G([-r, +\infty), E)$ by

$$(\mathcal{Q}u)(t) = \begin{cases} S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ + \sum_{k=1}^n \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0) + \varphi(0)] \\ + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s), \qquad t \geq 0, \\ \\ Q(u)(t) + \varphi(t), \qquad t \in [-r, 0]. \end{cases} \tag{3.1}$$

From (H2)(2), the integral $\int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s)$ is well defined. Clearly, if $\mathcal{Q}$ admit a fixed point in $G([-r, +\infty), E)$, then the system (1.1) admits a mild solution.

Now, we complete the proof by six steps.

Step.1. The set $\{\mathcal{Q}u : u(\cdot) \in \Omega\}$ is equiregulated.

For any $b \in (0, \infty)$, restrict $u(t)$ to interval $[-r, b)$. For any $t_0 \in [-r, b)$, we have

$$\begin{aligned} \|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\| &\leq \|(S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+))[Q(u)(0) + \varphi(0)]\| \\ &+ \|[(\varphi_1 * S_{\beta,\gamma_k})(t) - (\varphi_1 * S_{\beta,\gamma_k})(t_0^+)][\psi + Q_0(u)]\| \\ &+ \sum_{k=1}^n \frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)} \Big| \int_0^t (t-s)^{\beta-\gamma_k} ds - \int_0^{t_0^+} (t_0^+ - s)^{\beta-\gamma_k} ds \Big| \|[Q(u)(0) + \varphi(0)]\| \\ &+ \int_0^{t_0^+} \|[T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)]F(s, u(s), u_s)\|dg(s) \\ &+ \int_{t_0^+}^t \|T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)\|dg(s) \\ &\leq \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(\mathbb{E})} \cdot \|\varphi(0)\| + M|t - t_0^+| \cdot \|[\psi + Q_0(u)]\| \\ &+ \sum_{k=1}^n \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \|Q(u)(0) + \varphi(0)\| \\ &+ W(R) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{\mathcal{L}(E)} P(s)dg(s) \\ &+ CW(R) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} P(s)dg(s) \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t), \end{aligned} \tag{3.2}$$

where

$$I_1(t) = \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(E)} \cdot \|Q(u)(0) + \varphi(0)\|,$$

$$I_2(t) = M|t - t_0^+| \cdot \|\psi + Q_0(u)\|,$$

$$I_3(t) = \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \cdot \|Q(u)(0) + \varphi(0)\|,$$

$$I_4(t) = W(r) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} P(s) dg(s),$$

$$I_5(t) = CW(r) \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s).$$

From the expression of $I_2(t)$ and $I_3(t)$, we derive that $I_2(t) \to 0$ and $I_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Since the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$ yields the continuity in the sense of uniform operator topology. We dedude that $I_1(t) \to 0$ and applying dominated convergence theorem on $I_4(t)$ and , we can derive that $I_4(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H(t) = \int_0^t (t-s)^{\beta-\gamma_k} P(s) dg(s)$. Thanks to Lemma 2.4, we known that $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore, we have

$$\begin{aligned}
I_5(t) &= CW(r) \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s) \\
&\leq CW(r) \Big(\|H(t) - H(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} - (t_0^+ - s)^{\beta-\gamma_k}) P(s)\| dg(s) \Big) \\
&\to 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u.
\end{aligned}$$

Therefore, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$.

Similarly, one can demonstrate that for any $t_0 \in (-r, b]$, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. According to the arbitrariness of b, one can find that $u(t)$ is defined on $[-r, +\infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Hence, assert that $\{\mathcal{Q}u : u(\cdot) \in \Omega\}$ is equiregulated.

Step.2. $\mathcal{Q} : \Omega \to \Omega$ is continuous operator. Let $\{u^{(n)}\} \subset \Omega$ be a sequence such that $u^{(n)} \to u(t)$ in Ω as $n \to \infty$, then, $u^{(n)}(t) \to u(t)$ in E and $u_t^{(n)} \to u_t$ in $\mathcal{B}$ for every $t \geq 0$ as $n \to \infty$. For $t \in \mathbb{R}^+$, by the continuity of F and Q, Q_0, one can see that when $n \to \infty$,

$$F(t, u^{(n)}(t), u_t^{(n)}) \to F(t, u(t), u_t), \quad Q_0(u^{(n)}) \to Q_0(u), \quad Q(u^{(n)}) \to Q(u).$$

and

$$\|F(t, u^{(n)}(t), u_t^{(n)}) - F(t, u(t), u_t)\| \leq 2P(t)W(r) \tag{3.3}$$

and moreover for each $t \geq 0$, we have

$$\begin{aligned}
\|\mathcal{Q}(u^n)(t) - \mathcal{Q}(u)(t)\| \leq & S_{\beta,\gamma_k}(t)\|Q(u^{(n)}) - Q(u)\| \\
& + (\varphi_1 * S_{\beta,\gamma_k})(t)\|Q_0(u^{(n)}) - Q_0(u)\| \\
& + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)\|Q(u^{(n)}) - Q(u)\| \\
& + C\int_0^t (t-s)^{\beta-\gamma_k}\|F(s, u^{(n)}(s), u_s^{(n)}) - F(s, u(s), u_s)\|dg(s). \qquad (3.4)
\end{aligned}$$

By the inequalities (3.3)-(3.4) and the dominated convergence theorem for the Henstock-Lebesgue-Stieltjes integral, for each $t \geq 0$, we get that $\|\mathcal{Q}(u^{(n)})(t) - \mathcal{Q}(u)(t)\|_\Omega \to 0$ as $n \to \infty$.

Moreover, by Step 1, it can shown that $\{\mathcal{Q}(u^{(n)})\}_{n=1}^{\infty}$ is equiregulated. Therefore, taking account to Lemma 2.1, we derive that $\mathcal{Q}(u^{(n)})$ converge uniformly to $\mathcal{Q}(u)$. Hence, $\mathcal{Q}$ is a continuous operator.

Step.3. We prove that $\mathcal{Q}([v^{(0)}, w^{(0)}]) \subset \Omega$.

For any $u \in [v^{(0)}, w^{(0)}]$, it is clear that $(\mathcal{Q}u)$ is defined on $[-r, \infty)$, and because $\varphi \in \mathcal{B}$, we have $\mathcal{Q}u|_{[-r,0]} \in \mathcal{B}$. Thus it is easy to show that the function

$$\begin{aligned}
f : t \to & S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\
& + \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0) + \varphi(0)] \\
& + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s) \in SAP_\omega(E), \quad t \geq 0. \qquad (3.5)
\end{aligned}$$

Since $u|_{\mathbb{R}^+} \in SAP_\omega(E)$ and $u_t \in SAP_\omega(\mathcal{B})$ for all $t \geq 0$, $\|u(t+\omega) - u(t)\| \leq \epsilon$ and $\|u_{t+\omega} - u_t\|_\mathcal{B} \leq \epsilon$ become arbitrarily small by choosing t large enough. Hence, by the continuity of F, there exists a constant $t_{\epsilon,1} > 0$ such that, for every $t \geq t_{\epsilon,1}$, we obtain

$$\|F(t, u(t+\omega), u_{t+\omega}) - F(t, u(t), u_t)\| \leq \frac{\epsilon}{2} \qquad (3.6)$$

and we can find a positive constant $t_{\epsilon,2}$ sufficiently large such that for $t \geq t_{\epsilon,2}$, by (H1), we have

$$\|F(t+\omega, u(t+\omega), u_{t+\omega}) - F(t, u(t+\omega), u_{t+\omega})\| \leq \frac{\epsilon}{2}. \qquad (3.7)$$

Then for $t > t_\epsilon := \max\{t_{\epsilon,1}, t_{\epsilon,2}\}$, from (3.5), it follows that

$$\begin{aligned}
& f(t+\omega) - f(t) = S_{\beta,\gamma_k}(t+\omega)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t+\omega)[\psi + Q_0(u)] \\
& + \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(u)(0)+\varphi(0)] \\
& + \int_0^t T_{\beta,\gamma_k}(t+\omega-s) F(s,u(s),u_s) dg(s) \\
& - S_{\beta,\gamma_k}(t)\varphi(0) - (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\
& - \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)] \\
& - \int_0^t T_{\beta,\gamma_k}(t-s) F(s,u(s),u_s) dg(s) \\
& = S_{\beta,\gamma_k}(t+\omega)\varphi(0) - S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] \\
& + \Big((\varphi_1 * S_{\beta,\gamma_k})(t+\omega) - (\varphi_1 * S_{\beta,\gamma_k})(t) \Big)[\psi + Q_0(u)] \\
& + \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(u)(0)+\varphi(0)] \\
& - \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)] \\
& + \int_0^\omega T_{\beta,\gamma_k}(t+\omega-s) F(s,u(s),u_s) dg(s) \\
& + \int_0^t T_{\beta,\gamma_k}(t-s)(F(s+\omega,u(s+\omega),u_{s+\omega}) - F(s,u(s),u_s)) dg(s) \\
& + \int_0^t T_{\beta,\gamma_k}(t-s)(F(s,u(s+\omega),u_{s+\omega}) - F(s,u(s),u_s)) dg(s) \\
& := J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t).
\end{aligned} \tag{3.8}$$

Then

$$\|f(t+\omega) - f(t)\| \le \|J_1(t)\| + \|J_2(t)\| + \|J_3(t)\| + \|J_4(t)\| + \|J_5(t)\|. \tag{3.9}$$

Hence, we have

$$\begin{aligned}
\|J_1(t)\| &\le \|S_{\beta,\gamma_k}(t+\omega)[Q(u)(0)+\varphi(0)]\| + \|S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)]\| \\
&\le (\|S_{\beta,\gamma_k}(t+\omega)\| + \|S_{\beta,\gamma_k}(t)\|) \cdot \|Q(u)(0)+\varphi(0)\| \\
&\le \frac{2C\|Q(u)(0)+\varphi(0)\|}{1+|\kappa|(t^{\beta+1} + \sum_{k=1}^{n} \alpha_k t^{\gamma_k})},
\end{aligned}$$

it is implies that $\|J_1(t)\|$ tend to 0 as $t \to \infty$.

On the other hand, note that by (2.1) we have $\sup_{t>\tau}\|tS_{\beta,\gamma_k}(t)\|<\infty$, for each $\tau>0$. Since A is an ω-sectorial of angle $\gamma_k\frac{\pi}{2}$ then $\|\mathcal{L}[S_{\beta,\gamma_k}](\lambda)\|\to 0$ as $\lambda\to 0$. Thus, by the vector-valued Hardy-Littlewood theorem (see [2], Theorem 4.2.9) we conclude that

$$\|(\varphi_1 * S_{\beta,\gamma_k})(t)\|\to 0 \quad \text{as} \quad t\to\infty. \tag{3.10}$$

By (2.4), it is implies that

$$\|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\|\to 0 \text{ as } \ t\to\infty. \tag{3.11}$$

Hence, we have

$$\begin{aligned}\|J_2(t)\| &\le \|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega)[\psi+Q_0(u)]-(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi+Q_0(u)]\|\\ &\le \|(\varphi_1 * S_{\beta,\gamma_k})(t+\omega)-(\varphi_1 * S_{\beta,\gamma_k})(t)\|\cdot\|\psi+Q_0(u)\|\end{aligned}$$

By (3.10), we deduce that $\|J_2(t)\|$ tend to 0 as $t\to\infty$.

$$\begin{aligned}\|J_3(t)\| &\le \|\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)[Q(u)(0)+\varphi(0)]\\ &\quad -\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)]\|\\ &\le \sum_{k=1}^{n}\alpha_k\|[(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t+\omega)\\ &\quad -(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)]\|\cdot\|Q(u)(0)+\varphi(0)\|.\end{aligned}$$

By (3.11), we deduce that $\|J_3(t)\|$ tend to 0 as $t\to\infty$.

By (H1), we have

$$\begin{aligned}\|J_4\| &\le \int_0^{\omega}\|T_{\beta,\gamma_k}(t+\omega-s)\|\cdot\|F(s,u(s),u_s)\|dg(s)\\ &\le CW(r)\int_0^{\omega}(t+\omega-s)^{\beta-\gamma_k}P(s)dg(s),\end{aligned}$$

we deduce that $\|J_4(t)\|$ tend to 0 as $t \to \infty$. By (3.6), (3.7) and (H1), we have

$$\begin{aligned}
\|J_5\| &\leq \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s+\omega, u(s+\omega), u_{s+\omega}) - F(s, u(s+\omega), u_{s+\omega})\| dg(s) \\
&\quad + \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s, u(s+\omega), u_{s+\omega}) - F(s, u(s), u_s)\| dg(s) \\
&\leq 4CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} \|P\|_{\mathbb{HLS}_g^p} + \epsilon \int_{t_\epsilon}^t \|T_{\beta,\gamma_k}(t-s)\| dg(s) \\
&\quad + 4CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} \|P\|_{\mathbb{HLS}_g^p} + \epsilon \int_{t_\epsilon}^t \|T_{\beta,\gamma_k}(t-s)\| dg(s) \\
&\leq 8CW(r)\Big(\int_0^{t_\epsilon} (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} \|P\|_{\mathbb{HLS}_g^p} + 2\epsilon \int_0^t \|T_{\beta,\gamma_k}(t-s)\| dg(s),
\end{aligned}$$

which implies that $\|J_5(t)\|$ tends to 0 ad $t \to \infty$. Thus, from the above results, we can deduce that

$$\lim_{t\to\infty} \|f(t+\omega) - f(t)\| = 0.$$

Combining this with the definition $\mathcal{Q}$, we can conclude that $\mathcal{Q}(SPA_\omega(E)) \subset SPA_\omega(E)$, and combining this fact with Step 2, we can conclude that $(\mathcal{Q}u) \in \Omega$ for any $u \in [v^{(0)}, w^{(0)}]$, which implies that $\mathcal{Q}([v^{(0)}, w^{(0)}]) \subset \Omega$.

Step.4. We check that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonically increasing operator. Since $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ is positive, thus $(\varphi_1 * S_{\beta,\gamma_k})(t)$, $(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)$ and $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$ are also positive. On the one hand, in view of Definition 2.10, Definition 2.11, and the positivity of operators $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$, $(\varphi_1 * S_{\beta,\gamma_k})(t)$, $(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)$ and $T_{\beta,\gamma_k}(t) = (\varphi_\beta * S_{\beta,\gamma_k})(t)$, we can deduce that for any $t \in [0,\infty)$, $v^{(0)}(t) \leq (\mathcal{Q}v^{(0)})(t)$, together with $v^{(0)}(t) = \varphi(t) = (\mathcal{Q}v^{(0)})(t)$ for $t \in [-r, 0]$, we get $v^{(0)} \leq \mathcal{Q}v^{(0)}$. Similarly, $\mathcal{Q}w^{(0)} \leq w^{(0)}$ is available.

On the other hand, let $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(1)} \leq u^{(2)}$, we can see

$$v^{(0)}(t) \leq u^{(1)}(t) \leq u^{(2)}(t) \leq w^{(0)}(t), \quad t \in [-r, a],$$

$$v_t^{(0)} \leq u_t^{(1)} \leq u_t^{(2)} \leq w_t^{(0)}, \quad t \in [0, \infty).$$

Thus, by $(H1)$, $(H2)$, and the positivity of $T_1(t)(t \geq 0), T_2(t)(t \geq 0)$, we can get

$$\mathcal{Q}u^{(1)} \leq \mathcal{Q}u^{(2)}.$$

Consequently, $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonically increasing operator.

Now, we establish two iterative sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ in $[v^{(0)}, w^{(0)}]$ by

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad w^{(n)} = \mathcal{Q}w^{(n-1)}, \qquad n = 1, 2, \cdots. \tag{3.12}$$

Using the monotonicity of $\mathcal{Q}$, we can easily confirm that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ satisfy:

$$v^{(0)} \le v^{(1)} \le v^{(2)} \le \cdots \le v^{(n)} \le \cdots \le w^{(n)} \le \cdots \le w^{(2)} \le w^{(1)} \le w^{(0)}. \qquad (3.13)$$

Step.5. We prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent in Ω.

For any $a \in (0, +\infty)$, restrict $\{v^{(n)}\}$ to interval $[-r, a]$. Let $V = \{v^{(n)} | n \in \mathbb{N}\}$ and $V_0 = \{v^{(n-1)} | n \in \mathbb{N}\}$. Then $V(t) = (\mathcal{Q}V_0)(t)$ for $t \in [-r, a]$. In fact, $v^{(n)}(t) = \varphi(t)$ for $t \in [-r, 0]$, thus, $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [-r, 0]$. For $\forall \epsilon \in (0, t)$, we define a set $\{(\mathcal{Q}^\epsilon V_0)(t)\}$ by

$$\mathcal{Q}^\epsilon V_0(t) := \{Q^\epsilon v^{(n)}(t) | v^{(n)} \in V_0, t \in [0, a]\},$$

where

$$\begin{aligned}\mathcal{Q}^\epsilon v^{(n)}(t) &= S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t-\epsilon)[\psi + Q_0(v^{(n-1)})] \\ &+ \sum_{k=1}^{n} \alpha_k \int_0^{t-\epsilon} \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] ds \\ &+ \int_0^{t-\epsilon} T_{\beta,\gamma_k}(t-s) F(s, v^{(n-1)}(s), v_s^{(n-1)}) dg(s), \quad t \ge 0.\end{aligned}$$

Then from the compactness of $\{S_{\beta,\gamma_k}(t)\}_{t\ge 0}$, we obtain that the set $\mathcal{Q}^\epsilon V_0(t)$ is relatively compact in E for all $\epsilon \in (0, t)$. Moreover, for every $v^{(n)} \in V_0$ and $t \in [0, a]$, from the following inequality

$$\begin{aligned}\|\mathcal{Q}v^{(n)}(t) - \mathcal{Q}^\epsilon v^{(n)}(t)\| &\le \|((\varphi_1 * S_{\beta,\gamma_k}(t) - \varphi_1 * S_{\beta,\gamma_k}(t-\epsilon))[\psi + Q_0(v^{(n-1)})]\| \\ &+ \|\sum_{k=1}^{n} \alpha_k \int_{t-\epsilon}^{t} \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(v^{(n-1)})(0) + \varphi(0)] ds\| \\ &+ \|\int_{t-\epsilon}^{t} T_{\beta,\gamma_k}(t-s) F(s, v^{(n-1)}(s), v_s^{(n-1)}) dg(s)\| \\ &\le M\epsilon(\|\psi\| + c_0\|v^{(n)}\|_\infty + d_0) + \sum_{k=1}^{n} \frac{\alpha_k M \epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} (\|\varphi(0)\| + c_1\|v^{(n)}\|_\infty + d_1) \\ &+ CW(r) \int_{t-\epsilon}^{t} (t-s)^{\beta-\gamma_k} P(s) dg(s) \\ &\le M\epsilon(\|\psi\| + c_0\|v^{(n)}\|_\infty + d_0) + \sum_{k=1}^{n} \frac{\alpha_k M \epsilon^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} (\|\varphi(0)\| + c_1\|v^{(n)}\|_\infty + d_1) \\ &+ CW(r)\Big(\|H(t) - H(t-\epsilon)\| + \int_0^{t-\epsilon} |(t-s)^{\beta-\gamma_k} - (t-\epsilon-s)^{\beta-\gamma_k}| P(s) dg(s)\Big) \\ &\to 0 \quad \text{as} \quad \epsilon \to 0,\end{aligned}$$

thus, the set $\{(\mathcal{Q}V_0)(t)\}$ is relatively compact, which implies that $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [0, a]$. Thus, we have proved that $\{v^{(n)}(t)\}$ is relatively compact on E for $t \in [-r, a]$.

Therefore, $\{v^{(n)}\}$ is relatively compact in $G([-r, a], E)$ by the Arzelà-Ascoli Theorem, which implies that there is convergent subsequence in $v^{(n)}$. Combing with the monotonicity and the normality of the cone, it is clear that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ themselves are convergent, i.e., there exist $\underline{u}, \overline{u} \in G([-r, a], E)$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r, a].$$

According to the arbitrariness of a, one can find that $\underline{u}$ and $\overline{u}$ are defined on $[-r, \infty)$. On the other hand, it is easy to see $\lim_{t\longrightarrow\infty} \|\underline{u}(t+\omega) - \underline{u}(t)\| = 0$ and $\lim_{t\longrightarrow\infty} \|\overline{u}(t+\omega) - \overline{u}(t)\| = 0$. Hence, we can deduce that there exist $\underline{u}, \overline{u} \in \Omega$, such that

$$\underline{u}(t) = \lim_{n\to\infty} v^{(n)}(t), \quad \overline{u}(t) = \lim_{n\to\infty} w^{(n)}(t), \qquad t \in [-r, \infty). \tag{3.14}$$

Taking limit in (3.12), we have

$$\underline{u} = \mathcal{Q}\underline{u}, \qquad \overline{u} = \mathcal{Q}\overline{u}.$$

Therefore $\underline{u}, \overline{u} \in \Omega$ are fixed points of $\mathcal{Q}$ and they are the S-asymptotically ω-periodic mild solution of the problem (1.1).

Step.6. We claim that $\underline{u}$ and $\overline{u}$ are the minimal and maximal S-asymptotically ω-periodic mild solutions of the nonlocal problem (1.1), respectively.

Taking limit of both ends of (3.12), we can deduce from (3.14) that

$$\underline{u} = \mathcal{Q}\underline{u}, \quad \overline{u} = \mathcal{Q}\overline{u}. \tag{3.15}$$

Applying (3.13), we can get $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}] \subset \Omega$ that are fixed points of $\mathcal{Q}$ and $\underline{u} \le \overline{u}$. In fact, let $u \in [v^{(0)}, w^{(0)}]$ is an arbitrary fixed point of $\mathcal{Q}$, then for every $t \in [-r, \infty)$, we have $v^{(0)}(t) \le u(t) \le w^{(0)}(t)$, and

$$v^{(1)}(t) = (\mathcal{Q}v^{(0)})(t) \le (\mathcal{Q}u)(t) = u(t) \le (\mathcal{Q}w^{(0)})(t) = w^{(1)}(t),$$

namely,

$$v^{(1)} \le u \le w^{(1)}.$$

Repeat this process, we get

$$v^{(n)} \le u \le w^{(n)}, \qquad n = 1, 2, \cdots.$$

Let $n \to \infty$, we can see $\underline{u} \le u \le \overline{u}$. Therefore $\underline{u}$ and $\overline{u}$, respectively, are the minimal and maximal S-asymptotically ω-periodic mild solutions of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$, and $\underline{u}, \overline{u}$ can be obtained by the iterative sequences (3.12) starting from $v^{(0)}$ and $w^{(0)}$, respectively. The proof is finished. $\square$

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal. Let $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\alpha_k \ge 0, k = 1, 2, \dots n$ be given. A is an κ-sectorial operator of angle $\frac{\gamma_k \pi}{2}, k = 1, 2, \cdots, n$ with $\kappa < 0$, and A generates a positive and equicontinuous (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t \ge 0}$ on E. Assume that $\omega > 0$ is a constant and the nonlocal problem (1.1) has a lower mild solution $v^{(0)}$ and an upper mild solution $w^{(0)}$ with $v^{(0)} \le w^{(0)}$. If $\varphi \in K_{\mathcal{B}}$, $Q(u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times E \times \mathcal{B} \to E$ is continuous and satisfies the conditions (H1)-(H4) and the following conditions*

$(H5)$ For each $t \in \mathbb{R}^+$, and monotone sequence $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$, there exist constants $L_f \ge 0, 0 < ML_h(1 + \sum_{k=1}^{n} |\alpha_k| a^{\beta-\gamma_k+1}) + \widetilde{M} L_g < \frac{1}{2}$ such that

$$
\begin{aligned}
&\alpha(\{F(t, u^{(n)}(t), u_t^{(n)})\}) \le L_f \Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s \in [-r,0]} \alpha(\{u_t^{(n)}(s)\}) \Big), \\
&\alpha(\{Q_0(u^{(n)}(t))\}) \le L_g \alpha(\{u^{(n)}(t)\}),\ \alpha(\{Q(u^{(n)}(t))\}) \le L_h \alpha(\{u^{(n)}(t)\}).
\end{aligned}
$$

Then the nonlocal problem (1.1) has minimal and maximal S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedures starting from $v^{(0)}$ and $w^{(0)}$, respectively.

Proof. Let $\mathcal{Q}$ be defined by (3.1). From the proof of Theorem 3.1, we know that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a continuous increasing operator and $v^{(0)} \le \mathcal{Q}v^{(0)}$, $\mathcal{Q}w^{(0)} \le w^{(0)}$. Hence, the iterative sequences $v^{(n)}$ and $w^{(n)}$ defined by (3.12) satisfy (3.13). By $\{S_{\beta,\gamma_k}(t)\}_{t \ge 0}$ is an equicontinuous resolvent family, from the Step. 1 of proof of Theorem 3.1, we obtain that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are bounded and equiregulated in $t \in [-r, +\infty)$.

Next, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent in Ω.

For $\forall a > 0$, restrict $\{v^{(n)}\}$ to interval $[-r, a]$. Let $V = \{v^{(n)} | n \in \mathbb{N}\}$ and $V_0 = \{v^{(n-1)} | n \in \mathbb{N}\}$. Then $V = (\mathcal{Q}V_0)$. From $V_0 = V \cup \{v^{(0)}\}$ it follow that $\alpha(V_0(t)) = \alpha(V(t))$ for $t \in [-r, a]$.

For $t \in [-r, 0]$, in view of the fact that $v^{(n)}(t) = \mathcal{Q}v^{(n-1)}(t) = \varphi(t)$, we can see

$$
\alpha(V(t)) = 0, \quad t \in [-r, 0]. \tag{3.16}
$$

For $t \in [0, a]$, one can get

$$\sup_{s\in[-r,0]} \alpha(\{v_t^{(n)}(s)\}) = \sup_{s\in[-r,0]} \alpha(\{v^{(n)}(t+s)\}) \le \alpha(\{v^{(n)}(t)\}). \tag{3.17}$$

Thus, by Lemma 2.2, we have

$$\begin{aligned}
\alpha(V(t)) &= \alpha(\{\mathcal{Q}V_0(t)\}) \\
&= \alpha\Big(\Big\{S_{\beta,\gamma_k}(t)[Q(v^{(n-1)})(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(v^{(n-1)})] \\
&+ \sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(v^{(n-1)})(0) + \varphi(0)] \\
&+ \int_0^t T_{\beta,\gamma_k}(t-s)F(s, v^{(n-1)}(s), v_s^{(n-1)})dg(s)\Big\}\Big) \\
&\le \alpha(\{S_{\beta,\gamma_k}(t)[Q(v^{(n-1)})(0) + \varphi(0)]\}) + \alpha\Big(\Big\{\int_0^t S_{\beta,\gamma_k}(t)[\psi + Q_0(v^{(n-1)})]ds\Big\}\Big) \\
&+ \alpha\Big(\Big\{\sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(v^{(n-1)})(0) + \varphi(0)]\Big\}\Big) \\
&+ \alpha\Big(\Big\{\int_0^t T_{\beta,\gamma_k}(t-s)F(s, v^{(n-1)}(s), v_s^{(n-1)})dg(s)\Big\}\Big) \\
&= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.
\end{aligned}$$

Evidently,

$$\alpha_0 := \alpha(S_{\beta,\gamma_k}(t)[Q(v^{(n-1)})(0) + \varphi(0)]\}) \le 2M\alpha(\{Q(v^{(n-1)})(0)\}) = 2ML_h\alpha(V(t)).$$

According to Lemma 2.12, $(H5)$, (2.5) and (3.17), we have

$$\begin{aligned}
\alpha_1 &:= \alpha\Big(\Big\{(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(v^{(n-1)})]\Big\}\Big) \\
&\le 2\widetilde{M}\alpha(\{(\psi + Q_0(v^{(n-1)})(t))\}) \\
&\le 2\widetilde{M}\alpha(\{Q_0(v^{(n-1)})(t)\}) \\
&= 2\widetilde{M}L_g\alpha(V(t)),
\end{aligned}$$

$$\begin{aligned}
\alpha_2 &:= \alpha\Big(\Big\{\sum_{k=1}^{n} \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(v^{(n-1)})(0) + \varphi(0)]\Big\}\Big) \\
&\le 2\sum_{k=1}^{n} |\alpha_k| \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} \|S_{\beta,\gamma_k}(t-s)\|_{\mathcal{L}(E)} \cdot \alpha(\{Q(v^{(n-1)})(0)\})ds \\
&\le \frac{2ML_h \sum_{k=1}^{n} |\alpha_k| a^{\beta-\gamma_k+1}}{\Gamma(2+\beta-\gamma_k)} \alpha(V(t));
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 \quad &:= \quad \alpha\Big(\Big\{\int_0^t T_{\beta,\gamma_k}(t-s)F(s,v^{(n-1)}(s),v_s^{(n-1)})dg(s)\Big\}\Big) \\
&\leq \quad 2\int_0^t \|T_{\beta,\gamma_k}(t-s)\|_{\mathcal{L}(E)}\cdot \alpha(\{f(s,v^{(n-1)}(s),v_s^{(n-1)})\})ds \\
&\leq \quad 2C\int_0^t (t-s)^{\beta-\gamma_k}\alpha(\{f(s,v^{(n-1)}(s),v_s^{(n-1)})\})dg(s) \\
&\leq \quad \frac{2Ca^{\beta-\gamma_k+1}}{\beta-\gamma_k+1}L_f\int_0^t \alpha(V(s))dg(s).
\end{aligned}
$$

Consequently, we can deduce that for $t\in[0,a]$,

$$
\begin{aligned}
\alpha(V(t)) \leq 2ML_h\alpha(V(t)) &+ \frac{2ML_h\sum_{k=1}^n|\alpha_k|a^{\beta-\gamma_k+1}}{\Gamma(2+\beta-\gamma_k)}\alpha(V(t)) \\
&+ 2\widetilde{M}L_g\alpha(V(t)) + \frac{2Ca^{\beta-\gamma_k+1}}{\beta-\gamma_k+1}L_f\int_0^t \alpha(V(s))dg(s).
\end{aligned}
$$

Since $ML_h(1+\sum_{k=1}^n|\alpha_k|a^{\beta-\gamma_k+1})+\widetilde{M}L_g<\frac{1}{2}$, it gives that

$$
\alpha(V(t)) \leq \frac{2Ca^{\beta-\gamma_k+1}L_f}{(1+\beta-\gamma_k)[1-2ML_h(1+\sum_{k=1}^n|\alpha_k|a^{\beta-\gamma_k+1})-2\widetilde{M}L_g]}\int_0^t \alpha(V(s))dg(s).
$$

Hence, by Bellman inequality, $\alpha(V(t))\equiv 0$ in $[0,a]$. Combining with (3.16), we have $\alpha(V(t))\equiv 0$ in $[-r,a]$, which shows that $\{v^{(n)}(t)\}$ is precompact on E for any $t\in[-r,a]$. We can similarly show that $\{w^{(n)}(t)\}$ is also precompact on E for $t\in[-r,a]$. Hence, there are convergent subsequences in $\{v^{(n)}\}$ and $\{w^{(n)}\}$. Combining with the monotonicity and the normality of the cone, it is clear that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ themselves are convergent, i.e., there exist $\underline{u},\overline{u}\in C([-r,a],E)$, such that

$$
\underline{u}(t)=\lim_{n\to\infty}v^{(n)}(t), \quad \overline{u}(t)=\lim_{n\to\infty}w^{(n)}(t), \qquad t\in[-r,a].
$$

According to the arbitrariness of a, one can find that $\underline{u}$ and $\overline{u}$ are defined on $[-r,\infty)$. On the other hand, it is easy to see $\lim_{t\longrightarrow\infty}\|\underline{u}(t+\omega)-\underline{u}(t)\|=0$ and $\lim_{t\longrightarrow\infty}\|\overline{u}(t+\omega)-\overline{u}(t)\|=0$. Hence, we can deduce that there exist $\underline{u},\overline{u}\in\Omega$, such that

$$
\underline{u}(t)=\lim_{n\to\infty}v^{(n)}(t), \quad \overline{u}(t)=\lim_{n\to\infty}w^{(n)}(t), \qquad t\in[-r,\infty). \tag{3.18}
$$

Thus, from the Step.6 of proof of Theorem 3.1, $\underline{u},\overline{u}$ are the minimal and maximal S-asymptotically ω-periodic mild solutions of the problem (1.1), which can be obtained by monotone iterative sequences starting from $v^{(0)}$ and $w^{(0)}$. This completes the proof of Theorem 3.2.

In Theorem 3.2, the condition (H5) guarantees that the sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are precompact. However, according to [39, 52, 30], we can replace (H5) by adding appropriate conditions to the space E or positive cone K, see the following conclusions for details:

Corollary 3.3. *Without assuming (H5), the same conclusions as Theorem 3.2 hold if E is an ordered, weakly sequentially complete Banach space, whose positive cone $K \subset E$ is normal.*

Proof. Since E is weakly sequentially complete, Theorem 2.2 of [30] states that any monotonic and ordered bounded sequence in E is precompact. Let $u^{(n)}$ be a monotonic sequence in condition $(H5)$, then, $\{F(t, u^{(n)}(t), u_t^{(n)})\}$ and $\{g(u^{(n)}(t))\}$ are monotonic and ordered-bounded sequences. Based on the properties of measure of noncompactness, we can easily get that (H5) is valid. Hence, according to Theorem 3.2, Corollary 3.3 holds. $\Box$

Corollary 3.4. *Without assuming (H5), the same conclusions as Theorem 3.2 hold if E is an ordered, whose positive cone $K \subset E$ is regular.*

Proof. According to the proof Step.4. of Theorem 3.1, $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ defined by (3.1) is a continuous increasing operator, and $v^{(0)} \leq \mathcal{Q}v^{(0)},\ \mathcal{Q}w^{(0)} \leq w^{(0)}$. Since K is regular, then by the fixed point theorem of continuous increasing operator (see [39], Chapter 3 Theorem 2.2), $\mathcal{Q}$ has minimal and maximal fixed point $\underline{u}$ and $\overline{u}$ in $[v^{(0)}, w^{(0)}]$, and $\underline{u}, \overline{u}$ can be obtained by the iterative sequences (3.17) starting from $v^{(0)}$ and $w^{(0)}$, respectively. Consequently, Corollary 3.4 holds. $\Box$

Next, we discuss the uniqueness of S-asymptotically ω-periodic mild solution for the nonlocal problem (1.1).

Theorem 3.4. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal. Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given. A is an κ-sectorial operator of angle $\dfrac{\gamma_k \pi}{2}, k = 1, 2, \cdots, n$ with $\kappa < 0$, and A generates positive and equicontinuous (β, γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ on E. Assume that $\omega > 0$ is a constant and the nonlocal problem (1.1) has a lower mild solution $v^{(0)}$ and an upper mild solution $w^{(0)}$ with $v^{(0)} \leq w^{(0)}$. If $\varphi \in K_{\mathcal{B}},\ Q(u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K,\ F : \mathbb{R}^+ \times E \times \mathcal{B} \to E,\ Q, Q_0 : G([-r, +\infty), E) \to E$ are continuous as well as (A1)-(A3) and the following conditions are established:*

(H6) (1) For any $t \geq 0,\ x_1, x_2 \in E$ and $\phi_1, \phi_2 \in \mathcal{B}$ with $v^{(0)}(t) \leq x_1 \leq x_2 \leq w^{(0)}(t)$ and

$v_t^{(0)} \le \phi_1 \le \phi_2 \le w_t^{(0)}$, *there exist constant* $C_{f1}, C_{f2}, C_g \ge 0$ *such that*

$$F(t, x_2, \phi_2) - F(t, x_1, \phi_1) \le C_{f1}(x_2 - x_1) + C_{f2}(\phi_2 - \phi_1),$$

(2) *For any* $t \ge 0$, $u_1, u_2 \in C(\mathbb{R}^+, E)$ *with* $v^{(0)}(t) \le u_1 \le u_2 \le w^{(0)}(t)$, *there exist constant* $0 \le N(MC_h + \widetilde{M}C_g + C\sum_{k=1}^{n} a_k) < 1$ *such that*

$$Q_0(u_2) - Q_0(u_1) \le C_g(u_2 - u_1), \quad Q(u_2) - Q(u_1) \le C_h(u_2 - u_1).$$

(*H*7) *For any* $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ *with* $u^{(2)} \ge u^{(1)}$, *there exists a sufficiently small constant* $C_0 > 0$ *such that*

$$u^{(2)}(t) - u^{(1)}(t) \ge C_0(u_t^{(2)} - u_t^{(1)}), \qquad t \ge 0.$$

Then the nonlocal problem (1.1) has a unique S*-asymptotically* ω*-periodic mild solution in* $[v^{(0)}, w^{(0)}]$, *which can be obtained by the monotone iterative procedure starting from* $v^{(0)}$ *or* $w^{(0)}$. .

Proof. First of all, we verify that (H5) is valid. Actually, for each $t \ge 0$, let $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ be an increasing sequence, then $\{u_t^{(n)}\} \subset [v_t^{(0)}, w_t^{(0)}]$ is also an increasing sequence. For any $m, n \in \mathbb{N}^+$, from (H6) and the normality of positive cone K, it follows that

$$\|F(t, u^{(m)}, u_t^{(m)}) - F(t, u^{(n)}, u_t^{(n)})\| \le NC_{f1}\|u^{(m)} - u^{(n)}\|_C + NC_{f2}\|u_t^{(m)} - u_t^{(n)}\|_{\mathcal{B}},$$

$$\|Q_0(u^{(m)}) - Q_0(u^{(n)})\| \le NC_g\|u^{(m)} - u^{(n)}\|_{\mathcal{B}}, \ \|Q(u^{(m)}) - Q(u^{(n)})\| \le NC_h\|u^{(m)} - u^{(n)}\|_{\mathcal{B}}.$$

where N is the normal constant of K. Combing this with the definition of measure of noncompactness, one can see

$$\alpha(\{f(t, u^{(n)}(t), u_t^{(n)})\}) \le L_f\Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s\in[-r,0]} \alpha(\{u_t^{(n)}(s)\})\Big),$$

$$\alpha(\{Q_0(u^{(n)}(t))\}) \le L_g\alpha(\{u^{(n)}(t)\}), \ \alpha(\{Q(u^{(n)}(t))\}) \le L_h\alpha(\{u^{(n)}(t)\}),$$

where $L_f = \max\{NC_{f1}, NC_{f2}\}, L_g = NC_g$, $L_h = NC_h$. Thus, (H5) is valid. Consequently, from Theorem 3.1, the nonlocal problem (1.1) has minimal and maximal S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$. In what follows, we check that $\underline{u} = \overline{u}$.

Let $\mathcal{Q}$ be defined by (3.1), it is obvious that $\underline{u}(t) = \overline{u}(t) = \varphi(t)$ for $t \in [-r, 0]$. For any

$t \geq 0$, by (3.1) and (H6), (H7), we can calculate that

$$\begin{aligned}
\theta &\leq \overline{u}(t) - \underline{u}(t) = \mathcal{Q}\overline{u}(t) - \mathcal{Q}\underline{u}(t) \\
&= S_{\beta,\gamma_k}(t)(Q(\overline{u}) - Q(\underline{u})) \\
&+ (\varphi_1 * S_{\beta,\gamma_k})(t)(Q_0(\overline{u}) - Q_0(\underline{u})) \\
&+ \sum_{k=1}^{n} \alpha_k (\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)(Q(\overline{u}) - Q(\underline{u})) \\
&+ \int_0^t T_{\beta,\gamma_k}(t-s)(F(s,\overline{u}(s),\overline{u}_s) - F(s,\underline{u}(s),\underline{u}_s))dg(s) \\
&\leq MC_h(\overline{u}(t) - \underline{u}(t)) + \widetilde{M}C_g(\overline{u}(t) - \underline{u}(t)) + C\sum_{k=1}^{n} a_k(\overline{u}(t) - \underline{u}(t)) \\
&+ C\int_0^t (C_{f1}(\overline{u}(s) - \underline{u}(s)) + C_{f2}(\overline{u}_s - \underline{u}_s))dg(s) \\
&\leq MC_h(\overline{u}(t) - \underline{u}(t)) + \widetilde{M}C_g(\overline{u}(t) - \underline{u}(t)) \\
&+ C\sum_{k=1}^{n} a_k(\overline{u}(t) - \underline{u}(t)) + C(C_{f1} + \frac{C_{f2}}{C_0})\int_0^t (\overline{u}(s) - \underline{u}(s))dg(s).
\end{aligned}$$

Hence, by the normality of the cone K, it follows that for $t \geq 0$, we can obtain

$$\|\overline{u}(t)-\underline{u}(t)\| \leq N(MC_h+\widetilde{M}C_g+C\sum_{k=1}^{n} a_k)\|\overline{u}(t)-\underline{u}(t)\|+NC(C_{f1}+\frac{C_{f2}}{C_0})\int_0^t \|\overline{u}(s)-\underline{u}(s)\|dg(s).$$

Since < 1, it get that

$$\|\overline{u}(t) - \underline{u}(t)\| \leq \frac{NC(C_{f1} + \frac{C_{f2}}{C_0})}{1 - N(MC_h + \widetilde{M}C_g + C\sum_{k=1}^{n} a_k)} \int_0^t \|\overline{u}(s) - \underline{u}(s)\|dg(s).$$

By applying Bellman inequality Lemma 2.13, we can get $\overline{u}(t) = \underline{u}(t)$ for $t \geq 0$. Consequently, $\tilde{u} = \overline{u} = \underline{u}$ is the unique S-asymptotically ω-periodic mild solution of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$ and from the Theorem 3.1, we know that $\tilde{u}$ can be obtained by the monotone iterative procedure starting from $v^{(0)}$ or $w^{(0)}$. This completes the proof of Theorem 3.5. □

At the end of this section, we establish an existence result of S-asymptotically ω-periodic positive mild solutions for nonlocal problem (1.1) without assuming the existence of upper and lower S-asymptotically ω-periodic mild solutions.

Theorem 3.5. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal, Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1, 2, \ldots n$ be given. A is an κ-*

sectorial operator of angle $\dfrac{\gamma_k\pi}{2}, k = 1, 2, \cdots, n$ *with* $\kappa < 0$, *and* A *generates positive and equicontinuous* (β,γ_k)*-resolvent family* $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ *on* E. *Assume that* $\omega > 0$ *is a constant,* $\varphi \in K_{\mathcal{B}}$, $Q(u)(0) + \varphi(0) \in K \cap \mathcal{D}(A)$ *and* $\psi \in K$, $F : \mathbb{R}^+ \times K \times K_{\mathcal{B}} \to E$, $Q, Q_0 : G([-r,+\infty), K) \to K$ *are continuous and* $F(t,\theta,\theta) \geq \theta$ *for* $t \geq 0$. *If the condition (H2) and the following conditions are established:*

(H8) For any $R > 0,\ t \geq 0,\ x_1, x_2 \in K$ *with* $\theta \leq x_1 \leq x_2,\ \|x_i\| \leq R$ *and* $\phi_1, \phi_2 \in K_{\mathcal{B}}$ *with* $\theta \leq \phi_1 \leq \phi_2,\ \|\phi_i\|_{\mathcal{B}} \leq R$,

$$F(t, x_2, \phi_2) \geq F(t, x_1, \phi_1) \geq \theta;$$

(H9) For any $t \geq 0$, $x \in E$ *and* $\phi \in \mathcal{B}$, *there exist functions* $p_i(\cdot) \in \mathbb{HLS}_g^p(\mathbb{R}^+, \mathbb{R}^+)$ *for some* $p > 1$ *and nondecreasing functions* $\mathcal{F}_i \in C(\mathbb{R}^+, \mathbb{R}^+)(i = 1, 2)$ *as well as a positive constant* $\overline{K}$ *such that*

$$\|F(t, x, \phi)\| \leq p_1(t)\mathcal{F}_1(\|x\|) + p_2(t)\mathcal{F}_2(\|\phi\|_{\mathcal{B}}) + \overline{K},$$

where $\mathcal{F}_i$ *and* p_i *satisfy*

$$\liminf_{l\to+\infty} \frac{\mathcal{F}_i(l)}{l} := \zeta_i < +\infty,\ i = 1, 2;$$

(H10) The nonlocal functions $Q, Q_0(u)$ *are bounded and increasing for* $u \in G([-r,+\infty), K)$ *with* $\|u\|_C \leq R$ *such that*

$$\liminf_{R\to\infty} \frac{Q_0(R)}{R} := \eta < +\infty,\ \ \liminf_{R\to\infty} \frac{Q(R)}{R} := \eta_1 < +\infty.$$

(H11) For any $R > 0,\ t \geq 0$, *and the monotone increasing sequence* $\{u^{(n)}\} \subset \bar{B}(\theta, R)$, *there exist constants* $L_f, L_g, L_h \geq 0$ *such that*

$$\alpha(\{F(t, u^{(n)}(t), u^{(n)}_t)\}) \leq L_f\Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s\in[-r,0]} \alpha(\{u^{(n)}_t(s)\})\Big),$$
$$\alpha(\{Q_0(u^{(n)}(t))\}) \leq L_g\alpha(\{u^{(n)}(t)\}),\ \alpha(\{Q(u^{(n)}(t))\}) \leq L_h\alpha(\{u^{(n)}(t)\}).$$

(H12) The function $s \mapsto \int_0^{\cdot}(\cdot - s)^{\beta-\gamma_k}dg(s)$ *belongs to* $\mathbb{HLS}_g^q(\mathbb{R}^+, \mathbb{R}^+)$.

Then the nonlocal problem (1.1) has at least a S*-asymptotically* ω*-periodic positive mild solution* $u \in G([-r,\infty), K)$ *provided that*

$$(M + \sum_{k=1}^{n} \alpha_k M_1)\eta_1 + \widetilde{M}\eta + C\sup_{t\geq 0}\Big(\int_0^t (t-s)^{q(\beta-\gamma_k)}dg(s)\Big)^{\frac{1}{q}}(\zeta_1\|p_1\|_{\mathbb{HLS}_g^p} + \zeta_2\|p_2\|_{\mathbb{HLS}_g^p}) < 1, \tag{3.19}$$

and $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. Let a be any positive constant. For given $\varphi \in K_{\mathcal{B}},\ \|\varphi\|_{\mathcal{B}} \leq R$. Define

$$\Omega_R = \big\{u \in C([-r,\infty),K)\big|\|u(t)\| \leq R, t \in \mathbb{R}^+; u|_{[-r,0]} \in \mathcal{B}, u(t) = \varphi(t), t \in [-r,0]\big\},$$

and the operator $\mathcal{Q} : \Omega_R \to K$ by

$$(\mathcal{Q}u)(t) = \begin{cases} S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ + \sum_{k=1}^n \alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)] \\ + \int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s), \qquad t \in [0,a], \\ \\ Q(u)(t) + \varphi(t), \qquad t \in [-r,0]. \end{cases} \tag{3.20}$$

From the hypothesis (H8)-(H10), the positivity of $S_{\beta,\gamma_k}(t)(t \geq 0)$ and the definition of Ω_R, it follows that the positive mild solution of nonlocal problem (1.1) in $\mathbb{R}^+$ is equivalent to the fixed point of $\mathcal{Q}$.

Step.1. we check that there is a constant $R_0 > 0$ such that $\mathcal{Q}(\Omega_{R_0}) \subset \Omega_{R_0}$.

In view of (2.4), we observe that as $M_1 := \sup\limits_{t\geq 0} \|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| < +\infty$.

Indeed, if this were not so, it would follows that for any $R > 0$, there exists $u \in \Omega_R$ such that $\|\mathcal{Q}u\| > R$. In view of (2.5) and (3.20), for any $t \geq 0$, we have

$$\begin{aligned}
&\|(\mathcal{Q}u)(t)\| \leq \|S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\
&+ \sum_{k=1}^n \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0)+\varphi(0)]ds + \int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\| \\
&\leq \|S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)]\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\| \\
&+ \Big\|\sum_{k=1}^n \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0)+\varphi(0)]ds\Big\| + \Big\|\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s)\Big\| \\
&\leq \|S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)]\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\| \\
&+ \|\sum_{k=1}^n \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0)+\varphi(0)]ds\| \\
&+ \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s,u(s),u_s)\|dg(s) \\
&\leq M(\|\varphi\|_{\mathcal{B}} + Q(R)) + \widetilde{M}(\|\psi\| + Q_0(R)) + \Big(\sum_{k=1}^n \alpha_k M_1\Big)(\|\varphi\|_{\mathcal{B}} + Q(R))
\end{aligned}$$

$$
\begin{aligned}
&+C\Big(\mathcal{F}_1(R)\Big(\int_0^t [p_1(s)]^p dg(s)\Big)^{\frac{1}{p}} + \mathcal{F}_2(R)\Big(\int_0^t [p_2(s)]^p dg(s)\Big)^{\frac{1}{p}} + \overline{K}\Big) \cdot \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} \\
&\le (M + \sum_{k=1}^{n} \alpha_k M_1)(\|\varphi\|_{\mathcal{B}} + Q(R)) + \widetilde{M}(\|\psi\| + Q_0(R)) \\
&+ C \sup_{t\ge 0} \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} (\mathcal{F}_1(R)\|p_1\|_{\mathbb{HLS}_g^p} + \mathcal{F}_2(R)\|p_2\|_{\mathbb{HLS}_g^p} + \overline{K}). \qquad (3.21)
\end{aligned}
$$

Hence, according to the above calculation, we can see

$$
\begin{aligned}
R &< (M + \sum_{k=1}^{n} \alpha_k M_1)(\|\varphi\|_{\mathcal{B}} + Q(R)) + \widetilde{M}(\|\psi\| + Q_0(R)) \\
&\quad + C \sup_{t\ge 0} \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} (\mathcal{F}_1(R)\|p_1\|_{\mathbb{HLS}_g^p} + \mathcal{F}_2(R)\|p_2\|_{\mathbb{HLS}_g^p} + \overline{K}).
\end{aligned}
$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we can get

$$
(M+\sum_{k=1}^{n} \alpha_k M_1)\eta_1 + \widetilde{M}\eta + C \sup_{t\ge 0} \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s)\Big)^{\frac{1}{q}} (\zeta_1\|p_1\|_{\mathbb{HLS}_g^p} + \zeta_2\|p_2\|_{\mathbb{HLS}_g^p}) \ge 1,
$$

which is a contradiction (3.19). Thus, there is a constant $R_0 > 0$ such that $\mathcal{Q}(\Omega_{R_0}) \subset \Omega_{R_0}$.

Step.2. The set $\{Qu : u(\cdot) \in \Omega_R\}$ is equiregulated.

For any $b \in (0, \infty)$, restrict $u(t)$ to interval $[-r, b)$. For any $t_0 \in [-r, b)$, we have

$$
\begin{aligned}
\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\| &\le \|(S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+))[Q(u)(0) + \varphi(0)]\| \\
&+ \|[(\varphi_1 * S_{\beta,\gamma_k})(t) - (\varphi_1 * S_{\beta,\gamma_k})(t_0^+)][\psi + Q_0(u)]\| \\
&+ \sum_{k=1}^{n} \frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)} \Big| \int_0^t (t-s)^{\beta-\gamma_k} ds - \int_0^{t_0^+} (t_0^+ - s)^{\beta-\gamma_k} ds \Big| \|Q(u)(0) + \varphi(0)\| \\
&+ \int_0^{t_0^+} \|[T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)]F(s, u(s), u_s)\| dg(s) \\
&+ \int_{t_0^+}^{t} \|T_{\beta,\gamma_k(t-s)}(t-s)F(s, u(s), u_s)\| dg(s) \\
&\le \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(\mathbb{E})} \cdot \|Q(u)(0) + \varphi(0)\| + \widetilde{M}|t - t_0^+| \cdot \|\psi + Q_0(u)\| \\
&+ \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \|Q(u)(0) + \varphi(0)\|
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mathcal{F}_1(\|u\|) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} p_1(s) dg(s) \\
&+ \mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} p_2(s) dg(s) \\
&+ \overline{K} \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} dg(s) \\
&+ C\mathcal{F}_1(\|u\|) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_1(s) dg(s) \\
&+ C\mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_2(s) dg(s) \\
&+ C\overline{K} \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} dg(s) \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t) + I_8(t) + I_9(t), \qquad (3.21)
\end{aligned}
$$

where

$$I_1(t) = \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(E)} \cdot \|Q(u)(0) + \varphi(0)\|,$$

$$I_2(t) = \widetilde{M}|t - t_0^+| \cdot \|\varphi + Q_0(u)\|,$$

$$I_3(t) = \sum_{k=1}^n \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \cdot \|Q(u)(0) + \varphi(0)\|,$$

$$I_4(t) = (\mathcal{F}_1(\|u\|) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} p_1(s) dg(s),$$

$$I_5(t) = \mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} p_2(s) dg(s),$$

$$I_6(t) = \overline{K} \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} dg(s),$$

$$I_7(t) = C\mathcal{F}_1(\|u\|) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_1(s) dg(s),$$

$$I_8(t) = C\mathcal{F}_2(\|u_s\|_{\mathcal{B}}) \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} p_2(s) dg(s),$$

$$I_9(t) = C\overline{K} \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} dg(s).$$

From the expression of $I_2(t)$ and $I_3(t)$, we derive that $I_2(t) \to 0$ and $I_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Since the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$ yields the continuity in the sense of uniform operator topology. We dedude that $I_1(t) \to 0$ and applying dominated convergence theorem on $I_4(t), I_5(t), I_6(t)$ and $I_9(t)$, we can derive that $I_4(t), I_5(t), I_6(t), I_9(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H_1(t) = \int_0^t (t-s)^{\beta-\gamma_k} p_1(s)dg(s), H_2(t) = \int_0^t (t-s)^{\beta-\gamma_k} p_2(s)dg(s)$. Thanks to Lemma 2.4, we known that $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore, we have

$$
\begin{aligned}
I_7(t) &= C\mathcal{F}_1(R)\int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} p_1(s)dg(s) \\
&\leq C\mathcal{F}_1(R)\Big(\|H_1(t) - H_1(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} - (t_0^+ - s)^{\beta-\gamma_k})p_1(s)\|dg(s)\Big) \\
&\to 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u,
\end{aligned}
$$

and

$$
\begin{aligned}
I_8(t) &= C\mathcal{F}_2(R)\int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} p_2(s)dg(s) \\
&\leq C\mathcal{F}_2(R)\Big(\|H_2(t) - H_2(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} - (t_0^+ - s)^{\beta-\gamma_k})p_2(s)\|dg(s)\Big) \\
&\to 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u.
\end{aligned}
$$

Therefore, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. independently of $u \in \Omega_R$.

Similarly, one can demonstrate that for any $t_0 \in (-r, b]$, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. According to the arbitrariness of b, one can find that $u(t)$ is defined on $[0, \infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Hence, assert that $\{\mathcal{Q}u : u(\cdot) \in \Omega_R\}$ is equiregulated.

Step.3. We finally show that the operator $\mathcal{Q}$ has a positive fixed point on Ω_{R_0}.

We know that $\mathcal{Q} : \Omega_{R_0} \to \Omega_{R_0}$ is a monotonic increasing operator based on (H8)-(H10) and the proof Theorem 3.1.

Let $v^0 = \theta \in K$ and establish the iterative sequence $\{v^{(n)}\}$ by

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad n = 1, 2, \cdots . \tag{3.22}$$

Then according to the monotonicity of $\mathcal{Q}$, one can find $\{v^{(n)}\} \subset K$ and

$$\theta = v^{(0)} \leq v^{(1)} \leq \cdots \leq v^{(n)} \leq \cdots . \tag{3.23}$$

Similar to the proof of Theorem 3.2, we can get $\alpha(\{v^{(n)}(t)\}) \equiv 0$ in $[-r, a]$, that is, $\{v^{(n)}(t)\}$ is precompact, hence, it has a convergent subsequence $v^{(n_k)} \to u \in \Omega_1$, combined with its monotonicity (3.23) and the normality of cone K, it is easy to know that

$$v^{(n)} \to u \in G([-r, a], K), \qquad n \to \infty.$$

Taking limit of both ends of (3.22), and by the continuity of $\mathcal{Q}$, we can get $u = \mathcal{Q}u$, which shows that $u \in G([-r,a],K)$ is a positive mild solution of the nonlocal problem (1.1). According to the arbitrariness of a, one can find that $u(t)$ is defined on $[-r,\infty)$. On the other hand, by the method of Step.3 of Theorem 3.1, it is easy to see $\lim_{t\to\infty}\|u(t+\omega)-u(t)\| = 0$, which implies that $u(t)$ is a S-asymptotically ω-periodic mild solution for $t \geq 0$. Hence, we know that the nonlocal problem (1.1) has at least a S-asymptotically ω-periodic positive mild solution u in $G([-r,\infty),K)$. This completes the proof of the theorem. □

Theorem 3.6. *Let E be an ordered Banach space, whose positive cone $K \subset E$ is normal, Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_k \geq 0, k = 1,2,\dots n$ be given. A is an κ-sectorial operator of angle $\dfrac{\gamma_k \pi}{2}, k = 1,2,\cdots,n$ with $\kappa < 0$, and A generates positive and equicontinuous (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$ on E. Assume that $\omega > 0$ is a constant, $\varphi \in K_{\mathcal{B}}$, $Q(u)(0)+\varphi(0) \in K \cap \mathcal{D}(A)$ and $\psi \in K$, $F : \mathbb{R}^+ \times K \times K_{\mathcal{B}} \to E$, $Q, Q_0 : G([-r,+\infty),K) \to K$ are continuous and $F(t,\theta,\theta) \geq \theta$ for $t \geq 0$. If the condition (H2), (H8), (H11) and the following conditions are established:*

$(H13)$ For any bounded sets $D \subset E$, $\mathbb{D} \subset \mathcal{B}$, the set $\{F(t,x,\phi) | t \geq 0, x \in D, \phi \in \mathbb{D}\}$ is bounded, and

$$\lim_{t\to\infty}\|F(t+\omega,x,\phi)-F(t,x,\phi)\| = 0$$

for all $x \in E, \phi \in \mathcal{B}$,

$(H14)$ The nonlocal functions $Q, Q_0(u)$ are bounded increasing for $u \in G([-r,+\infty),K)$, that is, for any $v_1, v_2 \in G([0,+\infty),E)$ with $v_2 \geq v_1 \geq \theta$, $\|v_1\| \leq R$,

$$Q_0(v_2) \geq Q_0(v_1) \geq \theta,\ Q(v_2) \geq Q(v_1) \geq \theta.$$

Then the nonlocal problem (1.1) has at least a S-asymptotically ω-periodic positive mild solution $u \in G([-r,\infty),K)$.

Proof. Let R be any positive constant. For given $\varphi \in K_{\mathcal{B}}$, $\|\varphi\|_{\mathcal{B}} \leq R$. Define

$$\Omega_R = \big\{u \in C([-r,\infty),K) \big| \|u(t)\| \leq R, t \in \mathbb{R}^+; u|_{[-r,0]} \in \mathcal{B}, u(t) = \varphi(t), t \in [-r,0]\big\},$$

and the operator $\mathcal{Q} : \Omega_R \to K$ by

$$(\mathcal{Q}u)(t) = \begin{cases} S_{\beta,\gamma_k}(t)[Q(u)(0)+\varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\ +\sum_{k=1}^{n}\alpha_k(\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k})(t)[Q(u)(0)+\varphi(0)] \\ +\int_0^t T_{\beta,\gamma_k}(t-s)F(s,u(s),u_s)dg(s), \qquad t \in [0,a], \\ \\ Q(u)(t)+\varphi(t), \qquad t \in [-r,0]. \end{cases} \tag{3.24}$$

From the hypothesis (H8), (H13), the positivity of $S_{\beta,\gamma_k}(t)(t \geq 0)$ and the definition of Ω_R, it follows that $\mathcal{Q} : \Omega_R \to E$ is well defined. Hence, if u is a fixed point of $\mathcal{Q}$ on Ω_R, then u is undoubtedly a mild solution of nonlocal problem (1.1). Let

$$R_f = \max_{t\geq 0, \|u(t)\|, \|u\|_{\mathcal{B}} \leq R} \|F(t, u(t), u_t)\|, \quad R_g = \max_{\|u\|_\infty \leq R} \|Q_0(u)\|, \; R_h = \max_{\|u\|_\infty \leq R} \|Q(u)\|.$$

Step.1. we check that there is a constant $R_0 > 0$ such that $\mathcal{Q}(\Omega_{R_0}) \subset \Omega_{R_0}$.

In view of (2.4), we observe that as $M_1 := \sup_{t\geq 0} \|\varphi_{1+\beta-\gamma_k} * S_{\beta,\gamma_k}(t)\| < +\infty$.

Indeed, if this were not so, it would follows that for any $R > 0$, there exists $u \in \Omega_R$ such that $\|\mathcal{Q}u\| > R$. In view of (2.5) and (3.24), for any $t \geq 0$, we have

$$\begin{aligned}
\|(\mathcal{Q}u)(t)\| &\leq \|S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)] + (\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)] \\
&\quad + \sum_{k=1}^{n} \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0) + \varphi(0)]ds \\
&\quad + \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s)\| \\
&\leq \|S_{\beta,\gamma_k}(t)[Q(u)(0) + \varphi(0)]\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\| \\
&\quad + \Big\| \sum_{k=1}^{n} \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0) + \varphi(0)]ds \Big\| \\
&\quad + \Big\| \int_0^t T_{\beta,\gamma_k}(t-s)F(s, u(s), u_s)dg(s) \Big\| \\
&\leq \|S_{\beta,\gamma_k}(t)\varphi(0)\| + \|(\varphi_1 * S_{\beta,\gamma_k})(t)[\psi + Q_0(u)]\| \\
&\quad + \| \sum_{k=1}^{n} \alpha_k \int_0^t \frac{(t-s)^{\beta-\gamma_k}}{\Gamma(1+\beta-\gamma_k)} S_{\beta,\gamma_k}(s)[Q(u)(0) + \varphi(0)]ds \| \\
&\quad + \int_0^t \|T_{\beta,\gamma_k}(t-s)\| \cdot \|F(s, u(s), u_s)\|dg(s) \\
&\leq M(\|\varphi\|_{\mathcal{B}} + R_h) + \widetilde{M}(\|\psi\| + R_g) + \Big(\sum_{k=1}^{n} \alpha_k M_1 \Big)(\|\varphi\|_{\mathcal{B}} + R_h) \\
&\quad + CR_f \cdot \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s) \Big)^{\frac{1}{q}} \\
&\leq (M + \sum_{k=1}^{n} \alpha_k M_1)(\|\varphi\|_{\mathcal{B}} + R_h) + \widetilde{M}(\|\psi\| + R_g) \\
&\quad + CR_f \sup_{t\geq 0} \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s) \Big)^{\frac{1}{q}}.
\end{aligned}$$

Hence, according to the above calculation, we can see

$$R < (M + \sum_{k=1}^{n} \alpha_k M_1)(\|\varphi\|_{\mathcal{B}} + R_h) + \widetilde{M}(\|\psi\| + R_g) + CR_f \sup_{t\geq 0} \Big(\int_0^t (t-s)^{q(\beta-\gamma_k)} dg(s) \Big)^{\frac{1}{q}}.$$

Dividing both sides by R and taking the lower limit as $R \to \infty$, we can get the obvious contradiction of $1 < 0$. Thus, there is a constant $R_0 > 0$ such that $\mathcal{Q}(\Omega_{R_0}) \subset \Omega_{R_0}$.

Step.2. The set $\{\mathcal{Q}u : u(\cdot) \in \Omega_R\}$ is equiregulated.

For any $b \in (0, \infty)$, restrict $u(t)$ to interval $[-r, b)$. For any $t_0 \in [-r, b)$, we have

$$\begin{aligned}
\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\| &\leq \|(S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+))[Q(u)(0) + \varphi(0)]\| \\
&+ \|[(\varphi_1 * S_{\beta,\gamma_k})(t) - (\varphi_1 * S_{\beta,\gamma_k})(t_0^+)][\psi + Q_0(u)]\| \\
&+ \sum_{k=1}^{n} \frac{\alpha_k M}{\Gamma(1+\beta-\gamma_k)} \Big| \int_0^t (t-s)^{\beta-\gamma_k} ds - \int_0^{t_0^+} (t_0^+ - s)^{\beta-\gamma_k} ds \Big| \|Q(u)(0) + \varphi(0)\| \\
&+ \int_0^{t_0^+} \|[T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)]F(s, u(s), u_s)\| dg(s) \\
&+ \int_{t_0^+}^{t} \|T_{\beta,\gamma_k(t-s)}(t-s)F(s, u(s), u_s)\| dg(s) \\
&\leq \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(\mathbb{E})} \cdot (\|\varphi(0)\| + R_h) + M|t - t_0^+| \cdot (\|\psi\| + R_g) \\
&+ \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \|\varphi(0)\| + R_h\| \\
&+ R_f \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} dg(s) \\
&+ CR_f \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} dg(s) \\
&= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t), \qquad (3.25)
\end{aligned}$$

where

$$I_1(t) = \|S_{\beta,\gamma_k}(t) - S_{\beta,\gamma_k}(t_0^+)\|_{L(E)} \cdot \|\varphi(0) + R_h\|,$$

$$I_2(t) = M|t - t_0^+| \cdot (\|\psi\| + R_g),$$

$$I_3(t) = \sum_{k=1}^{n} \alpha_k M \Big| \frac{t^{1+\beta-\gamma_k} - (t_0^+)^{1+\beta-\gamma_k}}{\Gamma(2+\beta-\gamma_k)} \Big| \cdot \|\varphi(0) + R_h\|,$$

$$I_4(t) = R_f \int_0^{t_0^+} \|T_{\beta,\gamma_k}(t-s) - T_{\beta,\gamma_k}(t_0^+ - s)\|_{L(E)} dg(s),$$

$$I_5(t) = CR_f \int_{t_0^+}^{t} (t-s)^{\beta-\gamma_k} dg(s).$$

From the expression of $I_2(t)$ and $I_3(t)$, we derive that $I_2(t) \to 0$ and $I_3(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Since the compactness of $S_{\beta,\gamma_k}(t)$ and $T_{\beta,\gamma_k}(t)$ for $t > 0$ yields the continuity in the sense of uniform operator topology. We dedude that $I_1(t) \to 0$ and applying dominated convergence theorem on $I_4(t)$ and , we can derive that $I_4(t) \to 0$ as $t \to t_0^+$ independently of $u \in \Omega$. Let $H(t) = \int_0^t (t-s)^{\beta-\gamma_k} dg(s)$. Thanks to Lemma 2.4, we known that $H(t)$ is a regulated function on $\mathbb{R}^+$. Therefore, we have

$$\begin{aligned} I_5(t) &= CR_f \int_{t_0^+}^t (t-s)^{\beta-\gamma_k} dg(s) \\ &\le CR_f \Big(\|H(t) - H(t_0^+)\| + \int_0^{t_0^+} \|((t-s)^{\beta-\gamma_k} - (t_0^+ - s)^{\beta-\gamma_k})\| dg(s) \Big) \\ &\to 0 \ \text{ as } \ t \to t_0^+ \ \text{ independently of } \ u. \end{aligned}$$

Therefore, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. independently of $u \in \Omega$.

Similarly, one can demonstrate that for any $t_0 \in (-r, b]$, $\|(\mathcal{Q}u)(t) - (\mathcal{Q}u)(t_0^+)\|_\Omega \to 0$ as $t \to t_0^+$. According to the arbitrariness of b, one can find that $u(t)$ is defined on $[0, \infty)$. On the other hand, it is easy to see $\lim_{t\to\infty} \|u(t+\omega) - u(t)\| = 0$. Hence, assert that $\{\mathcal{Q}u : u(\cdot) \in \Omega_R\}$ is equiregulated.

Step.3. We finally show that the operator $\mathcal{Q}$ has a positive fixed point on Ω_{R_0}.

In view of (H1), (H8) and (H14), we verify that (H8)-(H10) hold, we know that $\mathcal{Q} : \Omega_{R_0} \to \Omega_{R_0}$ is a monotonic increasing operator based on the Step 3 in the proof Theorem 3.5.

Let $v^0 = \theta \in K$ and establish the iterative sequence $\{v^{(n)}\}$ by

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad n = 1, 2, \cdots . \tag{3.26}$$

Then according to the monotonicity of $\mathcal{Q}$, one can find $\{v^{(n)}\} \subset K$ and

$$\theta = v^{(0)} \le v^{(1)} \le \cdots \le v^{(n)} \le \cdots . \tag{3.27}$$

Similar to the proof of Theorem 3.2, we can get $\alpha(\{v^{(n)}(t)\}) \equiv 0$ in $[-r, a]$, that is, $\{v^{(n)}(t)\}$ is precompact, hence, it has a convergent subsequence $v^{(n_k)} \to u \in \Omega_1$, combined with its monotonicity (3.27) and the normality of cone K, it is easy to know that

$$v^{(n)} \to u \in G([-r, a], K), \qquad n \to \infty.$$

Taking limit of both ends of (3.26), and by the continuity of $\mathcal{Q}$, we can get $u = \mathcal{Q}u$, which shows that $u \in G([-r, a], K)$ is a positive mild solution of the nonlocal problem (1.1).

According to the arbitrariness of a, one can find that $u(t)$ is defined on $[-r,\infty)$. On the other hand, by the method of Step 3 of Theorem 3.1, it is easy to see $\lim_{t\to\infty}\|u(t+\omega)-u(t)\|=0$, which implies that $u(t)$ is a S-asymptotically ω-periodic mild solution for $t\geq 0$. Hence, we know that the nonlocal problem (1.1) has at least a S-asymptotically ω-periodic positive mild solution u in $G([-r,\infty),K)$. This completes the proof of the theorem. □

4 Applications

In this section, we give an example to illustrate our main results. Let

$\beta,\gamma_k>0(k=1,2,\ldots,n)$ be such that $0<\beta\leq\gamma_n\leq\cdots\leq\gamma_1\leq 1$. Consider the following measure driven differential equation:

$$\begin{cases} {}^cD_{0+}^{1+\beta}u(t,x)+\sum\limits_{k=1}^{n}\alpha_k{}^cD^{\gamma_k}u(t,x)=\Delta u(t,x)+\tau u(t,x)\\ +\frac{\sin(u(t+s))}{1+e^{2t}}dg(t),\quad (t,x)\in\mathbb{R}^+\times[0,\pi],\ s\in[-r,0],\\ u(t,0)=u(t,\pi)=0,\quad t\in\mathbb{R}^+,\\ u(t,x)=\int_0^a\rho(t,s)\log(1+|u(s,x)|)ds+\varphi(t,x),\quad (t,x)\in[-r,0]\times[0,\pi],\\ \frac{\partial u(t,x)}{\partial t}|_{t=0}=\frac{|u(t,x)|}{6+|u(t,x)|}+\psi(x),\quad x\in[0,\pi], \end{cases} \tag{4.1}$$

where Δ is Laplace operator, $a>0,\tau<0$ are constant, $g:[0,\pi]\to\mathbb{R}$ is a nondeacresing, left continuous function, $\rho(t,s)$ is a continuous function from $[0,\infty)\times[-r,0]$ to $\mathbb{R}^+$. Furthermore, define the operator $A:\mathcal{D}(A)\subset E\to E$ by $Au=\Delta u+\tau u$ and

$$\mathcal{D}(A)=\{u\in E: u,u'\quad \text{are absolutely continuous}, u''\in E, u(0)=u(\pi)=0\}.$$

Then it is well known that the operator A is κ-sectorial with $\kappa=\tau<0$ and angle $\frac{\pi}{2}$ (and hence of angle $\frac{\gamma_k\pi}{2}$) for all $\gamma_k\leq 1,k=1,2,\cdots,n$). Since $\beta,\gamma_k>0,k=1,2,\cdots,m$ be such that $0<\beta\leq\gamma_m\leq\cdots\leq\gamma_1\leq 1$, by Lemma 2.7 , we deduce that A generates a bounded (β,γ_k)-resolvent family $\{S_{\beta,\gamma_k}(t)\}_{t\geq 0}$.

We choose the workspace $E=L^2([0,\pi],\mathbb{R})$, which is an ordered Banach space with L^2-norm $\|\cdot\|_2$ and partial-order " $\leq$ ", $K=\{u\in L^2([0,\pi],\mathbb{R}):u(x)\geq 0,\ a.e.\ x\in[0,\pi]\}$ is a normal cone. Note $\mathcal{B}:=G([-r,0]\times[0,\pi],E)$ with the normal cone $K_{\mathcal{B}}=\{u\in\mathcal{B}: u(t,x)\in K, t\in[-r,0],\ a.e.\ x\in[0,\pi]\}$. We define

$$f(t,x,u(t,x),u(t+s,x))=\frac{\sin(u(t+s))}{1+e^{2t}},\quad t\in\mathbb{R}^+,\ s\in[-r,0].$$

$$Q_0(u(t,x))=\frac{|u(t,x)|}{6+|u(t,x)|},\ Q(u(t,x))=\int_0^a\rho(t,s)\log(1+|u(s,x)|)ds.$$

For $u \in [0,\pi]$, we set $\varphi(t) = \varphi(t,\cdot),\ \psi = \psi(\cdot),\ u(t) = u(t,\cdot),\ u_t(s) = u(t+s,\cdot)$ and

$$F(t,u(t),u_t) = f(t,\cdot,u(t,\cdot),u(t+s,\cdot)), \quad Q_0(u) = \frac{|u|}{6+|u|}, Q(u) = \int_0^a \rho(t,s)\log(1+|u|)ds.$$

Then, equation (4.1) can be transformed into the form of abstract nonlocal problem (1.1) in $L^2([0,\pi],\mathbb{R})$.

Further, from the definition of functions f and Q_0, we have

$$\|F(t,u(t),u_t)\| \le \frac{1}{2}\|u\|, \quad \|Q_0(u(t,x))\| \le \frac{1}{6}\|u\|, \|Q(u(t,x))\| \le \int_0^a \rho(t,s)ds\|u\|.$$

We deduce that condition (H4) is satisfied with $c_0 = \frac{1}{6}$ and $d_0 = 0$, $c_1 = \int_0^a \rho(t,s)ds$ and $d_1 = 0$. Additionally, (H1) is satisfied with $P(t) = \frac{1}{2}$ and $W(r) = r$.

Theorem 4.1. *Assume that $\omega > 0$, $f : \mathbb{R}^+ \times [0,\pi] \times K \times K_{\mathcal{B}} \to E$ be continuous and the conditions (H2) is satisfied. If the following conditions*

(A1) $f(t,x,0,0) \ge 0$ for $(t,x) \in \mathbb{R}^+ \times [0,\pi]$, and there is a function $0 \le w = w(t,x) \in G([-r,\infty) \times [0,\pi])$ satisfying $\lim_{t\to\infty} w(t+\omega,\cdot) - w(t,\cdot) = 0$, such that

$$\begin{cases} {}^cD_{0+}^{1+\beta} w(t,x) + \sum\limits_{k=1}^{n} \alpha_k {}^cD^{\gamma_k} w(t,x) \ge \Delta u(t,x) + \tau u(t,x) & \\ +f(t,x,w(t,x),w(t+s,x))dg(t), & (t,x) \in \mathbb{R}^+ \times [0,\pi], s \in [-r,0], \\ w(t,0) = w(t,\pi) = 0, & t \in \mathbb{R}^+, \\ w(t,x) \ge \int_0^a \rho(t,s)\log(1+|w(s,x)|)ds + \varphi(t,x), & (t,x) \in \mathbb{R}^+ \times [0,\pi], \\ \frac{\partial w(x,0)}{\partial t} \ge Q_0(w(t,x)) + \psi(x), & x \in [0,\pi]. \end{cases}$$

(A2) there exists a constant $l > 0$ such that for any $x \in [0,\pi], t \in \mathbb{R}^+$ and $0 \le x_1 \le x_2 \le w(\cdot,t), 0 \le \phi_1 \le \phi_2 \le w_t$,

$$f(t,x_2,\phi_2) - f(t,x_1,\phi_1) \ge \theta$$

hold, then all the conditions in Theorem 3.1 are satisfied, our results can be applied to system (4.1). Also, the problem (4.1) has minimal and maximal S-asymptotically ω-periodic solutions $\underline{u}, \overline{u} \in G([-r,\infty), L^2([0,\pi],\mathbb{R}) \cap SAP_\omega(L^2([0,\pi],\mathbb{R}))$ between 0 and w, which can be obtained by monotone iterative sequences starting from 0 and ω.

Proof. From the condition (A1), it follows that $v_0 \equiv 0$ and $w_0 = w(x,t) \ge 0$ are lower and upper S-asymptotically ω-periodic mild solutions of the problem (4.1), respectively.

Thus, by the condition (A2), one can find that the condition (H2) holds. Therefore, from Theorem 3.1, we can obtain that the problem (4.1) has minimal and maximal time S-asymptotically ω-periodic mild solutions $\underline{u}, \overline{u} \in G([-r,\infty), E) \cap SAP_\omega(E)$, which can be obtained by monotone iterative sequences starting from 0 and w, respectively. □

5 Conclusions

This paper has established some of results concerning the existence of maximal and minimal S-asymptotically ω-periodic mild solutions for this class of multi-term time-fractional measure differential equations in order Banach space, by means of the method of lower and upper solution. In the furture work, we study the existence of the S-asymptotically ω-periodic mild solutions for a class of multi-term time-fractional measure differential equations involving non-instantaneous in Banach spaces.

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Conflict of interest

The author declares that he has no conflict of interest.

Availability of data and materials

My manuscript has no associate data.

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