Solvability and Optimal Controls of Fractional Impulsive Stochastic Evolution Equations with Nonlocal Conditions

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ARTICLE HISTORY
Compiled December 5, 2023

ABSTRACT
This paper deals with the solvability and optimal controls of a class of impulsive fractional stochastic evolution equations with nonlocal initial conditions in a Hilbert space. Firstly, the existence and uniqueness of mild solutions for the considered system are investigated. Then, we derive the existence conditions of optimal pairs to the control systems. In the end, an example is presented to illustrate the effectiveness of our abstract results.

KEYWORDS
Nonlocal problem; Impulsive; Optimal control; Stochastic evolution equations

AMS CLASSIFICATION
49J15; 60H15; 47J35

1. Introduction

The purpose of this paper is to study the solvability and optimal controls to the following nonlinear time fractional evolution equations with impulsive and nonlocal initial conditions

$$\begin{cases} {}^cD_t^\alpha x(t)+Ax(t)=f(t,x(t))+\sigma(t,x(t))\frac{dW(t)}{dt} \\ \qquad +B(t)u(t), \quad t\in[0,\ b],\ t\neq t_i,\ i=1,2,\cdots,p, \\ x(0)=g(x), \\ \Delta x(t_i)=I_i(x(t_i)), \quad i=1,2,\cdots,p, \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivatives of order $\frac{1}{2}<\alpha<1$,

$$g(x)=\int_0^b h(s,x(s))ds. \tag{1.2}$$

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$0 < t_1 < t_2 < \cdots < t_p < b$, I_i is an impulsive function, $i = 1, 2, \cdots, p$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+)$, $x(t_i^-)$ denote the right and the left limit of x at t_i , respectively. The state $x(\cdot)$ takes values in the separable Hilbert space $\mathbb{H}$, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator and $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms of $\mathbb{H}$ and $\mathbb{K}$, and use $\langle \cdot, \cdot \rangle$ to denote inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. We are also using the same notation $\|\cdot\|$ for the norm of $L(\mathbb{K}, \mathbb{H})$, which denotes the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$. Suppose that $\{W(t) : t \geq 0\}$ is a given $\mathbb{K}$-valued Wiener process or Brownian motion with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The control function $u(\cdot)$ takes values in another separable reflexive Hilbert space $\mathbb{U}$, $B : \mathbb{U} \to \mathbb{H}$ is a linear operator. f, σ, I_i and h are appropriate functions to be given later.

In the past two decades, stochastic differential systems have attracted great interest because of their practical applications in many areas, such as economics, physics, population dynamics, chemistry, medicine biology, social sciences and other areas of science and engineering. For more details, we refer to the books by Mao[18], Da Prato and Zabczyk [20], Sobczyk [30], Grecksch and Tudor [31] and Liu [32]. One of the branches of stochastic differential equations is the theory of fractional stochastic evolution equations. Many researchers investigated the existence, uniqueness, controllability and asymptotic behavior of mild solutions to fractional stochastic evolution equations by using different approaches, see [1, 2, 4-10, 12-17, 19, 21-25, 28] and the references therein.

The theory of impulsive differential equations describes processes which experience a sudden change in their states at certain moments. For the basic theory on impulsive differential equations, the reader can refer to the monographs of Bainov and Simeonov [33], Benchohra et al. [34] and Lakshmikantham et al. [35]. Particularly, impulsive fractional evolution equations in Banach spaces has been emerging as an important area of investigation in the last few decades. For more details on this theory and its applications, we refer to the the references [3, 6, 7, 9-10, 12, 21, 36-44]. Some works [6, 7, 9, 10, 12, 21] considered fractional stochastic evolution equation with impulsive. For example, Balasubramaniam et al.[21] investigated a class of impulsive fractional stochastic integro-differential equations in Hilbert space of the form

$$\begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + J_t^{1-\alpha}[B(t)u(t) + f(t, x(t), x(a_1(t)), x(a_2(t)), \cdots, x(a_m(t)))] \\ \qquad +J_t^{1-\alpha}\big(\int_0^t g(s, x(s), x(b_1(s)), x(b_2(s)), \cdots, x(b_m(s))dws)\big) \quad t \in [0, b],\ t \neq t_i, \\ x(0) = x_0, \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \cdots, p, \end{cases}$$

they obtained the existence of mild solution and optimal controls for the considered system. Dhayal et al.[10] studied the existence of optimal multicontrol pairs for a class of noninstantaneous impulsive fractional stochastic differential systems driven by the Rosenblatt process with state-dependent delay.

It is well known that the study of abstract nonlocal Cauchy problem was initiated by Byszewski and Lakshmikantham [45]. Since the nonlocal initial condition have better effects in applications than the classical initial condition, differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [2, 3, 16, 23, 24, 27, 36, 37, 44, 46-48]. However, to the best of our knowledge, we have not seen the relevant papers to study the optimal control of system governed by fractional impulsive stochastic evolution equations with nonlocal conditions. Due to its importance in both theoretical and real-life applications point of view, it is significant to investigate its existence, controllability, and other quantitative properties.

Inspired by the above discussions, in this paper, we first prove the existence and uniqueness of mild solution for fractional impulsive stochastic evolution equations with nonlocal conditions (1.1). Secondly, the existence of fractional optimal controls for (1.1) is investigated. The obtained results are new and considered as a contribution to the theory of fractional impulsive stochastic optimal control problem.

The rest of this paper is organized as follows: In Section 2, we give some definitions and preliminary results to be used in this paper. In Section 3, the existence and uniqueness of mild solutions are proved. Existence of fractional optimal controls is shown in Section 4. Finally, In Section 5, an example is provided to illustrate the applications of the obtained results.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P-null sets. Let $\{e_k, k \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathbb{K}$. $\{W(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Brownian motion or Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a finite trace nuclear covariance operator $Q \geq 0$, we denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty$, which satisfies that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$. Let $\{W_k(t), k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ such that

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} W_k(t) e_k, \ t \geq 0.$$

Forthermore, we assume that $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\}$ is the σ-algebra generated by W and $\mathcal{F}_b = \mathcal{F}$. Let $L_2^0 = L_2(Q^{\frac{1}{2}}\mathbb{K}, \ \mathbb{H})$ denote the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle \phi, \varphi \rangle = Tr(\phi Q \varphi^*)$. It also turns out to be a separable Hilbert space. The collection of all $\mathcal{F}_b$-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted $L^2(\Omega, \mathbb{H})$, is a Banach space equipped with the norm $\|x\|_{L^2} = (\mathbb{E}\|x(\omega)\|^2)^{\frac{1}{2}}$, where $\mathbb{E}$ denotes the expectation with respect to the measure $\mathbb{P}$. For more details on stochastic integrals, see the books of [18, 20].

Let $C([0, b], L^2(\Omega, \mathbb{H}))$ be the Banach space of all continuous mappings from $[0, b]$ to $L^2(\Omega, \mathbb{H})$ with the norm $\|x\|_C = (\sup_{t\in[0,b]} \mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}$. Let

$$\begin{aligned} PC([0, b], L^2(\Omega, \mathbb{H})) = \{x : [0, b] \to L^2(\Omega, \mathbb{H}), x(t) \text{ is continuous at } t \neq t_i, \\ \text{left continuous at } t = t_i, \text{ and the right limit } x(t_i^+) \\ \text{exists for } i = 1, 2, \cdots, p\}. \end{aligned}$$

$\mathcal{PC}([0, b], L^2(\Omega, \mathbb{H}))$ be the space of all $\mathcal{F}_t$-adapted measurable stochastic processes $x \in PC([0, b], L^2(\Omega, \mathbb{H}))$ with the norm $\|x\|_{\mathcal{PC}} = (\sup_{t\in[0,b]} \mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}$. It is easy to see that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space. We suppose that $\mathbb{U}$ is a separable reflexive Hilbert space from which the controls u take the values. Let

$$L^2_{\mathcal{F}}(J,\ \mathbb{U}) = \{u : J \times \Omega \to \mathbb{U} :\ u \text{ is } \mathcal{F}_{\mathrm{t}} - \text{adapted measurable stochastic processes and } \mathbb{E}\int_0^{\mathrm{b}} \|\mathrm{u}(\mathrm{t})\|^2 \mathrm{dt} < \infty\}.$$

Let Y be a nonempty closed bounded convex subset of $\mathbb{U}$. Define the admissible control set

$$U_{ad} = \{u(\cdot) \in L^2_{\mathcal{F}}(J,\ \mathbb{U})|\ u(t) \in Y, t \in J\}.$$

We assume that the control function $u \in U_{ad}$ and $B \in L^\infty(J, L(\mathbb{U}, \mathbb{H}))$, $\|B\|_\infty$ stands for the norm of operator B on Banach space $L^\infty(J, L(\mathbb{U}, \mathbb{H}))$, where $L^\infty(J, L(\mathbb{U}, \mathbb{H}))$ is the space of operator valued functions which are measurable in the strong operator topology and uniformly bounded on the interval J.

In the rest of the manuscript, we suppose that A generates a compact C_0-semigroup $T(t)(t \geq 0)$ of uniformly bounded linear operator in $\mathbb{H}$. That is there exists a positive constant $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

The following result will be used in the sequel of this paper.

Lemma 2.1. *(see [20]) For any $p \geq 1$ and for arbitrary L^0_2-valued predictable process $\chi(\cdot)$ such that*

$$\sup_{s\in[0,t]} \mathbb{E}\Big\| \int_0^s \chi(\tau) dW(\tau)\Big\|^{2p} \leq (p(2p-1))^p \Big(\int_0^t (\mathbb{E}\big\|\chi(s)\big\|^{2p}_{L^0_2})^{\frac{1}{p}} ds\Big)^p,\ t \in [0,\ \infty).$$

Definition 2.2. (see [26]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\ +\infty) \to \mathbb{R}$ is given by

$$I^\alpha_0 y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.3. (see [26]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : [0,\ +\infty) \to \mathbb{R}$ is given by

$$D^\alpha_0 y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.4. (see [26]) The Caputo fractional derivative of order $\alpha > 0$ of a function $y : [0,\ +\infty) \to \mathbb{R}$ is given by

$${}^c D^\alpha_0 y(t) = D^\alpha_0 \big[y(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0)\big],$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Remark 2.5. (i) If $y(t) \in C^n[0, +\infty)$, then

$$ {}^cD_0^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds = I_0^{n-\alpha} y^{(n)}(t). $$

(ii) If $y(t)$ is an abstract function, then the integrals which appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

For $x \in \mathbb{H}$, define two operators $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$ as follows:

$$ \mathscr{T}(t)x = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta) x d\theta, \quad \mathscr{S}(t)x = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\alpha \theta) x d\theta, \tag{2.1} $$

where

$$ \zeta_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \rho_\alpha(\theta^{-1/\alpha}), $$

$$ \rho_\alpha(\theta) = \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty). $$

$\zeta_\alpha(\theta)$ is a probability density function defined on $(0, +\infty)$ which satisfies

$$ \zeta_\alpha(\theta) \geq 0, \ \theta \in (0, \infty), \quad \int_0^\infty \zeta_\alpha(\theta) d\theta = 1, \quad \int_0^\infty \theta \zeta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}. \tag{2.2} $$

The following lemma about the operators $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$, which can be found in [27], will be used throughout this paper.

Lemma 2.6. *The operators $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$ satisfy the following properties:*

(i) For any fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators in $\mathbb{H}$, i.e., for any $x \in \mathbb{H}$,

$$ \| \mathscr{T}(t)x \| \leq M \| x \|, \quad \| \mathscr{S}(t)x \| \leq \frac{M}{\Gamma(\alpha)} \| x \| . \tag{2.3} $$

(ii) For every $x \in \mathbb{H}$, $t \to \mathscr{T}(t)x$ and $t \to \mathscr{S}(t)x$ are continuous functions from $[0, \infty)$ into $\mathbb{H}$.

(iii) The operators $\mathscr{T}(t)(t \geq 0)$ and $\mathscr{S}(t)(t \geq 0)$ are strongly continuous.

(iv) If the semigroup $T(t)$ is compact, then $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators in $\mathbb{H}$ for $t > 0$, and hence they are norm continuous.

Lemma 2.7. *(Krasnoselskii's Fixed Point Theorem, see[27]). Let $\mathbb{X}$ be a Banach space, let Y be a bounded closed and convex subset of $\mathbb{X}$ and let F_1, F_2 be maps of Y into $\mathbb{X}$ such that $F_1x + F_2y \in Y$ for every pair x, $y \in Y$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on Y.*

According to [19, 42], we adopt the following definition of the mild solution of (1.1).

Definition 2.8. For any given $u \in U_{ad}$, a stochastic process x is said to be a mild solution of (1.1) on $[0,b]$ if $x \in \mathcal{PC}([0,b], L^2(\Omega,\mathbb{H}))$ and satisfies
(i) $x(t)$ is measurable and adapted to $\mathcal{F}_t$;
(ii)$x(t)$ satisfies the following integral equation

$$\begin{aligned}x(t) = \mathscr{T}(t)g(x) &+ \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)[f(s,\ x(s)) + B(s)u(s)]ds \\ &+ \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\sigma(s,\ x(s))dW(s) + \sum_{0<t_i<t} \mathscr{T}(t-t_i)I_i(x(t_i)).\end{aligned}$$

3. Existence and Uniqueness of Mild Solution

To prove the main results, we list some assumptions:
(H1) Let $f : [0,\ b] \times \mathbb{H} \to \mathbb{H}$ be a continuous function. Suppose also that the following assumptions are satisfied:
(i) There exists a constant L_f such that

$$\|f(t,\ x)\|^2 \le L_f(1+ \parallel x \parallel^2),\ t \in J, \quad x \in \mathbb{H}.$$

(ii) For some $r > 0$, there exists a constant $\overline{L}_f$ such that for all $t \in J$ and $x,\ y \in \mathbb{H}$ satisfying $\|x\|^2 \le r,\ \|y\|^2 \le r$,

$$\|f(t,\ x) - f(t,\ y)\|^2 \le \overline{L}_f \parallel x - y \parallel^2).$$

(H2) Let $\sigma : J \times \mathbb{H} \to L_2^0$ be a continuous function. Suppose also that the following assumptions are satisfied:
(i) There exists a constant L_σ such that

$$\|\sigma(t,\ x)\|_{L_2^0}^2 \le L_\sigma(1+ \parallel x \parallel^2),\ t \in J, \quad x \in \mathbb{H}.$$

(ii) For some $r > 0$, there exists a constant $\overline{L}_\sigma$ such that for all $t \in J$ and $x,\ y \in \mathbb{H}$ satisfying $\|x\|^2 \le r,\ \|y\|^2 \le r$,

$$\|\sigma(t,\ x) - \sigma(t,\ y)\|_{L_2^0}^2 \le \overline{L}_\sigma \parallel x - y \parallel^2 .$$

(H3) Let $h : J \times \mathbb{H} \to \mathbb{H}$ be a continuous function. Suppose also that the following assumptions are satisfied:
(i) There exists a constant L_h such that

$$\|h(t,\ x)\|^2 \le L_h(1+ \parallel x \parallel^2),\ t \in J, \quad x \in \mathbb{H}.$$

(ii) For some $r > 0$, there exists a constant $\overline{L}_h$ such that for all $t \in J$ and $x,\ y \in \mathbb{H}$ satisfying $\|x\|^2 \le r,\ \|y\|^2 \le r$,

$$\|h(t,\ x) - h(t,\ y)\|^2 \le \overline{L}_h \parallel x - y \parallel^2 .$$

(H4) Let $I_i : \mathbb{H} \to \mathbb{H}$ be a continuous function for every $i = 1, 2, \cdot\cdot\cdot, p$. Suppose also that the following assumptions are satisfied:

(i) There exist constants M_i $(i = 1, 2, \cdots, p)$ such that

$$\|I_i(x)\|^2 \le M_i \|x\|^2, \quad i = 1, 2, \cdots, p, \quad x \in \mathbb{H}.$$

(ii) For some $r > 0$, there exist constants $\overline{M}_i$ $(i = 1, 2, \cdots, p)$ such that for all $t \in J$ and $x,\ y \in \mathbb{H}$ satisfying $\|x\|^2 \le r,\ \|y\|^2 \le r$,

$$\|I_i(x) - I_i(y)\|^2 \le \overline{M}_i \|x - y\|^2 \ i = 1, 2, \cdots, p.$$

We are now ready to state our main results.

Theorem 3.1. *Assume that $-A$ generates a compact C_0-semigroup $T(t)(t \ge 0)$ of uniformly bounded operators in Hilbert space $\mathbb{H}$. If the assumptions (H1)(i), (H2)(i), (H3) and (H4) are satisfied, then impulsive fractional stochastic systems with nonlocal conditions (1.1) has at least one mild solution in $\mathcal{PC}([0, b], L^2(\Omega, \mathbb{H}))$ provided that*

$$N + M^2 p \sum_{i=1}^{p} M_i < \frac{1}{5} \tag{3.1}$$

and

$$M^2 b^2 \overline{L}_h + \ M^2 p \sum_{i=1}^{p} \overline{M}_i < \frac{1}{2} \tag{3.2}$$

are satisfied, where

$$N = M^2 b^2 L_h + c_0 b L_f + c_0 L_\sigma, \quad c_0 = \frac{M^2}{\Gamma^2(\alpha)} \cdot \frac{b^{2\alpha-1}}{2\alpha - 1}.$$

Proof. For any constant $r > 0$, let

$$B_r = \{x \in \mathcal{PC}([0, b], L^2(\Omega, \mathbb{H})) : \|x\|^2_{\mathcal{PC}} \le r\}.$$

It is easy to see that B_r is a bounded closed convex set in $\mathcal{PC}([0, b], L^2(\Omega, \mathbb{H}))$.

Define two operators F_1 and F_2 on B_r as follows:

$$\begin{aligned}
(F_1 x)(t) =& \mathscr{T}(t) g(x) + \sum_{0 < t_i < t} \mathscr{T}(t - t_i) I_i(x(t_i)), \quad t \in [0,\ b], \\
(F_2 x)(t) =& \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, x(s)) ds + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) B(s) u(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) \sigma(s, x(s)) dW(s), \quad t \in [0,\ b].
\end{aligned}$$

Obviously, x is a mild solution of (1.1) if and only if the operator equation $x = F_1 x + F_2 x$ has a solution.

Next we prove that $F_1 + F_2$ has a fixed point by Krasnoselskii's Fixed Point Theorem. For this, we proceed in several steps.

Step 1. We prove that there exists a positive number r_0 such that $F_1 x + F_2 y \in B_{r_0}$ whenever $x,\ y \in B_{r_0}$.

In fact, choose

$$r_0 \geq \frac{5\big(N + c_0\|B\|_\infty^2 \int_0^b \mathbb{E}\|u(s)\|^2 ds\big)}{1 - 5\big(N + M^2 p \sum_{i=1}^p M_i\big)},$$

then for every pair $x,\ y \in B_{r_0}$ and $t \in J$, by Lemma 2.1, Lemma 2.6, conditions (H1)(i), (H2)(i) and Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\|(F_1x)(t) + (F_2y)(t)\|^2 \\
\leq & 5\mathbb{E}\|\mathscr{T}(t)g(x)\|^2 + 5\mathbb{E}\bigg\|\sum_{0<t_i<t} \mathscr{T}(t-t_i)I_i(x(t_i))\bigg\|^2 \\
&+ 5\mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f(s,y(s))ds\bigg\|^2 \\
&+ 5\mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)B(s)u(s)ds\bigg\|^2 \\
&+ 5\mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\sigma(s,y(s))dW(s)\bigg\|^2 \\
\leq & 5M^2 b\int_0^b L_h(1+\mathbb{E}\|x(s)\|^2)ds + 5M^2 p\sum_{i=1}^p M_i(\mathbb{E}\|x(t_i)\|^2) \\
&+ 5c_0\int_0^t L_f(1+\mathbb{E}\|y(s)\|^2)ds + 5c_0\int_0^t \mathbb{E}\|B(s)u(s)\|^2 ds \\
&+ \frac{5M^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2\alpha-2}L_\sigma(1+\mathbb{E}\|y(s)\|^2)ds \\
\leq & 5M^2b^2L_h(1+r_0) + 5M^2pr_0\sum_{i=1}^p M_i + 5c_0bL_f(1+r_0) \\
&+ 5c_0\|B\|_\infty^2\int_0^b \mathbb{E}\|u(s)\|^2 ds + 5c_0L_\sigma(1+r_0) \\
= & 5\big(N + M^2 p\sum_{i=1}^p M_i\big)r_0 + 5\big(N + c_0\|B\|_\infty^2\int_0^b \mathbb{E}\|u(s)\|^2 ds\big) \\
\leq & r_0.
\end{aligned}$$

It then follows that $F_1 + F_2$ maps B_{r_0} to B_{r_0}.
Step 2. F_1 is a contraction on B_{r_0}.

For any $x,\ y \in B_{r_0}$ and $t \in J$, it follows from (H3) and (H4) that

$$\begin{aligned}
&\mathbb{E}\|(F_1x)(t) - (F_1y)(t)\|^2 \\
\leq & 2\mathbb{E}\bigg\|\mathscr{T}(t)\int_0^t [h(s,x(s)) - h(s,y(s))]ds\bigg\|^2 + 2\mathbb{E}\|\sum_{i=1}^p \mathscr{T}(t-t_i)[I_i(x(t_i) - I_i(y(t_i))]\|^2 \\
\leq & \big(2M^2b^2\overline{L}_h + 2M^2p\sum_{i=1}^p \overline{M}_i\big)\|x-y\|_{\mathcal{PC}}^2,
\end{aligned}$$

which implies that

$$\|F_1x - F_1y\|^2_{\mathcal{PC}} \leq \ 2\big(M^2b^2\overline{L}_h + \ M^2p\sum_{i=1}^{p}\overline{M}_i\big)\|x-y\|^2_{\mathcal{PC}}.$$

By (3.2), we easily see that F_1 is a contraction on B_{r_0}.

Step 3. F_2 is a completely continuous operator.

Firstly, we show that the mapping F_2 is continuous on B_{r_0}. For this purpose, let $x_m \to x$ in B_{r_0}, then we have

$$f(t,\ x_m(t)) \to f(t,\ x(t)),\quad \sigma(t,\ x_m(t)) \to \sigma(t,\ x(t)),\quad (m\to\infty).$$

Moreover, by Hölder inequality and Lebesgue dominated convergence theorem, we can get

$$\begin{aligned}
&\mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[f(s,\ x_m(s)) - f(s,\ x(s))\big]ds\bigg\|^2\\
\leq&\Big(\frac{M}{\Gamma(\alpha)}\Big)^2\int_0^t (t-s)^{2\alpha-2}ds\int_0^t \mathbb{E}\|f(s,\ x_m(s)) - f(s,\ x(s))\|^2 ds\\
\leq&\frac{b^{2\alpha-1}}{2\alpha-1}\Big(\frac{M}{\Gamma(\alpha)}\Big)^2\int_0^t \mathbb{E}\|f(s,\ x_m(s)) - f(s,\ x(s))\|^2 ds\\
\to&0\ (m\to\infty).
\end{aligned}$$

On the other hand, from Lemma 2.1, Hölder inequality and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[\sigma(s,\ x_m(s)) - \sigma(s,\ x(s))\big]dW(s)\bigg\|^2\\
\leq&\ \Big(\frac{M}{\Gamma(\alpha)}\Big)^2\int_0^t (t-s)^{2\alpha-2}\mathbb{E}\|\sigma(s,\ x_m(s)) - \sigma(s,\ x(s))\|^2 ds\\
\to&\ 0\ (m\to\infty).
\end{aligned}$$

By the above discuss, we obtain the following relation:

$$\begin{aligned}
&\mathbb{E}\|F_2(x_m) - F_2(x)\|^2\\
\leq&\ 2\ \mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[f(s,\ x_m(s)) - f(s,\ x(s))\big]ds\bigg\|^2\\
+&2\ \mathbb{E}\bigg\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)\big[\sigma(s,\ x_m(s)) - \sigma(s,\ x(s))\big]dW(s)\bigg\|^2\\
\to&\ 0\ (m\to\infty),
\end{aligned}$$

which means that $F_2(x)$ is continuous in B_{r_0}.

Secondly, we prove that for any $t\in J$, $V(t) = \{F_2(x)(t),\ x\in B_{r_0}\}$ is relatively compact in $\mathbb{H}$. It is obvious that $V(0)$ is relatively compact in $\mathbb{H}$. Let $0 < t \leq b$ be given. For any $\epsilon \in (0,\ t)$ and $\nu > 0$, define an operator $F^{\epsilon,\nu}$ on B_{r_0} by

$$
\begin{aligned}
&(F^{\epsilon,\nu}x)(t)\\
=&\alpha\int_0^{t-\epsilon}\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)[f(s,x(s))+B(s)u(s)]d\theta ds\\
&+\alpha\int_0^{t-\epsilon}\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)\sigma(s,x(s))d\theta dW(s)\\
=&T(\epsilon^\alpha\nu)\alpha\int_0^{t-\epsilon}\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta-\epsilon^\alpha\nu)[f(s,x(s))+B(s)u(s)]d\theta ds\\
&+T(\epsilon^\alpha\nu)\alpha\int_0^{t-\epsilon}\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta-\epsilon^\alpha\nu)\sigma(s,x(s))d\theta dW(s).
\end{aligned}
$$

Then the set $\{(F^{\epsilon,\nu}x)(t) :\ x \in B_r\}$ is relatively compact in $\mathbb{H}$ because $T(\epsilon^\alpha\nu)$ is compact. From (H1)(i), (H2)(i), Lemma 2.1, Lemma 2.6 and Hölder inequality, we get that

$$
\begin{aligned}
&\mathbb{E}\|(F_1x)(t)-(F^{\epsilon,\nu}x)(t)\|^2\\
\le\ & 4\mathbb{E}\left\|\alpha\int_0^t\int_0^\nu \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)[f(s,x(s))+B(s)u(s)]d\theta ds\right\|^2\\
&+4\mathbb{E}\left\|\alpha\int_{t-\epsilon}^t\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)[f(s,x(s))+B(s)u(s)]d\theta ds\right\|^2\\
&+4\mathbb{E}\left\|\alpha\int_0^t\int_0^\nu \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)\sigma(s,x(s))d\theta dW(s)\right\|^2\\
&+4\mathbb{E}\left\|\alpha\int_{t-\epsilon}^t\int_\nu^\infty \theta\zeta_\alpha(\theta)(t-s)^{\alpha-1}T((t-s)^\alpha\theta)\sigma(s,x(s))d\theta dW(s)\right\|^2\\
\le&\frac{4M^2\alpha^2b^{2\alpha-1}}{2\alpha-1}\int_0^t\mathbb{E}\|f(s,x(s))+B(s)u(s)\|^2ds\left(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\right)^2\\
&+\frac{4M^2\alpha^2\epsilon^{2\alpha-1}}{(2\alpha-1)\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t\mathbb{E}\|f(s,x(s))+B(s)u(s)\|^2ds\\
&+4M^2\alpha^2\int_0^t(t-s)^{2(\alpha-1)}\mathbb{E}\|\sigma(s,x(s))\|^2_{L_2^0}ds\left(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\right)^2\\
&+\frac{4M^2\alpha^2}{\Gamma^2(1+\alpha)}\int_{t-\epsilon}^t(t-s)^{2(\alpha-1)}\mathbb{E}\|\sigma(s,x(s))\|^2_{L_2^0}ds\\
\le&\frac{4M^2\alpha^2b^{2\alpha-1}}{2\alpha-1}\left(2bL_f(1+r_0)+2\|B\|^2_\infty\int_0^b\mathbb{E}\|u(s)\|^2ds\right)\left(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\right)^2\\
&+\frac{4M^2\alpha^2\epsilon^{2\alpha-1}}{(2\alpha-1)\Gamma^2(1+\alpha)}\left(2L_f(1+r_0)\epsilon+2\|B\|^2_\infty\int_0^b\mathbb{E}\|u(s)\|^2ds\right)\\
&+\frac{4M^2\alpha^2b^{2\alpha-1}}{2\alpha-1}L_\sigma(1+r_0)\left(\int_0^\nu\theta\zeta_\alpha(\theta)d\theta\right)^2\\
&+\frac{4M^2\alpha^2\epsilon^{2\alpha-1}}{(2\alpha-1)\Gamma^2(1+\alpha)}L_\sigma(1+r_0)\to 0\ (\epsilon,\nu\to 0).
\end{aligned}
$$

Hence, there are relatively compact sets arbitrarily close to the set $V(t)(t > 0)$ in $\mathbb{H}$.

Therefore, the set $V(t)$ is relatively compact in $\mathbb{H}$.

Finally, we prove that $F_1(B_{r_0})$ equicontinuous on J.

For any $x \in B_{r_0}$ and $0 \le t_1 < t_2 \le b$, we have

$$\begin{aligned}
&\mathbb{E}\|(F_1x)(t_2) - (F_1x)(t_1)\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\mathscr{S}(t_2-s)\big[f(s,\ x(s)) + B(s)u(s)\big]ds \bigg\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{0}^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]\mathscr{S}(t_2-s)\big[f(s,\ x(s)) + B(s)u(s)\big] \bigg\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{0}^{t_1} (t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s) - \mathscr{S}(t_1-s)\big]\big[f(s,\ x(s)) + B(s)u(s)\big] \bigg\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\mathscr{S}(t_2-s)\sigma(s,\ x(s))dW(s) \bigg\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{0}^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]\mathscr{S}(t_2-s)\sigma(s,\ x(s))dW(s) \bigg\|^2 \\
&+ 6\mathbb{E}\bigg\| \int_{0}^{t_1} (t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s) - \mathscr{S}(t_1-s)\big]\sigma(s,\ x(s))dW(s) \bigg\|^2 \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

In order to prove $\mathbb{E}\|(Fx)(t_2) - (Fx)(t_1)\|^2 \to 0(t_2 - t_1 \to 0)$, we only need to show $I_i \to 0$ independently of $x \in B_{r_0}$ when $t_2 - t_1 \to 0$ for $i = 1, 2, \cdots, 6$.

For I_1 and I_4, we obtain by (H2)(i), (H3)(i), Lemma 2.1 and Lemma 2.6 that

$$\begin{aligned}
I_1 &= 6\mathbb{E}\bigg\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\mathscr{S}(t_2-s)\big[f(s,\ x(s)) + B(s)u(s)\big]ds \bigg\|^2 \\
&\le \frac{6M^2}{\Gamma^2(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{2\alpha-2}ds \int_{t_1}^{t_2} \mathbb{E}\|f(s,\ x(s)) + B(s)u(s)\|^2 ds \\
&\le \frac{6M^2\big[2bL_f(1+r_0) + 2\|B\|_\infty^2 \int_0^b \mathbb{E}\|u(s)\|^2 ds\big]}{\Gamma^2(\alpha)} \cdot \frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1} \\
&\to 0\ (t_2 - t_1 \to 0). \\
I_4 &= 6\mathbb{E}\bigg\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\mathscr{S}(t_2-s)\sigma(s,\ x(s))dW(s) \bigg\|^2 \\
&\le \frac{6M^2}{\Gamma^2(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{2\alpha-2}\mathbb{E}\|\sigma(s,\ x(s))\|^2 ds \\
&\le \frac{6M^2L_\sigma(1+r_0)}{\Gamma^2(\alpha)} \cdot \frac{(t_2-t_1)^{2\alpha-1}}{2\alpha-1} \\
&\to 0\ (t_2 - t_1 \to 0).
\end{aligned}$$

In a similar way, for I_2 and I_5, we get

$$I_2 = 6\mathbb{E}\bigg\| \int_{0}^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]\mathscr{S}(t_2-s)\big[f(s,\ x(s)) + B(s)u(s)\big] \bigg\|^2$$

$$
\begin{aligned}
&\leq \frac{6M^2}{\Gamma^2(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2ds\int_0^{t_1}\mathbb{E}\|f(s,\ x(s))+B(s)u(s)\|^2ds\\
&\leq \frac{6M^2\big[2L_fb(1+r_0)+2\|B\|_\infty^2\int_0^b\mathbb{E}\|u(s)\|^2ds\big]}{\Gamma^2(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2ds\\
&\to 0\ (t_2-t_1\to 0).
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= 6\mathbb{E}\bigg\|\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathscr{S}(t_2-s)\sigma(s,\ x(s))\bigg\|^2\\
&\leq \frac{6M^2}{\Gamma^2(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2\mathbb{E}\|\sigma(s,\ x(s))\|^2ds\\
&\leq \frac{6M^2L_\sigma(1+r_0)}{\Gamma^2(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]^2ds\\
&\to 0\ (t_2-t_1\to 0).
\end{aligned}
$$

Further, for I_3 and I_6, if $t_1=0,\ 0<t_2<b$, it is easy to see $I_3=I_6=0$, so for $t_1>0$ and $0<\varepsilon<t_1$ small enough, we have that

$$
\begin{aligned}
I_3 &= 6\mathbb{E}\bigg\|\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\big[f(s,\ x(s))+B(s)u(s)\big]ds\bigg\|^2\\
&\leq 12\mathbb{E}\bigg\|\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\big[f(s,\ x(s))+B(s)u(s)\big]ds\bigg\|^2\\
&\quad+12\mathbb{E}\bigg\|\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\big[f(s,\ x(s))+B(s)u(s)\big]ds\bigg\|^2\\
&\leq 12\sup_{s\in[0,t_1-\varepsilon]}\|\mathscr{S}(t_2-s)-(t_1-s)\|^2\big(2L_fb(1+r_0)+2\|B\|_\infty^2\int_0^b\mathbb{E}\|u(s)\|^2ds\big)\\
&\qquad\times\frac{t_1^{2\alpha-1}-\varepsilon^{2\alpha-1}}{2\alpha-1}\\
&\quad+12\Big(\frac{2M}{\Gamma(\alpha)}\Big)^2\big(2L_fb(1+r_0)+2\|B\|_\infty^2\int_0^b\mathbb{E}\|u(s)\|^2ds\big)\frac{\varepsilon^{2\alpha-1}}{2\alpha-1}\\
&\to 0\ (t_2-t_1\to 0 \text{ and } \varepsilon\to 0).
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= 6\mathbb{E}\bigg\|\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\sigma(s,\ x(s))dW(s)\bigg\|^2\\
&\leq 12\mathbb{E}\bigg\|\int_0^{t_1-\varepsilon}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\sigma(s,\ x(s))ds\bigg\|^2\\
&\quad+12\mathbb{E}\bigg\|\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\big[\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\big]\sigma(s,\ x(s))ds\bigg\|^2\\
&\leq 12\sup_{s\in[0,t_1-\varepsilon]}\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|^2L_\sigma(1+r_0)\frac{t_1^{2\alpha-1}-\varepsilon^{2\alpha-1}}{2\alpha-1}\\
&\quad+12\Big(\frac{2M}{\Gamma(\alpha)}\Big)^2L_\sigma(1+r_0)\frac{\varepsilon^{2\alpha-1}}{2\alpha-1}\\
&\to 0\ (t_2-t_1\to 0 \text{ and } \varepsilon\to 0).
\end{aligned}
$$

This implies that $F_1(B_{r_0})$ is equicontinuous.

Hence by the Arzela-Ascoli theorem one has that F_2 is a completely continuous operator. Thus, by Lemma 2.7, $F_1 + F_2$ has at least a fixed point $x \in B_{r_0}$, which is just the mild solution of system (1.1).

This completes the proof of Theorem 3.1. $\square$

Furthermore, if conditions (H1)(ii) and (H2)(ii) also hold, we can obtain the uniqueness theorem for system (1.1).

Theorem 3.2. *Assume that $-A$ generates a compact C_0-semigroup $T(t)(t \geq 0)$ of uniformly bounded operators in Hilbert space $\mathbb{H}$. Suppose the assumptions (H1)-(H4) hold, then impulsive fractional stochastic systems with nonlocal conditions (1.1) has an unique mild solution in $\mathcal{PC}([0,b], L^2(\Omega,\mathbb{H}))$ provided that (3.1) and*

$$4\left(M^2 b^2 \overline{L}_h + \ M^2 p \sum_{i=1}^{p} \overline{M}_i + c_0 \overline{L}_f b + c_0 \overline{L}_\sigma b\right) < \ 1, \tag{3.3}$$

are satisfied.

Proof. Introduce the mapping $F : \mathcal{PC}([0,b], L^2(\Omega,\mathbb{H})) \to \mathcal{PC}([0,b], L^2(\Omega,\mathbb{H}))$ by

$$(Fx)(t) = (F_1 x)(t) + (F_2 x)(t), \quad t \in [0, \ b].$$

Clearly, the mild solution of system (1.1) is equivalent to the fixed point of the operator F. By Step 1 of Theorem 3.1, we know that $F(B_{r_0}) \subset B_{r_0}$. For any $x_1, x_2 \in B_{r_0}$ and $t \in J$, we have

$$\begin{aligned}
&\mathbb{E}\|(Fx_2)(t) - (Fx_1)(t)\|^2 \\
&\leq 4\mathbb{E}\|\mathscr{T}(t)[g(x_2) - g(x_1)]\|^2 + \ 4\mathbb{E}\Big\|\sum_{0<t_i<t} \mathscr{T}(t-t_i)[I_i(x_2(t_i)) - I_i(x_1(t_i))]\Big\|^2 \\
&\quad + \ 4\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)[f(s, \ x_2(s)) - f(s, \ x_1(s))]ds\Big\|^2 \\
&\quad + \ 4\mathbb{E}\Big\|\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)[\sigma(s, \ x_2(s)) - \sigma(s, \ x_1(s))]dW(s)\Big\|^2 \\
&\leq 4M^2 b \int_0^b \mathbb{E}\big\|h(s, \ x_2(s)) - h(s, \ x_1(s))\big\|^2 ds + \ 4M^2 p \sum_{i=1}^{p} \mathbb{E}\big\|I_i(x_2(t_i)) - I_i(x_1(t_i))\big\|^2 \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\big\|f(s, \ x_2(s)) - f(s, \ x_1(s))\big\|^2 ds \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\big\|\sigma(s, \ x_2(s)) - \sigma(s, \ x_1(s))\big\|^2 ds \\
&\leq 4M^2 b\overline{L}_h \int_0^b \mathbb{E}\big\|x_2(s) - x_1(s)\big\|^2 ds + \ 4M^2 p \sum_{i=1}^{p} \overline{M}_i \mathbb{E}\big\|x_2(t_i) - x_1(t_i)\big\|^2 \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_f \int_0^t \mathbb{E}\big\|x_2(s) - x_1(s)\big\|^2 ds \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_\sigma \int_0^t \mathbb{E}\big\|x_2(s)) - x_1(s)\big\|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 4M^2b^2\overline{L}_h\big\|x_2-x_1\big\|_{\mathcal{PC}}^2+\ 4M^2p\sum_{i=1}^{p}\overline{M}_i\big\|x_2-x_1\big\|_{\mathcal{PC}}^2\\
&\quad+\ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_f b\big\|x_2-x_1\big\|_{\mathcal{PC}}^2+\ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_\sigma b\big\|x_2-x_1\big\|_{\mathcal{PC}}^2\\
&=4\bigg(M^2b^2\overline{L}_h+\ M^2p\sum_{i=1}^{p}\overline{M}_i+c_0\overline{L}_f b+c_0\overline{L}_\sigma b\bigg)\big\|x_2-x_1\big\|_{\mathcal{PC}}^2\\
&:=\kappa\big\|x_2-x_1\big\|_{\mathcal{PC}}^2.
\end{aligned}
$$

Hence

$$\|(Fx_2)-(Fx_1)\|_{\mathcal{PC}}^2\leq\kappa\|x_2-x_1\|_{\mathcal{PC}}^2.$$

We have by (3.3) that F is a contraction mapping on B_{r_0}. Thus, by the well known contraction mapping principle we know that F has a unique fixed point $x\in B_{r_0}$, that is, $x(t)$ is the unique mild solution of system (1.1).

This completes the proof of Theorem 3.2. □

4. Existence of Optimal Controls

In this section, we investigate the existence of optimal controls.

Let x^u denote the mild solution of system (1.1) corresponding to the control $u\in U_{ad}$. Consider the Lagrange problem $(\mathscr{P})$:

Find an optimal pair $(x^0,u^0)\in\mathcal{PC}([0,b],L^2(\Omega,\mathbb{H}))\times U_{ad}$ such that

$$\mathcal{J}(x^0,u^0)\leq\mathcal{J}(x^u,u),\quad \textit{for all}\ (x^u,u)\in\mathcal{PC}([0,b],L^2(\Omega,\mathbb{H}))\times U_{ad},\tag{4.1}$$

where the cost function

$$\mathcal{J}(x^u,u)=\mathbb{E}\bigg(\int_0^b\mathscr{L}(t,x^u(t),u(t))dt\bigg).$$

Assume that

(L1) The functional $\mathscr{L}:J\times\mathbb{H}\times\mathbb{U}\to\mathbb{R}\cup\{\infty\}$ is $\mathcal{F}_t$ measurable.

(L2) For any $t\in J$, $\mathscr{L}(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $\mathbb{H}\times\mathbb{U}$.

(L3) For any $t\in J$ and $x\in\mathbb{H}$, $\mathscr{L}(t,x,\cdot)$ is convex on $\mathbb{U}$.

(L4) There exist two constants $d_1\geq 0,\ d_2>0$, ξ is nonnegative and $\xi\in L^1(J,\mathbb{R})$ such that

$$\mathscr{L}(t,x,u)\geq\xi(t)+d_1\mathbb{E}\|x\|^2+d_2\mathbb{E}\|u\|^2.$$

Now we are in a position to present the existence of optimal controls for problem $(\mathscr{P})$.

Theorem 4.1. *Let hypothesis of Theorem 3.2 and (L1)-(L4) hold. Suppose that B is a strongly continuous operator, then Lagrange problem $(\mathscr{P})$ admits at least one optimal pair, that is, there exists an admissible state-control pair*

$$(x^0,u^0)\in\mathcal{PC}([0,b],L^2(\Omega,\mathbb{H}))\times U_{ad},$$

such that

$$\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x^u, u), \quad \textit{for all} \quad (x^u, u) \in \mathcal{PC}([0,b], L^2(\Omega, \mathbb{H})) \times U_{ad}. \tag{4.2}$$

Proof. Without loss of generality, we suppose that

$$\inf\{\mathcal{J}(x^u, u) | u \in U_{ad}\} = \varepsilon < +\infty.$$

Otherwise, there is nothing to prove. It follows from (L4) that $\varepsilon > -\infty$. We obtain by definition of infimum that there is a minimizing sequence of feasible pairs $(x^m, u^m) \in \mathcal{PC}([0,b], L^2(\Omega, \mathbb{H})) \times U_{ad}$ such that

$$\mathcal{J}(x^m, u^m) \rightarrow \ \varepsilon, \quad m \rightarrow \infty,$$

where x^m is a mild solution of system (1.1) corresponding to $u^m \in U_{ad}$.

Note that $\{u^m\} \subset U_{ad}(m = 1, 2, \cdots)$, which implies that $\{u^m\} \in L^2_{\mathcal{F}}(J, \ \mathbb{U})$ is bounded. Thus, there exists $u^0 \in L^2_{\mathcal{F}}(J, \ \mathbb{U})$ and a subsequence extracted from $\{u^m\}$ (still denoted $\{u^m\}$) such that

$$u^m \xrightarrow{w} u^0 \ \ (m \rightarrow \infty).$$

Since U_{ad} is convex and closed, from the Marzur theorem[49], we deduce that $u^0 \in U_{ad}$.

Let x^0 be the mild solution of (1.1) corresponding to u^0. It follows the boundedness of $\{u^m\}$, $\{u^0\}$, one can check that there exists a positive number r_0 such that $\|x^m\|^2_{\mathcal{PC}} \leq r_0, \ \ \|x^0\|^2_{\mathcal{PC}} \leq r_0$. For $t \in J$, we have

$$\begin{aligned}
&\mathbb{E}\|x^m(t) - x^0(t)\|^2 \\
&\leq 4\mathbb{E}\|\mathscr{T}(t)[g(x^m) - g(x^0)]\|^2 + \ 4\mathbb{E}\Big\| \sum_{0<t_i<t} \mathscr{T}(t-t_i)[I_i(x_2(t_i)) - I_i(x^0(t_i))]\Big\|^2 \\
&\quad + \ 4\mathbb{E}\Bigg\| \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) \Bigg[\big(f(s, \ x^m(s)) - f(s, \ x^0(s))\big) \\
&\quad + \ \big(B(s)u^m(s) - B(s)u^0(s)\big) \Bigg] ds \Bigg\|^2 \\
&\quad + \ 4\mathbb{E}\Bigg\| \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s)[\sigma(s, \ x^m(s)) - \sigma(s, \ x^0)] dW(s) \Bigg\|^2 \\
&\leq 4M^2 b \int_0^b \mathbb{E}\big\|h(s, \ x^m(s)) - h(s, \ x^0(s))\big\|^2 ds + \ 4M^2 p \sum_{i=1}^p \mathbb{E}\big\|I_i(x^m(t_i)) - I_i(x^0(t_i))\big\|^2 \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^t 2\mathbb{E}\big\|f(s, \ x^m(s)) - f(s, \ x^0(s))\big\|^2 \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^t 2\mathbb{E}\big\|B(s)u^m(s) - B(s)u^0(s)\big\| ds \\
&\quad + \ \frac{4M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^t \mathbb{E}\big\|\sigma(s, \ x^m(s)) - \sigma(s, \ x^0(s))\big\|^2 ds \\
&\leq 4M^2 b^2 \overline{L}_h \big\|x^m - x_1\big\|^2_{\mathcal{PC}} + \ 4M^2 p \sum_{i=1}^p \overline{M}_i \big\|x^m - x^0\big\|^2_{\mathcal{PC}}
\end{aligned}$$

$$
\begin{aligned}
&+\ \frac{8M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_f b\big\|x^m-x^0\big\|^2_{\mathcal{PC}}+\ \frac{4M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\overline{L}_\sigma b\big\|x^m-x^0\big\|^2_{\mathcal{PC}}\\
&+\ \frac{8M^2}{\Gamma^2(\alpha)}\frac{b^{2\alpha-1}}{2\alpha-1}\big\|Bu^m-Bu^0\big\|^2_{L^2_{\mathcal{F}}(J,\ \mathbb{U})}\\
&=4\bigg(M^2b^2\overline{L}_h+\ M^2p\sum_{i=1}^{p}\overline{M}_i+2c_0\overline{L}_f b+c_0\overline{L}_\sigma b\bigg)\big\|x^m-x^0\big\|^2_{\mathcal{PC}}\\
&+\ 8c_0\big\|Bu^m-Bu^0\big\|^2_{L^2_{\mathcal{F}}(J,\ \mathbb{U})},
\end{aligned}
$$

which means

$$
\big\|x^m-x^0\big\|^2_{\mathcal{PC}}\leq\frac{8c_0\big\|Bu^m-Bu^0\big\|^2_{L^2_{\mathcal{F}}(J,\ \mathbb{U})}}{1-4\bigg(M^2b^2\overline{L}_h+\ M^2p\sum_{i=1}^{p}\overline{M}_i+2c_0\overline{L}_f b+c_0\overline{L}_\sigma b\bigg)}.
$$

Since B is strongly continuous, we get

$$
\big\|Bu^m-Bu^0\big\|^2_{L^2_{\mathcal{F}}(J,\ \mathbb{U})}\xrightarrow{s}0\ (m\to\infty).
$$

Consequently,

$$
\big\|x^m-x^0\big\|^2_{\mathcal{PC}}\xrightarrow{s}0\ (m\to\infty).
$$

Thus, by (L1)-(L4)and Balders theorem (see Theorem 2.1 [29]), we can deduce that $(x,\ u)\to\mathbb{E}\bigg(\int_0^b\mathscr{L}(t,x(t),u(t))dt\bigg)$ is sequentially lower semicontinuous in the strong topology of $L^1_{\mathcal{F}}(J,\ \mathbb{H})$ and weak topology of $L^2_{\mathcal{F}}(J,\ \mathbb{U})\subset L^1_{\mathcal{F}}(J,\ \mathbb{U})$. Hence, $\mathcal{J}$ is weakly lower semicontinuous on $L^2_{\mathcal{F}}(J,\ \mathbb{U})$. Therefore, we obtain

$$
\begin{aligned}
\varepsilon\ &=\ \lim_{m\to\infty}\mathbb{E}\bigg(\int_0^b\mathscr{L}(t,x^m(t),u^m(t))dt\bigg)\\
&\geq\ \mathbb{E}\bigg(\int_0^b\mathscr{L}(t,x^0(t),u^0(t))dt\bigg)=\mathcal{J}(x^0,u^0)\geq\ \varepsilon,
\end{aligned}
$$

which implies that $u^0\in U_{ad}$ is a minimum of $\mathcal{J}$.

This completes the proof of Theorem 4.1. □

Remark 4.2. The result of Theorem 4.1 can be extended to the noninstantaneous impulsive fractional stochastic evolution equations with nonlocal conditions. The corresponding result that appear are also new.

Remark 4.3. In recent paper [10], Dhayal et al. studied the existence of optimal multicontrol pairs for a class of noninstantaneous impulsive fractional stochastic differential systems. In [11], Dhayal et al. obtained the optimal pair for a nonlinear system governed by the fractional differential equation by using the resolvent family and approximation techniques. In [12], Dhayal et al. discussed the approximate and trajectory controllability for a class of fractional stochastic differential equations with noninstantaneous impulses. Inspired by [10, 11, 12], in the future, we will investigate the fractional stochastic evolution equations with nonlocal initial conditions and noninstantaneous impulsive.

Remark 4.4. The uniqueness of the solution is a prerequisite for discussing optimal control, so it is necessary that the mild solution of (1.1) should be unique.

5. Application

To illustrate the main result, we consider the following fractional stochastic control system

$$\begin{cases} \frac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}}x(z,t)-\frac{\partial^2}{\partial s^2}x(z,t)=\left(\frac{\sin t}{10}+\frac{x(z,t)}{t+10}\right)+\frac{1}{10}\left(\frac{1}{1+e^t}+\frac{|x(z,t)|}{1+|x(z,t)|}\right)\frac{dW(t)}{dt} \\ \qquad +\int_0^1 \mathcal{K}(z,\ s)u(s,t)ds, \quad z\in[0,1],\, t\in[0,1],\, t\neq\frac{1}{2}, \\ \Delta x(\frac{1}{2},z)=\frac{|x(z,t)|}{5+|x(z,t)|}, \quad z\in[0,1], \\ x(0,t)=x(\pi,t)=0, \quad t\in[0,1], \\ x(z,0)=\int_0^1\frac{1}{8}\left(e^{-t}+\sin(x(z,s))\right)ds, \quad z\in[0,1], \end{cases} \tag{5.1}$$

where $W(t)$ is a standard one dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$.

In order to write the above system (5.1) into the abstract form of (1.1), let $\mathbb{H}=\mathbb{U}=L^2[0,\ 1]$ with the norm $\|w\|=\left(\int_0^1|w(z)|^2dz\right)^{\frac{1}{2}}$. Define the operator $A:D(A)\subset\mathbb{H}\to\mathbb{H}$ by

$$D(A)=\{w\in\mathbb{H}\mid w',w''\in X, w(0)=w(1)=0\}, \qquad Aw=-\frac{\partial^2 w}{\partial z^2}.$$

We know that $-A$ generates a compact, analytic semigroup $T(t)(t\geq 0)$ in $\mathbb{H}$ and

$$T(t)v=\Sigma_{n=1}^{\infty}e^{-n^2t}(v,v_n)v_n, \quad \|T(t)\|\leq e^{-t}<1,\ t>0,$$

where $v_n=\sqrt{2}\sin(ns)$, $n=1,2,\cdots$ is the orthogonal set of eigenvectors in A. Moreover, we assume that $\mathcal{K}:[0,1]\times[0,1]\to\mathbb{R}$ is continuous, and the admissible control set

$$U_{ad}=\{u\in\mathbb{U}\mid \|u\|_{L^2_{\mathcal{F}}}\leq 1\}.$$

For any $t\in[0,1]$, let

$$x(t)(z)=x(z,t), \quad B(t)u(t)(z)=\int_0^1\mathcal{K}(z,\ s)u(s,t)ds,$$

$$f(t,x(t))(z)=\frac{\sin t}{10}+\frac{x(z,t)}{t+10}, \quad \sigma(t,x(t))(z)=\frac{1}{10}\left(\frac{1}{1+e^t}+\frac{|\, x(z,t)\,|}{1+|\, x(z,t)\,|}\right),$$

$$I_1(x(t))(z) = \frac{\mid x(z,t) \mid}{5+ \mid x(z,t) \mid}, \quad h(t,x(t))(z) = \frac{1}{8}\big(e^{-t} + \sin(x(z,s))\big).$$

Then the problem (5.1) can be rewritten into the abstract form of (1.1) with the cost function

$$\mathcal{J}(x,u) = \mathbb{E}\bigg(\int_0^b \int_0^1 |x(z,t)|^2 dzdt + \int_0^b \int_0^1 |u(z,t)|^2 dzdt\bigg).$$

We can easily check that the assumptions (H1)-(H4) holds with $Ł_f = Ł_\sigma = \frac{1}{50}$, $\overline{L}_f = \overline{L}_\sigma = \frac{1}{100}$, $Ł_h = \overline{L}_h = \frac{1}{32}$ and $M_1 = \overline{M}_1 = \frac{1}{25}$. In addition,

$$N + M^2 p \sum_{i=1}^{p} M_i < \frac{1}{32} + 1.68 \cdot \frac{1}{25} \approx 0.09 < \frac{1}{5},$$

$$M^2 b^2 \overline{L}_h + \ M^2 p \sum_{i=1}^{p} \overline{M}_i = \frac{1}{32} + \frac{1}{25} < \frac{1}{2},$$

$$M^2 b^2 \overline{L}_h + \ M^2 p \sum_{i=1}^{p} \overline{M}_i + c_0 \overline{L}_f b + c_0 \overline{L}_\sigma b < \frac{1}{32} + \frac{1}{25} + 1.68 \cdot \frac{1}{50} \approx 0.1 < 0.25.$$

Hence, by Theorem 4.1, system (5.1) has at least one optimal pair.

6. Conclusions

In this paper, the optimal controls for a class of impulsive stochastic fractional evolution equations with nonlocal initial conditions in a Hilbert space is studied. More precisely, by utilizing the fractional calculus, stochastic analysis theory, and fixed point theorems, we obtained the existence and uniqueness of mild solutions and optimal pairs for these equations. Finally, an example is provided to show the effectiveness of the proposed results. There are two direct issues which require further study. We will investigate the fractional stochastic evolution equations for order $\alpha \in (1, 2]$ with nonlocal initial conditions and noninstantaneous impulsive. Also, we will be devoted to study the optimal controls problem for fractional stochastic partial differential inclusions with nonlocal initial conditions.

Statements and Declarations

No potential conflict of interest was reported by the authors.

Funding

Research is supported by NNSF of China (12261078), NSF of Gansu Province (21JR11RE031, 22JR11RE193), Innovation Foundation of Tianshui Normal University (CXJ2020-02, CXT2019-03) and Innovation Team of Tianshui Normal University (TDJ2022-03).

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