The smooth solutions of a class of coupled KdV equations

Boling Guo[†] Qi Guo[‡,*]

[†]Institute of Applied Physics and Computational Mathematics, Beijing, 100088, P. R. China

[‡] The Graduate School of China Academy of Engineering Physics, Beijing, 100088, P. R. China

Abstract

This paper is devoted to the study of the periodic initial boundary value problem and Cauchy problem for the coupled KdV equations. By the Galerkin method and sequential approximation, we get a series of a priori estimates and establish the existence of classical local solution to the periodic problem for the system. Then we obtain the existence and uniqueness of global smooth solution when the coefficients of the system satisfy certain conditions by energy method, conserved quantities and nonconservative quantity $I(u,v)$.

Key words: KdV equations, periodic solutions, conserved quantities.

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1 Introduction

In this paper, we consider a class of coupled KdV (Korteweg-de Vries) equations as follows:

$$\begin{cases} u_t + 2bu_x + au_{xxx} = -2b(uv)_x, \quad x \in \mathbb{R}, t > 0 \\ v_t + bv_x + bvv_x + cv_{xxx} = -b(|u|^2)_x, \quad x \in \mathbb{R}, t > 0 \end{cases} \tag{1.1}$$

where $u(x,t)$ is a complex value function, and $v(x,t)$ is real-valued, the coefficients a, b, c are real constants which are not zero. Deconinck and Nguyen [4] derived the system (1.1) in the process of deriving the NLS-KdV (nonlinear Schrödinger-Korteweg-de Vries) system with the traditional ansatz used in [6] from a generic system which has nonlinearities that are quadratic, cubic, etc. And in this paper, it has been proved that the system (1.1) has at least the following conserved quantities.

$$\begin{aligned} &H_0(u) = \int_{-\infty}^{\infty} u \, \mathrm{d}x, \quad H_1(v) = \int_{-\infty}^{\infty} v \, \mathrm{d}x, \\ &H_2(u,v) = \int_{-\infty}^{\infty} (|u|^2 + v^2) \, \mathrm{d}x, \\ &H_3(u,v) = \int_{-\infty}^{\infty} (\frac{a}{2}|u_x|^2 + \frac{c}{2}v_x^2 - \frac{b}{6}v^3 - b|u|^2 v - b|u|^2 - \frac{b}{2}v^2) \, \mathrm{d}x. \end{aligned}$$

*Corresponding author

E-mail addresses: *gbl@iapcm.ac.cn(B. Guo), guoqi20@gscaep.ac.cn(Q. Guo)* .

KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

is a unidirectional shallow water partial differential equation discovered by Dutch mathematicians Korteweg and De-Vries when they studied the small amplitude long wave motion in shallow water, and they gave the solitary wave solution. Then, Zabusky and Kruskal [27] considered the periodic boundary conditions for the KdV equation, and simulated the initial and exact solutions, finally obtained some special properties of the KdV isolated wave. The KdV equation is widely used in solid state physics, plasma, quantum theory and so on [3, 7]. So far, various properties of individual KdV equation, including infinite symmetry, infinite multiconserved quantities, inverse scattering transformation, Painlevè property, Bäcklund property and Darboux transformation have been well known [17, 25, 26]. In addition to the individual KdV equation, researchers have recently discovered a series of different forms of coupled KdV systems in the practical physics and mathematics fields, which are commonly used to describe the near-resonance interaction between hierarchical fluid internal waves [8], interstellar near-resonance wave interaction [9], etc. And how to use the known theories to explore the various properties of these nonlinear systems and their solutions has gradually become one of the attention topics of researchers. Later, the first coupled KdV equation

$$\begin{cases} u_t + 6\alpha uu_x - 2bvv_x + \alpha u_{xxx} = 0, \\ v_t + 3\beta uv_x + \beta v_{xxx} = 0, \end{cases} \tag{1.2}$$

was proposed by Hirota, which were derived to model the interaction of water waves. Here α, β, b are constants. And Hirota and Satsuma [14] obtained the isolated subsolutions and three fundamental conserved quantities of the coupled system (1.2). In [15], using the iterative Darboux transformation, the authors firstly obtained the analytical solution and non-singular complex solution of the following coupled system

$$\begin{cases} u_t + 6vv_x - 6uu_x + u_{xxx} = 0, \\ v_t - 6uv_x - 6vu_x + v_{xxx} = 0. \end{cases}$$

In [1], Basakoğlu and Gürel proved the existence and smoothness of a global attractor in the energy space of the following coupled system

$$\begin{cases} u_t + au_{xxx} + 3a(u^2)_x + \beta(v^2)_x = 0, \quad x \in \mathbb{T} \\ v_t + v_{xxx} + 3uv_x = 0, \end{cases}$$

by smoothing estimates, where $a \in (\frac{1}{4}, 1), \beta \in \mathbb{R}$. Similar works can also be referred to [2, 16, 19, 20, 23, 24].

The literature proving the existence of solutions for the KdV equation or its derived system by the Galerkin finite element method can be referred to [13, 18, 21]. In particular, Guo [11] researched the existence of periodic solution of the KdV system as follows

$$\begin{cases} u_t + f(u)_x + u_{xxx} = 0, \qquad x \in \mathbb{R}, t > 0 \\ u(x,0) = u(x,T), u(x+2l,t) = u(x,t), \qquad l > 0 \end{cases}$$

and obtained the weak solution space as $L^\infty(0,T;H^4(-l,l))\cap C^1(0,T;L^2(-l,l))$, where $l>0$. By using the Galerkin finite element method and a priori estimation, Yang [22] studied the initial boundary value problem of a generalized KdV system

$$\begin{cases} u_t+f\left(u\right)_x=\alpha u_{xx}+\beta u_{xxx}, \qquad x\in\mathbb{R}^+,t>0\\ u(x,t)|_{t=0}=u_0(x),\\ u(x,t)|_{x=0}=0, \qquad u(x,t)\to 0 \quad (x\to\infty)\end{cases}$$

and obtained the weak solution space as $L^\infty(0,T;H_0^2(\mathbb{R}^+))$, where $\alpha\geq 0,\beta<0$. Ding and Wei [5] investigated the existence of the periodic solution for the coupled system as follows

$$\begin{cases} u_t+\alpha vv_x+\alpha\sigma v_x+\beta_1 uu_x+\beta_2 u^2u_x+\beta u_{xxx}=k_1u_{xx}, \qquad x\in\mathbb{R},t>0\\ v_t+\delta(uv)_x+\epsilon vv_x+\epsilon\sigma v_x+\delta\sigma u_x=k_2v_{xx}, \qquad x\in\mathbb{R},t>0\\ u|_{t=0}=u_0(x),v|_{t=0}=v_0(x), \qquad 0\leq x\leq 1\\ u(x+1,t)=u(x,t),v(x+1,t)=v(x,t),\end{cases} \tag{1.3}$$

and obtained the weak solution space as $L^\infty(0,T;H^3([0,1]))$, where $\alpha,\sigma,\beta_1,\beta_2,\beta,\delta,\epsilon$ are real constants. The unknown functions u,v are all real-valued functions. In addition, we can prove the similar problems by other methods. By the conserved quantities and priori estimates, Guo and Tan [10] researched the global existence and uniqueness of smooth solution to the initial value problem of the following coupled system

$$\begin{cases} u_t=u_{xxx}+6uu_x+2vv_x,\\ v_t=2(uv)_x.\end{cases}$$

In [12], He established the existence of smooth solution to the system of coupled non-linear KdV equations

$$\begin{cases} u_t=a(u_{xxx}+6uu_x)+2bvv_x,\\ v_t=-v_{xxx}-3uv_x,\end{cases}$$

where a and b are constants. His proof depended on the presence of dispersive terms in both components and did not extend to the system of non-linear KdV equations with a hyperbolic partial differential equations.

In this paper, we concern with the coupled system (1.1):

$$\begin{cases} u_t+2bu_x+au_{xxx}=-2b(uv)_x, \qquad x\in[-l,l],t>0\\ v_t+bv_x+bvv_x+cv_{xxx}=-b(|u|^2)_x, \qquad x\in[-l,l],t>0\end{cases} \tag{1.4}$$

with the initial value conditions

$$u|_{t=0}=u_0(x),\, v|_{t=0}=v_0(x), \qquad x\in[-l,l] \tag{1.5}$$

and the periodic conditions

$$u(x+l,t)=u(x-l,t),\, v(x+l,t)=v(x-l,t). \qquad l>0 \tag{1.6}$$

It can be seen that the system $(1.4)-(1.6)$ is a dispersion system when $k_1 = k_2 = 0$ in the system (1.3) and the unknown functions include a complex value function, which leads that the highest derivative in the calculation process cannot be controlled by the lower derivatives, thus furtherly increasing the difficulty of proving the existence of the global smooth solution. Firstly, we construct the existence of classical local solution of the system $(1.4)-(1.6)$ by the Galerkin finite element method and sequential approximation. Next, through the conserved quantities and nonconservative quantity $I(u,v)$ of the system, we obtain a series of priori estimates and then we achieve the existence of global smooth solution. Finally, we prove the uniqueness of the smooth solution.

Now, we state our main results as follows:

Theorem 1.1. *If the following conditions are met,*
(1)$ac > 0, \frac{c}{a} > \frac{3\sqrt{5}-5}{10}$;
(2)$u_0, v_0 \in H^m([-l,l]), m \geq 4$, *and they are periodic functions with period* $2l$*;*
then the periodic initial value problem $(1.4)-(1.6)$ *admits a unique global periodic smooth solution with* $u_0(x), v_0(x)$ *as initial values, and there holds*

$$u(x,t), v(x,t) \in L^\infty(\mathbb{R}^+; H^m([-l,l])).$$

If only the condition (2) is met, then there exists a constant $T_0 > 0$ *such that the system* $(1.4)-(1.6)$ *admits a unique local periodic smooth solution with* $u_0(x), v_0(x)$ *as initial values, and there holds*

$$u(x,t), v(x,t) \in L^\infty(0, T_0; H^m([-l,l])).$$

For the priori estimates of the solution to the system (1.4) are unconcerned with the period parameter l, we can derive the global smooth solution as $l \to \infty$, a.e. $x \in \mathbb{R}$. Theorem 1.1 is the global smooth solutions to the periodic initial boundary value problem for the system (1.4) and Theorem 1.2 is the global smooth solition for the Cauchy problem.

Theorem 1.2. *Assumed that* $u_0(x), v_0(x) \in H^m(\mathbb{R}), m \geq 4$*, then there exists a constant* $T_0 > 0$ *such that the system* $(1.4)-(1.5)$ *admits a unique local smooth solution with* $u_0(x), v_0(x)$ *as initial values, and there holds*

$$u(x,t), v(x,t) \in L^\infty(0, T_0; H^m(\mathbb{R})).$$

If $ac > 0, \frac{c}{a} > \frac{3\sqrt{5}-5}{10}$ *are satisfied on this basis , then there is a unique global smooth solution of the problem* $(1.4)-(1.5)$ *satisfying*

$$u(x,t), v(x,t) \in L^\infty(\mathbb{R}^+; H^m(\mathbb{R})).$$

Remark. We define the generalized solution of the coupled system $(1.4)-(1.6)$ as follows. Here we let $\Omega = [-l,l]$ or $\mathbb{R}$.

Definition 1.1. *The set of solution* $u(x,t), v(x,t) \in L^\infty(\mathbb{R}^+; H^m(\Omega)), m \geq 2$ *is called a generalized periodic solution of the coupled KdV system* $(1.4)-(1.6)$ *or* $(1.4)-(1.5)$ *if for any test function* $\psi(x,t) \in \Phi := \{\psi : \psi \in C^\infty([0,T] \times \Omega), \psi(x,T) \equiv 0, \forall T > 0\}$*, there hold the following integral identities:*

$$\int_0^T \int_\Omega (u\psi_t + 2bu\psi_x + au_{xx}\psi_x + 2buv\psi_x)\,dx\,dt + \int_\Omega u_0\psi(x,0)\,dx = 0,$$

$$\int_0^T \int_\Omega (v\psi_t + bv\psi_x + \frac{b}{2}v^2\psi_x + cv_{xx}\psi_x + |u|^2\psi_x)\,dx\,dt + \int_\Omega v_0\psi(x,0)\,dx = 0,$$

with the periodic and initial conditions $(1.5)-(1.6)$.

Theorem 1.3. *Assumed that* $ac > 0, \frac{c}{a} > \frac{3\sqrt{5}-5}{10}$ *and* $u_0, v_0 \in H^m(\Omega), m \geq 2$, *then there exists a unique generalized solution of the system* $(1.4)-(1.6)$ *or* $(1.4)-(1.5)$ *satisfying*

$$u(x,t), v(x,t) \in L^\infty(\mathbb{R}^+; H^m(\Omega)).$$

Proof. Taking the sequences of the initial values $\{u_0^i\}, \{v_0^i\} \in H^m(\Omega), m \geq 2$, when $i \to \infty$, $\{u_0^i\}, \{v_0^i\}$ are strongly converge in $H^m(\Omega)$ to u_0 and v_0. Then we can prove that $\{u^i(x,t)\}, \{v^i(x,t)\}$ are strongly converge to $u(x,t)$ and $v(x,t)$ respectively in $L^\infty(\mathbb{R}^+; H^m(\Omega))$. Thus $u(x,t), v(x,t) \in L^\infty(\mathbb{R}^+; H^m(\Omega))$. From the standard method, we can prove that $u(x,t), v(x,t)$ is the unique generalized solution of the system $(1.4)-(1.6)$ or $(1.4)-(1.5)$ satisfying the Definition 1.1, here we omit the details.

Notations. Throughout the paper, C stands for a generic positive constant, which may be different from line to line. We will use the notation $A \lesssim B$ to denote the relation $A \leq CB$ for conciseness.

This paper is organized as follows. In section 2, we present several function spaces and symbols, which will be frequently used throughout the rest of the paper. In section 3, we construct the approximate solutions by the Galerkin finite element method and prove the existence of classical local solution by sequential approximation. In section 4, we give some priori estimates by the conserved quantities and nonconservative quantity $I(u,v)$, then we obtain the existence of global smooth solution. The uniqueness of smooth solution will be proved in section 5.

2 Preliminaries

In this preliminaries section, we introduce some function spaces, symbols and a lemma which play an important role in our proofs.

$C^k([-l,l])$ denotes a complex valued function space which is continuously differentiable k times on the interval $[-l,l]$.

$L^p([-l,l])$ denotes that the Lebesque measurable complex valued function $f(x)$ on the interval $[-l,l]$ has a p^{th}-integrable space, and its norm is expressed as

$$\|f\|_p = (\int_{-l}^l |f|^p\,\mathrm{d}x)^{1/p}.$$

Denote the inner product as follows:

$$(f,g) = \int_{-l}^l f(x,t)\overline{g(x,t)}\,\mathrm{d}x.$$

where $\overline{g(x,t)}$ represents the complex conjugate of $g(x,t)$, then $L^2([-l,l])$ is a complete complex Hilbert space.

$L^\infty([-l,l])$ denotes a space where Lebesgue measurable function $f(x)$ is almost bounded on the interval $[-l,l]$, and its norm is expressed as

$$\|f\|_\infty = \operatorname*{esssup}_{x\in[-l,l]} |f(x)|.$$

$H^s([-l,l])$ denotes a complex valued function space with generalized derivatives $D^k u(|k| \le s) \in L^2([-l,l])$, and its norm is expressed as

$$\|u\|_{H^s}^2 = \sum_{|k|\le s} \|D^k u\|_2^2.$$

$H_0^s([-l,l])$ denotes the closure of an infinitely differentiable function with compact support $C_0^\infty([-l,l])$ in the norm sense of H^s on the interval $[-l,l]$.

$W_p^m([-l,l])$ represents the function space composed of $D^k u(|k| \le m) \in L^p([-l,l])$, where $D^k u$ is the weak partial derivative of u, and its norm is expressed as

$$\|u\|_{W_p^m}^p = \sum_{|k|\le m} \|D^k u\|_p^p.$$

$L^\infty(0,T;H^s)$ indicates that the complex valued function $u(x,t)$ belongs to the H^s space as a function of x, and there holds

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{H^s} < \infty.$$

Lemma 2.1. *(Sobolev inequality) Given $\epsilon > 0, n$, there exist a constant C which depends on ϵ and n, such that*

$$\|\frac{\partial^k u}{\partial x^k}\|_\infty \le C\|u\|_2 + \epsilon\|\frac{\partial^n u}{\partial x^n}\|_2, \quad k < n$$
$$\|\frac{\partial^k u}{\partial x^k}\|_2 \le C\|u\|_2 + \epsilon\|\frac{\partial^n u}{\partial x^n}\|_2. \quad k < n$$

3 Existence of local solutions

In this section, we prove that $(1.4)-(1.6)$ admits at least one classical local solution by using the Galerkin finite element method and sequential approximation. Firstly, we construct the Galerkin finite element solution.

Choosing $\{w_j(x)\}$ is the basis function of N-dimensional space $S^4 \subset H^4(R)$, where $S^4 = \{p(x), x \in [-l,l]$; $p(x)$ periodically expand to $C^3(\mathbb{R})$, $p(x)$ is the quintic polynomial on the interval $[ih,(i+1)h]$, $i = -m, 1-m, \cdots, 0, 1, \cdots, m-1$, $mh = l, \mathbb{R}$ is the real axis $\}$, and the basis function $\{w_j(x)\}$ is a characteristic function class of the ordinary differential equation $y'' = -\lambda y$ with the boundary conditions $y(-l) = y(l)$. Let the approximate solutions be:

$$u^h(x,t) = \sum_{j=-m}^{m-1} \eta_j(t) w_j(x), \quad v^h(x,t) = \sum_{j=-m}^{m-1} \zeta_j(t) w_j(x),$$

and the coefficient functions $\eta_j(t), \zeta_j(t)$ satisfy the periodic problems of nonlinear ordinary differential equations:

$$(u_t^h, w_j) + 2b(u_x^h, w_j) + a(u_{xxx}^h, w_j) + 2b((u^h v^h)_x, w_j) = 0, \qquad j = -m, 1-m, \cdots, m-1 \tag{3.1}$$

$$(v_t^h, w_j) + b(v_x^h, w_j) + b(v^h v_x^h, w_j) + c(v_{xxx}^h, w_j) + b((|u^h|^2)_x, w_j) = 0, \qquad j = -m, 1-m, \cdots, m-1 \tag{3.2}$$

$$u^h(x,0) = \sum_{j=-m}^{m-1} \eta_j(0) w_j(x) = u_0^h(x), \quad v^h(x,0) = \sum_{j=-m}^{m-1} \zeta_j(0) w_j(x) = v_0^h(x). \tag{3.3}$$

where $\{\eta_j(t)\}$ are complex value functions,while $\{\zeta_j(t)\}$ are real-valued functions. Because of the linear independency for $\{w_j(x)\}$ and the denseness of $\{w_j(x)\} \in H^4(\mathbb{R}) \subset H^1(\mathbb{R})$, there exists a complex constant c_j and a real constant d_j such that

$$u_0^h(x) \overset{H^1(\mathbb{R})}{\longrightarrow} u_0(x), \quad v_0^h(x) \overset{H^1(\mathbb{R})}{\longrightarrow} v_0(x), \tag{3.4}$$

$$\eta_j(0) = c_j, \quad \zeta_j(0) = d_j. \tag{3.5}$$

If $u^h(x,t)$ and $v^h(x,t)$ satisfy $(3.1)-(3.5)$, then $u^h(x,t), v^h(x,t)$ are a set of finite element solutions of the problem (1.4).

The solutions of Cauchy problem of nonlinear ordinary differential equations $(3.1)-(3.5)$ exist, for

$$(\frac{\partial u^h}{\partial t}, w_j) = (\frac{\partial}{\partial t} \sum_{k=-m}^{m-1} \eta_k w_k, w_j) = \sum_{k=-m}^{m-1} \eta_k'(t)(w_k, w_j),$$

$$(\frac{\partial v^h}{\partial t}, w_j) = (\frac{\partial}{\partial t} \sum_{k=-m}^{m-1} \zeta_k w_k, w_j) = \sum_{k=-m}^{m-1} \zeta_k'(t)(w_k, w_j).$$

Since the basis functions $\{w_j(x)\}$ are linearly independent, $\det(w_k, w_j) \neq 0$. And from the priori estimation of u^h and v^h by the following lemmas, we can know that the solutions $\eta_j(t), \zeta_j(t)$ of the problem $(3.1)-(3.5)$ exist.

Lemma 3.1. *Let $u_0^h(x), v_0^h(x) \in L^2([-l,l])$, then there exists a constant $C > 0$ such that*

$$\|u^h\|_2^2 + \|v^h\|_2^2 \leq C,$$

where the constant C is only related to $\|u_0^h\|_2, \|v_0^h\|_2$.

Proof. Multipling $\overline{\eta_j(t)}$ by (3.1), and summing about j we get

$$(u_t^h, u^h) + 2b(u_x^h, u^h) + a(u_{xxx}^h, u^h) + 2b((u^h v^h)_x, u^h) = 0, \tag{3.6}$$

where

$$Re(u_t^h, u^h) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^h\|_2^2, \quad Re(2bu_x^h, u^h) = Re(au_{xxx}^h, u^h) = 0, \quad 2b((u^h v^h)_x, u^h) = -2b(u^h v^h, u_x^h).$$

Multipling $\zeta_j(t)$ by (3.2), and summing about j we arrive at

$$(v_t^h, v^h) + b(v_x^h, v^h) + b(v^h v_x^h, v^h) + c(v_{xxx}^h, v^h) + b((|u^h|^2)_x, v^h) = 0, \tag{3.7}$$

where

$$(v_t^h, v^h) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v^h\|_2^2, \quad b(v_x^h, v^h) = b(v^h v_x^h, v^h) = c(v_{xxx}^h, v^h) = 0,$$

$$b((|u^h|^2)_x, v^h) = b(u_x^h v^h, u^h) + b(u^h v^h, u_x^h).$$

Combining (3.6) and (3.7) and taking the real part we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u^h\|_2^2 + \|v^h\|_2^2) = 0.$$

Integrating the above equality with respect to $t \in [0, T]$, one gets

$$\|u^h(T)\|_2^2 + \|v^h(T)\|_2^2 = \|u^h(0)\|_2^2 + \|v^h(0)\|_2^2 \le C.$$

Lemma 3.2. *Under the conditions in Lemma* 3.1*, and* $u_0^h(x), v_0^h(x) \in H^2([-l, l])$*, then there exists a constant* $T_0 > 0$ *such that for any* $t \in [0, T_0]$*, there holds*

$$\|u_x^h\|_2^2 + \|v_x^h\|_2^2 + \|u_{xx}^h\|_2^2 + \|v_{xx}^h\|_2^2 \le C.$$

Proof. Multiplying $\overline{\eta_j(t)}$ by $(u_t^h + 2bu_x^h + au_{xxx}^h + 2b(u^h v^h)_x, -\lambda w_j) = 0$ and summing about j we arrive at

$$(u_t^h + 2bu_x^h + au_{xxx}^h + 2b(u^h v^h)_x, u_{xx}^h) = 0, \tag{3.8}$$

where

$$Re(u_t^h, u_{xx}^h) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^h\|_2^2, \quad Re(2bu_x^h + au_{xxx}^h, u_{xx}^h) = 0,$$

and in the above relations we have used $w_j''(x) = -\lambda w_j$.

Multiplying $\zeta_j(t)$ by $(v_t^h + bv_x^h + bv^h v_x^h + cv_{xxx}^h + b(|u^h|^2)_x, -\lambda w_j) = 0$ and summing about j we obtain

$$(v_t^h + bv_x^h + bv^h v_x^h + cv_{xxx}^h + b(|u^h|^2)_x, v_{xx}^h) = 0, \tag{3.9}$$

where

$$(v_t^h, v_{xx}^h) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_x^h\|_2^2, \quad (bv_x^h + cv_{xxx}^h, v_{xx}^h) = 0, \quad (bv^h v_x^h, v_{xx}^h) = -\frac{b}{2}\int_{-l}^{l} (v_x^h)^3 \,\mathrm{d}x.$$

Combining (3.8) and (3.9) and taking the real part we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_x^h\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|v_x^h\|_2^2 + b\int_{-l}^{l} (v_x^h)^3 \,\mathrm{d}x + 6b\int_{-l}^{l} |u_x^h|^2 v_x^h \,\mathrm{d}x = 0.$$

Thus, we can get the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_x^h\|_2^2 + \|v_x^h\|_2^2) \lesssim \|v_x^h\|_2^2\|v_x^h\|_\infty + \|u_x^h\|_2^2\|v_x^h\|_\infty. \tag{3.10}$$

Similarly, multiplying $\overline{\eta_j(t)}, \zeta_j(t)$ by $(u_t^h + 2bu_x^h + au_{xxx}^h + 2b(u^h v^h)_x, \lambda^2 w_j) = 0$ and $(v_t^h + bv_x^h + bv^h v_x^h + cv_{xxx}^h + b(|u^h|^2)_x, \lambda^2 w_j) = 0$ respectively, and summing about j we arrive at

$$(u_t^h + 2bu_x^h + au_{xxx}^h + 2b(u^h v^h)_x, u_{xxxx}^h) = 0, \tag{3.11}$$

$$(v_t^h + bv_x^h + bv^h v_x^h + cv_{xxx}^h + b(|u^h|^2)_x, v_{xxxx}^h) = 0. \tag{3.12}$$

By calculation, combining (3.11) and (3.12) and taking the real part we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{xx}^h\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|v_{xx}^h\|_2^2 + 5b\int_{-l}^{l} v_x^h (v_{xx}^h)^2 \,\mathrm{d}x + 5b\int_{-l}^{l} |u_{xx}^h|^2 v_x^h \,\mathrm{d}x + Re2b\int_{-l}^{l}(4u_x^h\overline{u_{xx}^h}v_{xx}^h + \overline{u_x^h}u_{xx}^h v_{xx}^h)\,\mathrm{d}x = 0.$$

Thus, we can get estimate as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_{xx}^h\|_2^2 + \|v_{xx}^h\|_2^2) \lesssim \|v_{xx}^h\|_2^2\|v_x^h\|_\infty + \|u_{xx}^h\|_2^2\|v_x^h\|_\infty + \|u_{xx}^h\|_2\|v_{xx}^h\|_2\|u_x^h\|_\infty. \tag{3.13}$$

Finally combining (3.10) and (3.13) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_x^h\|_2^2 + \|v_x^h\|_2^2 + \|u_{xx}^h\|_2^2 + \|v_{xx}^h\|_2^2) \lesssim \|u_x^h\|_2^3 + \|v_x^h\|_2^3 + \|u_{xx}^h\|_2^3 + \|u_{xx}^h\|_2^3.$$

Thus by the above inequality we get, if $u_0^h, v_0^h \in H^2([-l,l])$, there exist constants $T_0, C > 0$, such that for any $t \in [0, T_0]$, there holds

$$\|u_x^h\|_2^2 + \|v_x^h\|_2^2 + \|u_{xx}^h\|_2^2 + \|v_{xx}^h\|_2^2 \le C.$$

Lemma 3.3. *Under the conditions in Lemma 3.2, and $u_0^h(x), v_0^h(x) \in H^3([-l,l])$, then there exists a constant $T_0 > 0$, such that for any $t \in [0, T_0]$, there holds*

$$\|u_t^h\|_2^2 + \|v_t^h\|_2^2 \le C.$$

Proof. Differentiating (3.1) and (3.2) with respect to t, then multiplying by $\overline{\eta_j'(t)}$ and $\zeta_j'(t)$ respectively and summing about j we can get

$$(E_t + 2bE_x + aE_{xxx} + 2b(E_x v^h + u_x^h F + E v_x^h + u^h F_x), E) = 0, \tag{3.14}$$

$$(F_t + bF_x + b(F v_x^h + v^h F_x) + cF_{xxx} + b(E_x\overline{u^h} + u_x^h\overline{E} + \overline{E}_x u^h + \overline{u_x^h}E), F) = 0, \tag{3.15}$$

where $E := u_t^h, F := v_t^h$.

Applying integration by parts and taking the real part, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|E\|_2^2 + \|F\|_2^2) = &- \frac{b}{2}\int_{-l}^{l} v_x^h(2|E|^2 + |F|^2)\,\mathrm{d}x - 2bRe\int_{-l}^{l} u_x^h\overline{E}F\,\mathrm{d}x + 2bRe\int_{-l}^{l}\overline{u_x^h}FE\,\mathrm{d}x \\ &- b\int_{-l}^{l}(u_x^h\overline{E}F + \overline{u_x^h}EF)\,\mathrm{d}x \\ \le\, & b(\|u_x^h\|_\infty + \|v_x^h\|_\infty)(\|E\|_2^2 + \|F\|_2^2).\end{aligned}$$

Combining Lemma 3.2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|E\|_2^2 + \|F\|_2^2) \lesssim \|E\|_2^2 + \|F\|_2^2.$$

Thus, if $u_0^h(x), v_0^h(x) \in H^3([-l,l])$, there exist constants $T_0, C > 0$, such that for any $t \in [0, T_0]$, there holds

$$\|E\|_2^2 + \|F\|_2^2 \leq C.$$

Lemma 3.4. *Under the conditions in Lemma* 3.3*, and* $u_0^h(x), v_0^h(x) \in H^3([-l,l])$*, then there exists a constant* $T_0 > 0$*, such that for any* $t \in [0, T_0]$*, there holds*

$$\|u_{xxx}^h\|_2^2 + \|v_{xxx}^h\|_2^2 \leq C.$$

Lemma 3.5. *Under the conditions in Lemma* 3.4*, and* $u_0^h(x), v_0^h(x) \in H^4([-l,l])$*, then there exists a constant* $T_0 > 0$*, such that for any* $t \in [0, T_0]$*, there holds*

$$\|u_{xt}^h\|_2^2 + \|v_{xt}^h\|_2^2 \leq C.$$

Proof. Differentiating (3.1) and (3.2) with respect to t, then multiplying them by $\overline{\eta_j'(t)}$ and $\zeta_j'(t)$ respectively and summing about j we can get

$$(E_t + 2bE_x + aE_{xxx} + 2b(E_x v^h + u_x^h F + E v_x^h + u^h F_x), E_{xx}) = 0, \tag{3.16}$$

$$(F_t + bF_x + b(F v_x^h + v^h F_x) + cF_{xxx} + b(E_x \overline{u^h} + u_x^h \overline{E} + \overline{E_x} u^h + \overline{u_x^h} E), F_{xx}) = 0. \tag{3.17}$$

Similarly to the estimations of Lemma 3.1 − 3.3, combining the above two and taking the real part we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|E_x\|_2^2 + \|F_x\|_2^2) \lesssim \|E_x\|_2^2 + \|F_x\|_2^2 + 1.$$

Thus, by the Gronwall inequality we obtain that, if $u_0^h(x), v_0^h(x) \in H^4([-l,l])$, there exist constants $T_0, C > 0$, such that $\forall t \in [0, T_0]$, there holds

$$\|E_x\|_2^2 + \|F_x\|_2^2 \leq C.$$

Lemma 3.6. *Under the conditions in Lemma* 3.5*, and* $u_0^h(x), v_0^h(x) \in H^4([-l,l])$*, then there exists a constant* $T_0 > 0$*, such that for any* $t \in [0, T_0]$*, there holds*

$$\|u_{xxxx}^h\|_2^2 + \|v_{xxxx}^h\|_2^2 \leq C.$$

Thanks to the Lemma 3.1−3.6, we obtain the result about the existence of classical local solution as follows.

Theorem 3.1. *If* $u_0(x), v_0(x) \in H^4([-l,l]))$ *and they are periodic functions with period* $2l$*, then there exists a constant* $T_0 > 0$ *such that the periodic initial value problem* (1.4) − (1.6) *admits at least one classical local solution with* $u_0(x), v_0(x)$ *as initial values satisfying*

$$u(x,t), v(x,t) \in L^\infty(0, T_0; C^3([-l,l])).$$

Proof. Thanks to Lemmas 3.1 − 3.6, we obtain that there exist a constant T_0 such that for any $0 \leq t \leq T_0$, $\{u^h\}$ and $\{v^h\}$ are uniformly bounded in $H^4([-l,l])$, and the upper bound continuously depends on the initial values, therefore we can select subsequences (still recorded as) $\{u^h\}, \{v^h\}$ such that when $h \to 0$, $\{u^h\}, \{v^h\}$ are weakly star converge in $L^\infty(0, T_0; H^4([-l,l]))$ to u and v respectively; and $\{u_t^h\}, \{v_t^h\}$ are weakly star converge in $L^\infty(0, T_0; H^1([-l,l]))$ to u_t and v_t respectively. Especially, in $L^\infty(0, T_0; L^2([-l,l]))$, $\{(u^h v^h)_x\}, \{v^h v_x^h\}$ and $\{(|u^h|^2)_x\}$ are weakly star converge to $(uv)_x, vv_x$ and $(|u|^2)_x$ respectively.

Therefore let $h \to 0$, we can obtain that the classical local solutions of the coupled problem (1.4) − (1.6) exist and satisfy $u(x,t), v(x,t) \in L^\infty(0, T_0; C^3([-l,l]))$.

4 Existence of global solution

In this section, we prove the existence of global smooth solution.

The set of solution $u = u(x,t), v = v(x,t)$ for the periodic initial value problem (1.4) − (1.6) satisfy $(1.4)_1, (1.4)_2, u(\cdot,t), v(\cdot,t) \in C^3([-l,l]), u_{xxx}(\cdot,t), v_{xxx}(\cdot,t) \in H^1([-l,l])$ satisfy the initial value condition (1.5) and periodic condition (1.6) and we know that, $\forall x, t$, the initial functions $u_0(x), v_0(x)$ should be the periodic functions with period $2l$. In the following lemmas, it is assumed that $u(x,t), v(x,t)$ are periodic solutions with $u_0(x), v_0(x)$ as initial values respectively.

Lemma 4.1. *Assume that $ac > 0$ and $u_0(x), v_0(x) \in H^1([-l,l])$, then for any $T > 0, t \in [0,T]$ there holds*

$$\|u\|_{H^1}^2 + \|v\|_{H^1}^2 \le C,$$

where the constant C depends on $a, b, c, \|u_0\|_{H^1}, \|v_0\|_{H^1}$.

Proof. Taking the inner product of $(1.4)_1, (1.4)_2$ with $\overline{u}$ and v on the interval $[-l,l]$ respectively, we have

$$(u_t + 2bu_x + au_{xxx} + 2b(uv)_x, u) = 0, \tag{4.1}$$

$$(v_t + bv_x + bvv_x + cv_{xxx} + b(|u|^2)_x, v) = 0, \tag{4.2}$$

Then summing (4.1) and (4.2), taking the real part and using integration by parts we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_2^2 + \|v\|_2^2) = 2bRe\int_{-l}^{l} uv\overline{u_x}\,\mathrm{d}x - b\int_{-l}^{l}(|u|^2)_x v\,\mathrm{d}x.$$

Finally, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_2^2 + \|v\|_2^2) = 0,$$

and integrating it in $t \in [0,T], \forall T > 0$, we have

$$\|u(\cdot,T)\|_2^2 + \|v(\cdot,T)\|_2^2 = \|u_0\|_2^2 + \|v_0\|_2^2.$$

Through the conserved quantity $H_3(u,v)$, we get

$$H_3(u,v)|_{[-l,l]} = H_3(u_0,v_0)|_{[-l,l]},$$

where

$$H_3(u,v)|_{[-l,l]} = \int_{-l}^{l}(\frac{a}{2}|u_x|^2 + \frac{c}{2}v_x^2 - \frac{b}{6}v^3 - b|u|^2v - b|u|^2 - \frac{b}{2}v^2)\,\mathrm{d}x.$$

Thus by Lemma 2.1, if $ac > 0$, we have

$$\begin{aligned}\frac{|a|}{2}\|u_x\|_2^2 + \frac{|c|}{2}\|v_x\|_2^2 \le& |H_3(u_0,v_0)|_{[-l,l]}| + |\int_{-l}^{l}(\frac{b}{6}v^3 + b|u|^2v + b|u|^2 + \frac{b}{2}v^2)\,\mathrm{d}x| \\ \le& C + \frac{|b|}{6}\|v\|_2^2\|v\|_\infty + |b|\|u\|_2^2\|v\|_\infty + |b|\|u\|_2^2 + \frac{|b|}{2}\|v\|_2^2\end{aligned}$$

$$
\begin{aligned}
&\leq C+\frac{|b|}{6}\|v\|_2^{\frac{5}{2}}\|v_x\|_2^{\frac{1}{2}}+|b|\|u\|_2^2\|v\|_2^{\frac{1}{2}}\|v_x\|_2^{\frac{1}{2}}+|b|\|u\|_2^2+\frac{|b|}{2}\|v\|_2^2\\
&\leq C+\frac{|c|}{8}\|v_x\|_2^2+C(b,c)\|v\|_2^{\frac{10}{3}}+\frac{|c|}{8}\|v_x\|_2^2+C(b,c)\|u\|_2^{\frac{8}{3}}\|v\|_2^{\frac{2}{3}}\\
&\quad+|b|\|u\|_2^2+\frac{|b|}{2}\|v\|_2^2\\
&\leq C(b,c,\|u\|_2,\|v\|_2)+\frac{|c|}{4}\|v_x\|_2^2.
\end{aligned}
$$

where $C(b,c,\|u\|_2,\|v\|_2)$ represents a constant related to $b,c,\|u\|_2,\|v\|_2$, and $C(b,c)$ also has a similar definition. Thus,

$$
\frac{|a|}{2}\|u_x\|_2^2+\frac{|c|}{4}\|v_x\|_2^2\leq C(b,c,\|u\|_2,\|v\|_2).
$$

And we complete the proof of Lemma 4.1.

Lemma 4.2. *If the following conditions are met,*
(1)$ac>0,\frac{c}{a}>\frac{3\sqrt{5}-5}{10}$;
(2)$u_0,v_0\in H^2([-l,l])$;
then for any $T>0,t\in[0,T]$ there holds

$$
\|u_{xx}\|_2^2+\|v_{xx}\|_2^2\leq C,
$$

where the constant C is related to $a,b,c,\|u_0\|_{H^2},\|v_0\|_{H^2}$.

Proof. Before proving this lemma, we give a claim, that is $I(u,v)$ is bounded, and

$$
\begin{aligned}
I(u,v)=\int_{-l}^{l}&(27a^2c|u_{xx}|^2+(15c^3+15ac^2-3a^2c)v_{xx}^2-90abc|u_x|^2v+(25bc^2+25abc-5a^2b)v_x^2v\\
&-(60abc+30bc^2)(|u|^2)_xv_x)\,\mathrm{d}x.
\end{aligned}
$$

Let

$$
I_1(u,v)=\int_{-l}^{l}(\gamma_1|u_{xx}|^2+\gamma_2v_{xx}^2+\gamma_3|u_x|^2v+\gamma_4v_x^2v+\gamma_5(|u|^2)_xv_x)\,\mathrm{d}x,
$$

where $\gamma_i(i=1,2,3,4)$ are undetermined coefficients and γ_1,γ_2 are the same sign and not zero, that is, the same positive or negative numbers, thus there holds

$$
\begin{aligned}
\|u_{xx}\|_2^2+\|v_{xx}\|_2^2\leq&\, C+|\gamma_3|\|u_x\|_2^2\|v\|_\infty+|\gamma_4|\|v_x\|_2^2\|v\|_\infty+|\gamma_5|\|u_x\|_2\|v_x\|_2\|u\|_\infty\\
\leq&\, C.
\end{aligned}\tag{4.3}
$$

where the constant C is related to $\gamma_i(i=1,2,3,4,5),\|u\|_{H^1},\|v\|_{H^1}$.

Next, we prove the claim. Firstly, differenting $I_1(u,v)$ with respect to t we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}I_1(u,v)=\int_{-l}^{l}&\gamma_1(u_{xx}\overline{u_{xxt}}+\overline{u_{xx}}u_{xxt})+2\gamma_2v_{xx}v_{xxt}+\gamma_3(|u_x|^2v_t+u_x\overline{u_{xt}}v+\overline{u_x}u_{xt}v)\\
&+\gamma_4(v_x^2v_t+2vv_xv_{xt})+\gamma_5((|u|^2)_{xt}v_x+(|u|^2)_xv_{xt})\,\mathrm{d}x.
\end{aligned}\tag{4.4}
$$

where

$$\int_{-l}^{l}(u_{xx}\overline{u_{xxt}}+\overline{u_{xx}}u_{xxt})\,\mathrm{d}x=\int_{-l}^{l}-10b|u_{xx}|^2v_x+6b|u_x|^2v_{x^3}-2b(\overline{u}u_{xx}v_{x^3}+u\overline{u_{xx}}v_{x^3})\,\mathrm{d}x, \tag{4.5}$$

$$\int_{-l}^{l}2v_{xx}v_{xxt}\,\mathrm{d}x=\int_{-l}^{l}-5bv_xv_{xx}^2+4b|u_x|^2v_{x^3}+2b(\overline{u}u_{xx}v_{x^3}+u\overline{u_{xx}}v_{x^3})\,\mathrm{d}x, \tag{4.6}$$

$$\begin{aligned}\int_{-l}^{l}(|u_x|^2v_t+u_x\overline{u_{xt}}v+\overline{u_x}u_{xt}v)\,\mathrm{d}x=\int_{-l}^{l}&b|u_x|^2v_x-3a|u_{xx}|^2v_x+(a-c)|u_x|^2v_{x^3}-5b|u_x|^2vv_x\\&+2b|u|^2v_xv_{xx}+2b|u|^2vv_{x^3}-b|u_x|^2(|u|^2)_x\,\mathrm{d}x,\end{aligned} \tag{4.7}$$

$$\int_{-l}^{l}(v_x^2v_t+2vv_xv_{xt})\,\mathrm{d}x=\int_{-l}^{l}-3cv_xv_{xx}^2-4b|u|^2v_xv_{xx}-2b|u|^2vv_{x^3}-bvv_x^3\,\mathrm{d}x, \tag{4.8}$$

$$\begin{aligned}\int_{-l}^{l}((|u|^2)_{xt}v_x+(|u|^2)_xv_{xt})\,\mathrm{d}x=\int_{-l}^{l}&(2c+a)|u_x|^2v_{x^3}+(c-a)(\overline{u}u_{xx}v_{x^3}+u\overline{u_{xx}}v_{x^3})-b|u|^2v_{x^3}\\&-2b|u_x|^2vv_x+8b|u|^2v_xv_{xx}-b(u\overline{u_{xx}}vv_x+\overline{u}u_{xx}vv_x)\,\mathrm{d}x.\end{aligned} \tag{4.9}$$

Combining the above equalities (4.4) – (4.9), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_1(u,v)\le&(-10\gamma_1b-3\gamma_3a)|u_{xx}|^2v_x-(5\gamma_2b+3\gamma_4c)v_xv_{xx}^2\\&+(6\gamma_1b+4\gamma_2b+\gamma_3(a-c)+\gamma_5(2c+a))|u_x|^2v_{x^3}\\&+(-2b\gamma_1+2b\gamma_2+\gamma_5(c-a))(\overline{u}u_{xx}v_{x^3}+u\overline{u_{xx}}v_{x^3})+C(\|u_{xx}\|_2^2+\|v_{xx}\|_2^2)+C,\end{aligned} \tag{4.10}$$

where the constant C is related to $\gamma_i(i=1,\cdots,5),a,b,\|u\|_{H^1},\|v\|_{H^1}$. Therefore in order to enable the right side of the (4.10) to be controlled by $\|u_{xx}\|_2^2+\|v_{xx}\|_2^2+C$, the coefficients of the top four items on the right end of the (4.10) should be 0, that is

$$-10\gamma_1b-3\gamma_3a=0,\quad 5\gamma_2b+3\gamma_4c=0,$$
$$6\gamma_1b+4\gamma_2b+\gamma_3(a-c)+\gamma_5(2c+a)=0,\quad -2b\gamma_1+2b\gamma_2+\gamma_5(c-a)=0.$$

From these we can obtain the relationship between $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ and γ_5 as

$$\gamma_2=\frac{5c^2+5ac-a^2}{9a^2}\gamma_1,\quad \gamma_3=-\frac{10b}{3a}\gamma_1,\quad \gamma_4=\frac{25bc^2+25abc-5a^2b}{27a^2c}\gamma_1,\quad \gamma_5=-\frac{20ab+10bc}{9a^2}\gamma_1. \tag{4.11}$$

At the same time, in order to guarantee the coefficients γ_1,γ_2 are the same sign and not zero, there must hold $5c^2+5ac-a^2>0$, thus the relationship between the parameters a,c is obtained as follows:

$$ac>0,\quad \frac{c}{a}>\frac{3\sqrt{5}-5}{10}.$$

Combining (4.10) and (4.11), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I_1(u,v)\lesssim I_1(u,v)+C.$$

Applying the Gronwall inequality, we know that $I_1(u,v)$ is bounded. And combining (4.3), we finally obtain that desired reslut.

Lemma 4.3. *Under the conditions in Lemma 4.2, and $u_0, v_0 \in H^3([-l,l])$, then for any $T>0, t\in [0,T]$ there holds*

$$\|u_t\|_2^2 + \|v_t\|_2^2 \le C,$$

where the constant C is related to $a, b, c, \|u_0\|_{H^3}, \|v_0\|_{H^3}$.

Proof. Differenting the equations $(1.4)_1, (1.4)_2$ with respect to t and taking the inner product with $\overline{u_t}, v_t$ respectively on the interval $x\in[-l,l]$ we have

$$(u_{tt} + 2bu_{xt} + au_{x^3t} + 2b(u_{xt}v + u_xv_t + u_tv_x + uv_{xt}), u_t) = 0, \tag{4.12}$$

$$(v_{tt} + bv_{xt} + b(v_tv_x + vv_{xt}) + cv_{x^3t} + b(u_{xt}\overline{u} + u_x\overline{u_t} + \overline{u_x}u_t + \overline{u_{xt}}u), v_t) = 0, \tag{4.13}$$

Summing (4.12) and (4.13), taking the real part and using integration by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|_2^2 + \|v_t\|_2^2) =& -\frac{b}{2}\int_{-l}^{l} v_x(2|u_t|^2 + v_t^2)\,\mathrm{d}x - 2bRe\int_{-l}^{l} u_xv_t\overline{u_t}\,\mathrm{d}x + 2bRe\int_{-l}^{l}\overline{u_x}v_tu_t\,\mathrm{d}x\\ &- b\int_{-l}^{l}(u_x\overline{u_t}v_t + \overline{u_x}u_tv_t)\,\mathrm{d}x\\ \le& \, b(\|u_x\|_\infty + \|v_x\|_\infty)(\|u_t\|_2^2 + \|v_t\|_2^2).\end{aligned}$$

Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_t\|_2^2 + \|v_t\|_2^2) \lesssim \|u_t\|_2^2 + \|v_t\|_2^2.$$

Combining the Gronwall inequality, we derive that for any $T>0, t\in[0,T]$, there holds

$$\|u_t\|_2^2 + \|v_t\|_2^2 \le C.$$

Lemma 4.4. *Under the conditions in Lemma 4.3, then for any $T>0, t\in[0,T]$ there holds*

$$\|u_{xxx}\|_2^2 + \|v_{xxx}\|_2^2 \le C.$$

where the constant C is depends on $a, b, c, \|u_0\|_{H^3}, \|v_0\|_{H^3}$.

Proof. Taking the inner product of $(1.4)_1, (1.4)_2$ with $\overline{u_{xxx}}$ and v_{xxx} respectively we have

$$(u_t + 2bu_x + au_{xxx} + 2b(uv)_x, u_{xxx}) = 0, \tag{4.14}$$

$$(v_t + bv_x + bvv_x + cv_{xxx} + b(|u|^2)_x, v_{xxx}) = 0. \tag{4.15}$$

Thus, taking the real part of (4.14) we obtain

$$\begin{aligned}a\|u_{xxx}\|_2^2 =& -Re\int_{-l}^{l}(u_t + 2bu_x + 2bu_xv + 2buv_x)\overline{u_{xxx}}\,\mathrm{d}x\\ \le& (\|u_t\|_2 + 2b\|u_x\|_2 + 2b\|v\|_\infty\|u_x\|_2 + 2b\|u\|_\infty\|v_x\|_2)\|u_{xxx}\|_2,\end{aligned}$$

and applying the lemmas 4.1 − 4.3, we have

$$\|u_{xxx}\|_2 \le C. \qquad \forall T>0, t\in[0,T]$$

In the same way, we can get the following estimate

$$\|v_{xxx}\|_2 \le C. \qquad \forall T>0, t\in[0,T]$$

And we complete the proof of Lemma 4.4.

Similar to the proof of the Lemmas $4.1-4.4$, if $u_0(x), v_0(x) \in H^m([-l,l]), m\ge 0$, we obtain the following lemma by the induction argument.

Lemma 4.5. *Assumed that $u_0(x), v_0(x) \in H^s([-l,l]), s\ge 0$, then for any $T>0, t\in[0,T]$ there holds*

$$\|u_{x^s}\|_2^2 + \|v_{x^s}\|_2^2 \le C. \tag{4.16}$$

where the constant C is depends on $a,b,c,\|u_0\|_{H^s}, \|v_0\|_{H^s}$.

Proof. This lemma will be proved by the induction for s. According to the Lemmas $4.1-4.4$, we can know that the estimate holds for $0\le s\le 3$.

Now we assume that the estimate holds for $s=M-1\ge 3$, and we will prove that (4.16) holds for $s=M$.

Using the integration by parts we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{x^M}\|_2^2 =& -2bRe\int_{-l}^{l}(uv)_{x^{M+1}}\overline{u_{x^M}}\,\mathrm{d}x\\ &\le \|u_{x^M}\|_2^2+\|v_{x^M}\|_2^2 + C + b\int_{-l}^{l}|u_{x^M}|^2 v_x\mathrm{d}x - b\int_{-l}^{l}(uv_{x^{M+1}}\overline{u_{x^M}} + \overline{u}u_{x^M}v_{x^{M+1}})\,\mathrm{d}x,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{x^M}\|_2^2 =& -b\int_{-l}^{l}(vv_x)_{x^M}v_{x^M}\mathrm{d}x - b\int_{-l}^{l}(|u|^2)_{x^{M+1}}v_{x^M}\,\mathrm{d}x\\ &\le \|u_{x^M}\|_2^2+\|v_{x^M}\|_2^2 + C + \frac{b}{2}\int_{-l}^{l}(v_{x^M})^2 v_x\,\mathrm{d}x - b\int_{-l}^{l}(\overline{u}u_{x^{M+1}}v_{x^M} + u\overline{u_{x^{M+1}}}v_{x^M})\,\mathrm{d}x.\end{aligned} \tag{4.18}$$

Combining (4.17) and (4.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_{x^M}\|_2^2 + \|v_{x^M}\|_2^2) \lesssim \|u_{x^M}\|_2^2 + \|v_{x^M}\|_2^2 + 1,$$

then combining the Gronwall inequality, we derive that

$$\|u_{x^M}\|_2^2 + \|v_{x^M}\|_2^2 \le C.$$

Thus by the induct method, we can obtain the estimate (4.16).

5 Uniqueness of smooth solution

In this section, we prove the uniqueness of global smooth solution in Theorem 1.1.

The proof of Theorem 1.1 Supposed that u_1, v_1 and u_2, v_2 are two sets of solutions to the system $(1.4)-(1.6)$. Let $\varphi = u_1-u_2, \phi = v_1 - v_2$, we have

$$\varphi_t + 2b\varphi_x + a\varphi_{xxx} = -2b(\varphi_x v_2 + \varphi v_{2x} + \phi_x u_1 + \phi u_{1x}) \tag{5.1}$$

$$= -2b(u_{2x}\phi + u_2\phi_x + \varphi_x v_1 + \varphi v_{1x}), \tag{5.2}$$

$$\phi_t + b\phi_x + c\phi_{xxx} = -b(v_2\phi_x + v_{1x}\phi) - b(u_{2x}\overline{\varphi} + u_2\overline{\varphi_x} + \overline{u_1}\varphi_x + \overline{u_{1x}}\varphi) \tag{5.3}$$

$$= -b(v_2\phi_x + v_{1x}\phi) - b(\varphi\overline{u_{2x}} + \varphi_x\overline{u_2} + u_1\overline{\varphi_x} + u_{1x}\overline{\varphi}), \tag{5.4}$$

and $\varphi(x,0) = \phi(x,0) = 0, \varphi(x+l,t) = \varphi(x-l,t), \phi(x+l,t) = \phi(x-l,t)$.

Firstly, taking the inner product of (5.1) and (5.2) with φ respectively, and taking the real part after the summation we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|_2^2 =& - b\int_{-l}^{l}|\varphi|^2 v_{2x}\,\mathrm{d}x - b\int_{-l}^{l}|\varphi|^2 v_{1x}\,\mathrm{d}x - b\int_{-l}^{l}(u_1\phi_x\overline{\varphi} + \overline{u_1}\phi_x\varphi + u_2\phi_x\overline{\varphi} + \overline{u_2}\phi_x\varphi)\,\mathrm{d}x\\ &- b\int_{-l}^{l}(u_{1x}\phi\overline{\varphi} + \overline{u_{1x}}\phi\varphi + u_{2x}\phi\overline{\varphi} + \overline{u_{2x}}\phi\varphi)\,\mathrm{d}x\\ &\le b\sum_{k=1}^{2}(\|u_{kx}\|_\infty + \|v_{kx}\|_\infty)(\|\varphi\|_2^2 + \|\phi\|_2^2) - b\int_{-l}^{l}(u_1\phi_x\overline{\varphi} + \overline{u_1}\phi_x\varphi + u_2\phi_x\overline{\varphi} + \overline{u_2}\phi_x\varphi)\,\mathrm{d}x.\end{aligned} \tag{5.5}$$

Secondly, taking the inner product of (5.3) and (5.4) with ϕ respectively and summing them we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_2^2 =& b\int_{-l}^{l} v_{2x}\phi^2\,\mathrm{d}x - 2b\int_{-l}^{l} v_{1x}\phi^2\,\mathrm{d}x - b\int_{-l}^{l}(u_{2x}\phi\overline{\varphi} + \overline{u_{1x}}\phi\varphi + \overline{u_{2x}}\phi\varphi + u_{1x}\phi\overline{\varphi})\,\mathrm{d}x\\ &- b\int_{-l}^{l}(u_2\phi\overline{\varphi_x} + \overline{u_1}\phi\varphi_x + \overline{u_2}\phi\varphi_x + u_1\phi\overline{\varphi_x})\,\mathrm{d}x\\ &\le b\sum_{k=1}^{2}(\|u_{kx}\|_\infty + \|v_{kx}\|_\infty)(\|\varphi\|_2^2 + \|\phi\|_2^2) - b\int_{-l}^{l}(u_2\phi\overline{\varphi_x} + \overline{u_1}\phi\varphi_x + \overline{u_2}\phi\varphi_x + u_1\phi\overline{\varphi_x})\,\mathrm{d}x.\end{aligned} \tag{5.6}$$

Combining (5.5) – (5.6) and applying integration by parts we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|_2^2 + \|\phi\|_2^2) &\le b\sum_{k=1}^{2}(\|u_{kx}\|_\infty + \|v_{kx}\|_\infty)(\|\varphi\|_2^2 + \|\phi\|_2^2) + b\int_{-l}^{l}\sum_{k=1}^{2}(u_{kx}\phi\overline{\varphi} + \overline{u_{kx}}\phi\varphi)\,\mathrm{d}x\\ &\le b\sum_{k=1}^{2}(\|u_{kx}\|_\infty + \|v_{kx}\|_\infty)(\|\varphi\|_2^2 + \|\phi\|_2^2).\end{aligned} \tag{5.7}$$

Thus, by the Gronwall inequality we obtain $\varphi = \phi = 0$ when $u_1, u_2, v_1, v_2 \in L^\infty(0,T;H^3[-l,l])$. This completes the proof of Theorem 1.1.

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