Pseudo almost periodic solution of fractional-order Clifford-valued high-order Hopfield neural networks

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Abstract

In this work, based on the principle of contraction mapping, we deduce sufficient conditions ensuring the existence of pseudo almost periodic solutions of fractional-order Clifford-valued high-order Hopfield neural networks (FCHHNNs). In addition, we employ a kind of Gronwall inequality to study the finite-time stability of pseudo almost periodic solutions of FCHHNNs. The results and methods of our paper are new. Finally, we give a numerical example to illustrate the effectiveness of the results obtained.

Keywords: Fractional-order calculus; High-order Hopfield neural network; Clifford algebra; Pseudo almost periodic solution; Finite-time stability.

1 Introduction

Clifford-valued neural networks (CVNNs) have greater advantages in high-dimensional signal processing and storage capacity because they require fewer connection weight functions compared to real-valued, complex-valued, and quaternion-valued neural networks [1–4]. In recent years, more and more scholars have devoted themselves to the theoretical and practical application research of CVNN, and have achieved many results [5–12]. Since the multiplication of Clifford algebra does not satisfy the commutative law, many results have been obtained by decomposing CVNNs into real-valued neural networks [7, 8, 13, 14]. However, the results

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obtained by the decomposition method are essentially about the real-valued system, which is not easy to be directly applied to the CVNN system under consideration. Therefore, it is of great theoretical significance and potential application value to further explore the direct method for studying the qualitative behavior of CVNNs, namely, the non decomposition method.

In addition, it is well known that fractional order differential equation models can better describe some real processes with heredity, memory and nonlocality than integer order differential equation models in many aspects. Therefore, in recent decades, researchers have proposed a large number of fractional differential equation models, and the application fields of fractional differential equations are also expanding. In particular, the study of fractional order neural networks has received extensive attention and achieved some good results [15–30]. However, the research results on fractional order CVNNs are few. Especially, up to now, no results of almost periodic oscillation of fractional-order CVNNs have been published, but nevertheless, as is well known, almost periodic oscillations are one of the crucial dynamics of neural networks [11, 24, 25, 31–33]. Therefore, the study of almost periodic oscillations of fractional-order CVNNs has important theoretical and practical implications.

Moreover, it is well known that high-order Hopfield neural networks (HHNNs) have advantages over their corresponding low-order ones in approximation, convergence speed, storage capacity and fault tolerance. Therefore, HHNNs have always been the focus of research [34–37]. In addition, time delay is ubiquitous and inevitable in practical systems. Consequently, it is more reasonable to consider the neural network systems with time delay.

Inspired by the above observations, and noted that in a certain sense, it is more practical to consider the finite-time stability than the stability in Lyapunov's sense. Therefore, the main purpose of this paper is to study the existence and finite-time stability of pseudo almost periodic solutions of FCHHNN with time-varying delays.

The contributions of this work are

(1) The result obtained in this paper is the first result regarding the pseudo almost periodic solutions of fractional-order CVNN.

(2) The approaches used in the paper can be applied to study almost periodic and almost automorphic solutions to other types of fractional-order CVNNs.

(3) Even when the network considered in this paper degenerates into a real-valued one, the results of this paper still remain new.

The remaining part of the paper is structured in this way: in Sect. 2, we introduce some concepts, notations and preliminary results, as well as the description of the model. In Sect. 3, we discuss the existence and finite time stability of pseudo almost periodic solutions for the network under consideration. In Sect. 4, we present a numerical example and computer simulation. Finally, we draw a brief conclusion in Sect. 5.

2 Preliminaries and model description

Let $\mathcal{A} = \left\{ \sum_{A \in \mathcal{P}} a^A e_A, a^A \in \mathbb{R} \right\}$ be a real Clifford algebra over $\mathbb{R}^m$ (see [38]), where $\mathcal{P} = \{\emptyset, 1, 2, \dots, A, \dots, 12\cdots m\}$ and $e_A = e_{h_1} e_{h_2} \cdots e_{h_\nu}$, $A \in \mathcal{P}$, $e_\emptyset = e_0 = 1$ and $e_h, h = 1, 2, \dots, m$ are its generators and satisfy $e_i^2 = 1, i = 1, 2, \dots, s$, $e_i^2 = -1, i = s+1, s+2, \dots, m$, $e_i e_j + e_j e_i = 0, i \neq j$, where $i, j = 1, 2, \dots, m$.

For $x = \sum_{A \in \mathcal{P}} x^A e_A \in \mathcal{A}$, define $|x|_{\mathcal{A}} = \sqrt{\sum_{A \in \mathcal{P}} (x^A)^2}$ and for $y = (y_1, y_2, \dots, y_n) \in \mathcal{A}^n$, define $\|y\|_{\mathcal{A}^n} = \max_{1 \leq p \leq n} \{|y_p|_{\mathcal{A}}\}$. Obviously, $(\mathcal{A}, |\cdot|_{\mathcal{A}})$ and $(\mathcal{A}^n, |\cdot|_{\mathcal{A}^n})$ are Banach spaces, respectively.

Definition 2.1. *[39] The fractional integral of order α for a function $f \in L^1([a, b], \mathbb{R})$ is defined as*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s) ds, \ t \in [a, b], $$

where $\alpha > 0$ and Γ is the Gamma function, a and b may take $-\infty$ and $+\infty$ as their values.

Definition 2.2. *[39] The Caputo derivative of order α for a function $f \in C^{n-1}([a, b], \mathbb{R})$ and $f^{(n)} \in L^1([a, b], \mathbb{R})$ is defined by*

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^n(s)}{(t - s)^{\alpha + 1 - n}} ds, \ t \in [a, b], $$

where n is a nature number such that $n - 1 < \alpha < n$ and Γ is the Gamma function, a and b may take $-\infty$ and $+\infty$ as their values.

Definition 2.3. *The fractional integral with fractional order $\alpha > 0$ of function $f = \sum_{A \in \mathcal{P}} f^A e_A \in L^1([a, b]), \mathcal{A})$ is defined as ${}_aI_t^\alpha(t) f(t) = \sum_{A \in \mathcal{P}} {}_aI_t^\alpha f^A(t) e_A$ and the Caputo fractional-order derivative of order α for function $f \in C^{n-1}([a, b], \mathcal{A})$ and $f^{(n)} \in L^1([a, b], \mathcal{A})$ is defined as ${}_aD_t^\alpha f(t) = \sum_{A \in \mathcal{P}} {}_aD_t^\alpha f^A(t) e_A$, where n is a nature number such that $n - 1 < \alpha < n$.*

Let $BC(\mathbb{R}, \mathcal{A}^l)$ be the set of bounded continuous functions from $\mathbb{R}$ to $\mathcal{A}^l$, where l is a positive integer. Then $(BC(\mathbb{R}, \mathcal{A}^l), \|\cdot\|_\infty)$ is a Banach space, where $\|f\|_\infty := \sup_{t \in \mathbb{R}} \|f(t)\|_{\mathcal{A}^l}$ for $f \in BC(\mathbb{R}, \mathcal{A}^l)$.

Definition 2.4. *[40] Function $f \in BC(\mathbb{R}, \mathcal{A}^l)$ is called to be almost periodic if for every $\varepsilon > 0$, there is a positive number $l = l(\varepsilon)$ such that in each interval with length l, there is a τ satisfying*

$$ \|f(t + \tau) - f(t)\|_{\mathcal{A}^l} < \varepsilon, \ \textit{for all } t \in \mathbb{R}. $$

The space of all such functions will be denoted by $AP(\mathbb{R}, \mathcal{A}^l)$.

Define

$$ PAP_0(\mathbb{R}, \mathcal{A}^l) := \left\{ f \in BC(\mathbb{R}, \mathcal{A}^l) : \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^L \|f(s)\|_{\mathcal{A}^l} ds = 0 \right\}. $$

Definition 2.5. *[40] Let $f \in BC(\mathbb{R}, \mathcal{A}^l)$, then f is called pseudo almost periodic if there exist $f_1 \in AP(\mathbb{R}, \mathcal{A}^l)$ and $f_2 \in PAP_0(\mathbb{R}, \mathcal{A}^l)$ such that $f = f_1 + f_2$. The collection of all such functions will be denoted by $PAP(\mathbb{R}, \mathcal{A}^l)$.*

Lemma 2.1. *[40] The set $PAP(\mathbb{R}, \mathcal{A})$ with the supremum norm is a Banach space.*

Lemma 2.2. *[40] Let $\alpha \in \mathbb{R}$, f, $g \in PAP(\mathbb{R}, \mathcal{A})$, then αf, $f + g$, $f \cdot g \in PAP(\mathbb{R}, \mathcal{A})$.*

Lemma 2.3. *Let $f \in PAP(\mathbb{R}, \mathcal{A})$, $\tau \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$ with $\inf\limits_{t\in\mathbb{R}} |1 - \dot{\tau}(t)| > 0$, then $f(\cdot - \tau(\cdot)) \in PAP(\mathbb{R}, \mathcal{A})$.*

Proof. In view of Definition 2.5, f can be expressed as $f = f_1 + f_0$, in which $f_1 \in AP(\mathbb{R}, \mathcal{A}^n)$ and $f_0 \in PAP_0(\mathbb{R}, \mathcal{A}^n) \subset BC(\mathbb{R}, \mathcal{A}^n)$, then one has

$$f(t - \tau(t)) = f_1(t - \tau(t)) + f_0(t - \tau(t)).$$

Similar to the proof of Lemma 5 in [41], one can easily get $f_1(\cdot - \tau(\cdot)) \in AP(\mathbb{R}, \mathcal{A}^n)$.

In addition, noting that

$$\begin{aligned}
&\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|f_0(t - \tau(t))\|_{\mathcal{A}^n} dt \\
\le& \lim_{T\to+\infty} \frac{1}{\inf\limits_{t\in R} |1 - \dot{\tau}(t)|} \frac{1}{2T} \left| \int_{-T-\tau(-T)}^{T-\tau(T)} \|f_0(s)\|_{\mathcal{A}^n} ds \right| \\
=& \lim_{T\to+\infty} \frac{1}{\inf\limits_{t\in R} |1 - \dot{\tau}(t)|} \frac{1}{2T} \left| \left(\int_{-T-\tau(-T)}^{-T+\tau(T)} + \int_{-T+\tau(T)}^{T-\tau(T)} \right) \|f_0(s)\|_{\mathcal{A}^n} ds \right| \\
=& \lim_{T\to+\infty} \frac{1}{\inf\limits_{t\in R} |1 - \dot{\tau}(t)|} \frac{T - \tau(T)}{T} \frac{1}{2(T - \tau(T))} \int_{-T+\tau(T)}^{T-\tau(T)} \|f_0(s)\|_{\mathcal{A}^n} ds = 0,
\end{aligned}$$

we arrive at $f_0(\cdot - \tau(\cdot)) \in PAP_0(\mathbb{R}, \mathcal{A}^n)$. Thus, $f(\cdot - \tau(\cdot)) \in PAP(\mathbb{R}, \mathcal{A}^n)$. This completes the proof. ∎

The model we are concerned in this paper is the following Caputo FCHHNN with time-varying delays:

$$\begin{aligned}
{}_{t_0}D_t^{\alpha} x_p(t) =& - a_p x_p(t) + \sum_{q=1}^{n} a_{pq}(t) f_q(x_q(t - \tau_{pq}(t))) + \sum_{q=1}^{n} \sum_{l=1}^{n} b_{pql}(t) g_q(x_q(t - \sigma_{pql}(t))) \\
&\times g_l(x_l(t - \nu_{pql}(t))) + I_p(t), \ t > t_0, \ p = 1, 2, \ldots, n, \qquad (2.1)
\end{aligned}$$

where $0 < \alpha < 1$ is a constant, n denotes the number of units in the network; $x_p(t) \in \mathcal{A}$ is the state of the pth unit at time t; $a_p \ge 0$ is the self feedback connection weight; $a_{pq}(t) \in \mathcal{A}$ and

$b_{pql}(t) \in \mathcal{A}$ are connection weights of the network; $\tau_{pq}(t) \geq 0$, $\sigma_{pql}(t) \geq 0$ and $\nu_{pql}(t) \geq 0$ are transmission delays at time t; $I_p(t) \in \mathcal{A}$ represents the external inputs at time t; $f_q, g_q : \mathcal{A} \to \mathcal{A}$ are activation functions of signal transmission.

We will use the following notations:

$$\hat{\tau} = \max_{1\leq p,q\leq n}\left\{\sup_{t\in\mathbb{R}} \tau_{pq}(t)\right\},\ \hat{\sigma} = \max_{1\leq p,q,l\leq n}\left\{\sup_{t\in\mathbb{R}} \sigma_{pql}(t)\right\},\ \hat{\nu} = \max_{1\leq p,q,l\leq n}\left\{\sup_{t\in\mathbb{R}} \nu_{pql}(t)\right\},$$

$$\rho = \max\{\hat{\tau}, \hat{\sigma}, \hat{\nu}\},\ \hat{a}_{pq} = \sup_{t\in\mathbb{R}} |a_{pq}(t)|_{\mathcal{A}},\ \hat{b}_{pql} = \sup_{t\in\mathbb{R}} |b_{pql}(t)|_{\mathcal{A}},\ \hat{I}_p = \sup_{t\in\mathbb{R}} |I_p(t)|_{\mathcal{A}}.$$

System (2.1) is supplemented with the initial values:

$$x_p(s) = \psi_p(s),\ s \in [t_0 - \rho, t_0],\ p = 1, 2, \ldots, n,$$

where $\psi_p \in C([t_0 - \rho, t_0], \mathcal{A})$.

In the next section, in order to get our main results, we need the following conditions:

(S_1) For $p, q, l = 1, \ldots, n$, a_{pq}, b_{pql}, $I_p \in PAP(\mathbb{R}, \mathcal{A})$, $\tau_{pq}, \sigma_{pql}, \nu_{pql} \in AP(\mathbb{R}, \mathbb{R}^+) \cap C^1(\mathbb{R}, \mathbb{R})$ satisfying $\inf_{t\in\mathbb{R}} |1 - \tau_{pq}(t)| > 0, \inf_{t\in\mathbb{R}} |1 - \sigma_{pql}(t)| > 0, \inf_{t\in\mathbb{R}} |1 - \nu_{pql}(t)| > 0$.

(S_2) There exist positive numbers L_q^f, L_q^g, M_q^g such that, for all $x, y \in \mathcal{A}, q = 1, \ldots, n$,

$$|f_q(x) - f_q(y)|_{\mathcal{A}} \leq L_q^f |x - y|_{\mathcal{A}},\ |g_q(x) - g_q(y)|_{\mathcal{A}} \leq L_q^g |x - y|_{\mathcal{A}},\ |g_q(x)|_{\mathcal{A}} \leq M_q^g,$$

$f_q, h_q \in C(\mathcal{A}, \mathcal{A})$; in addition, $f_q(0) = g_q(0) = 0$.

(S_3)

$$\max_{1\leq p\leq n}\left\{\frac{C_p}{a_p} + \frac{\hat{I}_p}{a_p}\right\} \leq r, \quad \xi = \max_{1\leq p\leq n}\left\{\frac{D_p}{a_p}\right\} < 1,$$

where for $p = 1, 2, \ldots, n$,

$$C_p = \Big(\sum_{q=1}^{n} \hat{a}_{pq} L_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} \hat{b}_{pql} L_q^g M_l^g\Big) r,$$

$$D_p = \sum_{q=1}^{n} \hat{a}_{pq} L_q^f + \sum_{q=1}^{n}\sum_{l=1}^{n} \hat{b}_{pql}\Big(L_q^g M_l^g + L_l^g M_q^g\Big).$$

3 Main results

Based on Definition 2.3 and Clifford algebra's multiplication rule as well as Definition 3.1 in [42] of solutions for real-valued equations, we can introduce the following definition:

Definition 3.1. *A function* $x = (x_1, x_2, \cdots, x_n)^T \in C([t_0 - \rho, +\infty), \mathcal{A}^n)$ *is called a mild solution of system* (2.1)*, if it meets*

$$\begin{cases} x_p(t) = U_p(t - t_0)x_0 + \int_{t_0}^{t} (t-s)^{\alpha-1}\varphi_p(t-s)[\Theta_p(s,x) + I_p(s)]ds, \;\; t > t_0, \\ x_p(t) = \psi_p(t), \;\; t \in [t_0 - \rho, t_0], \end{cases} \tag{3.1}$$

where

$$\Theta_p(s,x) = \sum_{q=1}^{n} \left[a_{pq}(s) f_q(x_q(s - \tau_{pq}(s))) + \sum_{l=1}^{n} b_{pql}(s) g_q(x_q(s - \sigma_{pql}(s))) g_l(x_l(s - \nu_{pql}(s))) \right],$$

$$x_p(t_0) = \psi_p(t_0) = x_0, \;\; \varrho_\alpha(\gamma) = \frac{1}{\alpha}\gamma^{-1-\frac{1}{\alpha}}\vartheta_\alpha(\gamma^{\frac{-1}{\alpha}}),$$

$$U_p(t) = \int_0^{+\infty} \varrho_\alpha(\gamma) e^{-a_p t^\alpha \gamma} d\gamma, \;\; \varphi_p(t) = \alpha \int_0^{+\infty} \gamma \varrho_\alpha(\gamma) e^{-a_p t^\alpha \gamma} d\gamma,$$

$$\vartheta_\alpha(\gamma) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \gamma^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \;\; \gamma \in (0, +\infty),$$

and ϱ_α *satisfies*

$$\varrho_\alpha(\gamma) \geq 0, \quad \gamma \in (0, +\infty), \quad \int_0^{+\infty} \varrho_\alpha(\gamma) d\gamma = 1, \quad \int_0^{+\infty} \gamma \varrho_\alpha(\gamma) d\gamma = \frac{1}{\Gamma(\alpha+1)}.$$

Letting $t_0 \longrightarrow -\infty$*, we gain*

$$x_p(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s)[\Theta_p(s,x) + I_p(s)]ds,$$

which is a solution of system (2.1)*.*

Let $\mathbb{B} = \{x = (x_1, x_2, \ldots, x_n)^T \in BC(\mathbb{R}, \mathcal{A}^n) : x_p \in PAP(\mathbb{R}, \mathcal{A}),\ p = 1, 2, \ldots, n\}$, then $(\mathbb{B}, \|\cdot\|_\infty)$ is a Banach space.

Theorem 3.1. *Let* (S_1)*-*(S_3) *be fulfilled, then system* (2.1) *possesses unique one pseudo almost periodic mild solution in* $\mathbb{B}_r = \{x \mid x \in \mathbb{B}, \|x\|_\infty \leq r\}$*.*

Proof. Define a mapping $\Phi : \mathbb{B} \to BC(\mathbb{R}, \mathcal{A}^n)$ as follow:

$$\Phi x = ((\Phi x)_1, (\Phi x)_2, \cdots, (\Phi x)_n)^T,$$

where for $p = 1, 2, \ldots, n$, $t \in \mathbb{R}$,

$$(\Phi x)_p(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s)(\Theta_p(s,x) + I_p(s))ds.$$

Firstly, we will show that $\Phi : \mathbb{B} \to \mathbb{B}$. For every $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{B}_r$, from (S_1)-(S_3) it follows that $f_q, g_q, \tau_{pq}, \sigma_{pql}, \nu_{pql}$ meet all the conditions of Lemma 2.3, and hence, by Lemmas 2.2 and 2.3, we can derive that

$$\phi_p(\cdot) = \Theta_p(\cdot, x) + I_p(\cdot) \in PAP(\mathbb{R}, \mathcal{A}).$$

So, there are $\phi_p^1 \in AP(\mathbb{R}, \mathcal{A})$ and $\phi_p^0 \in PAP_0(\mathbb{R}, \mathcal{A})$ such that $\phi_p = \phi_p^1 + \phi_p^0$ where $p = 1, 2, \dots, n$. Consequently, we have

$$\begin{aligned}(\Phi x)_p(t) &= \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p(s) ds \\ &= \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p^1(s) ds + \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p^0(s) ds \\ &:= (\Phi x)_p^1(t) + (\Phi x)_p^0(t), \ p = 1, 2, \dots, n. \end{aligned} \tag{3.2}$$

We will prove that $(\Phi x)_p^1 \in AP(\mathbb{R}, \mathcal{A})$ and $(\Phi x)_p^0 \in PAP_0(\mathbb{R}, \mathcal{A})$, for $p = 1, 2, \dots, n$.

Since $\phi_p^1 \in AP(\mathbb{R}, \mathcal{A})$, for given $\varepsilon > 0$, there corresponds an $l = l(\varepsilon) > 0$ such that every interval of length l contains a point $\zeta \in (a, a+l)$ such that

$$|\phi_p^1(t+\zeta) - \phi_p^1(t)|_{\mathcal{A}} < \varepsilon, \ t \in \mathbb{R}, \ p = 1, 2, \dots, n.$$

From this and (3.2) it follows that

$$\begin{aligned}|(\Phi x)_p^1(t+\zeta) - (\Phi x)_p^1(t)|_{\mathcal{A}} &= \bigg| \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p^1(s+\zeta) ds \\ &\qquad - \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p^1(s) ds \bigg|_{\mathcal{A}} \\ &\le \sup_{t \in \mathbb{R}} \bigg| \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) (\phi_p^1(s+\zeta) - \phi_p^1(s)) ds \bigg|_{\mathcal{A}} \\ &\le \varepsilon \int_{-\infty}^{t} (t-s)^{\alpha-1} \alpha \int_0^{+\infty} \gamma \varrho_\alpha(\gamma) e^{-a_p (t-s)^\alpha \gamma} d\gamma ds \\ &\le \varepsilon \int_0^{+\infty} \int_0^{+\infty} \alpha \gamma \varrho_\alpha \sigma^{\alpha-1} e^{-a_p \sigma^\alpha \gamma} d\sigma d\gamma = \frac{\varepsilon}{a_p}, \ t \in \mathbb{R}, \end{aligned}$$

which implies $(\Phi x)_p^1 \in AP(\mathbb{R}, \mathcal{A})$ for $p = 1, 2, \dots, n$.

Noting that

$$\begin{aligned} &\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \bigg| \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi_p(t-s) \phi_p^0(s) ds \bigg|_{\mathcal{A}} dt \\ =& \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \bigg| \int_0^{\infty} \sigma^{\alpha-1} \varphi_p(\sigma) \phi_p^0(t-\sigma) d\sigma \bigg|_{\mathcal{A}} dt \\ \le& \frac{1}{a_p} \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |\phi_p^0(t-\sigma)|_{\mathcal{A}} dt \end{aligned}$$

$$=0,\ p=1,2,\ldots,n,$$

we have $(\Phi x)_p^0 \in PAP_0(\mathbb{R},\mathcal{A})$. Hence, for $p=1,2,\ldots,n$, $(\Phi x)_p \in PAP(\mathbb{R},\mathcal{A})$, this yields that $\Phi:\mathbb{B}\to\mathbb{B}$.

Next, we show that $\Phi:\mathbb{B}_r\to\mathbb{B}_r$.

For any $x\in\mathbb{B}_r$, we have

$$\begin{aligned}|\Theta_p(t,x)|_{\mathcal{A}} \leq& \sum_{q=1}^{n}\hat{a}_{pq}\big(|f_q(x_q(t-\tau_{pq}(t)))-f_q(0)|_{\mathcal{A}}+|f_q(0)|_{\mathcal{A}}\big)\\ &+\sum_{q=1}^{n}\sum_{l=1}^{n}\hat{b}_{pql}(|g_q(x_q(t-\sigma_{pql}(t)))-g_q(0)|_{\mathcal{A}}+|g_q(0)|_{\mathcal{A}})M_l^g\\ \leq&\Bigg(\sum_{q=1}^{n}\hat{a}_{pq}L_q^f+\sum_{q=1}^{n}\sum_{l=1}^{n}\hat{b}_{pql}L_q^gM_l^g\Bigg)\|x\|_\infty\leq C_p,\ p=1,2,\ldots,n,\end{aligned}\tag{3.3}$$

which combined with (3.3) and (S_3) leads to

$$\begin{aligned}|(\Phi x)_p(t)|_{\mathcal{A}}\leq&\int_{-\infty}^{t}(t-s)^{\alpha-1}\varphi_p(t-s)\big(C_p+\hat{I}_p\big)ds\\ \leq&\frac{C_p}{a_p}+\frac{\hat{I}_p}{a_p}\leq r,\ p=1,2,\ldots,n.\end{aligned}$$

Hence,

$$\|\Phi x\|_\infty=\sup_{t\in\mathbb{R}}\{\|(\Phi x)(t)\|_{\mathcal{A}^n}\}\leq r,$$

that is, $\Phi x\in\mathbb{B}_r$. Therefore, $\Phi:\mathbb{B}_r\to\mathbb{B}_r$.

Finally, we prove that Φ is a contraction mapping. For any $x,y\in\mathbb{B}_r$, we find

$$\begin{aligned}&|(\Phi x)_p(t)-(\Phi y)_p(t)|_{\mathcal{A}}\\ \leq&\|x-y\|_\infty\int_{-\infty}^{t}(t-s)^{\alpha-1}\varphi_p(t-s)\sum_{q=1}^{n}\Bigg[\hat{a}_{pq}L_q^f+\sum_{l=1}^{n}\hat{b}_{pql}\Big(L_q^gM_l^g+L_l^gM_q^g\Big)\Bigg]ds\\ \leq&\frac{D_p}{a_p}\|x-y\|_\infty\leq\xi\|x-y\|_\infty,\ p=1,2,\ldots,n,\end{aligned}$$

which implies that Φ is a contraction.

Consequently, Φ possesses a unique fixed point in $\mathbb{B}_r$. The proof is completed. ∎

Remark 3.1. *From the proof process of Theorem 3.1, it is easy to see that the proof method of Theorem 3.1 is also applicable to studying the existence of pseudo almost automorphic solutions for system* (2.1).

In the following, we take the initial moment $t_0 = 0$ to discuss the finite-time stability of system (2.1), that is, we consider system (2.1) supplemented with the following initial condition:

$$x_p(\theta) = \psi_p(\theta),\ \theta \in [-\rho, 0],\ p = 1, 2, \ldots, n,$$

where $\psi_p \in C([-\rho, 0], \mathcal{A})$.

Definition 3.2. *A mild solution x^* of system* (2.1) *with initial value ψ^* is called finite-time stable with respect to $\{\delta, \varepsilon, T\}$, here $0 < \delta < \varepsilon$ and $T > 0$, if every mild solution x of system* (2.1) *with initial value ψ satisfies that if $\|\psi - \psi^*\| \le \delta$, then, for $t \in [0, T]$,*

$$\|x(t) - x^*(t)\|_{\mathcal{A}^n} \le \varepsilon,$$

where $\|\psi - \psi^*\|_\rho = \sup\limits_{t\in[-\rho,0]} \|\psi(t) - \psi^*(t)\|_{\mathcal{A}^n}$.

Theorem 3.2. *Assume* (S_1)-(S_3) *hold. If*

$$\delta E_\alpha\big(MT^\alpha\big) \le \varepsilon, \tag{3.4}$$

where $M = \max\limits_{1\le p\le n}\{D_p\}$ and $E_\alpha(\cdot)$ is the Mittag-Leffler function of one parameter, then system (2.1) *possesses exactly one pseudo almost periodic mild solution that is finite-time stable with respect to $\{\delta, \varepsilon, T\}$.*

Proof. Denote by x^* the pseudo almost periodic mild solution of system (2.1) with initial value ψ^* and let x be any mild solution to system (2.1) with initial value ψ. Set $z = x - x^*$, invoking (3.1), one gets

$$\begin{aligned} z_p(t) =& U_p(t)(\psi_p(0) - \psi_p^*(0)) + \int_0^t (t-s)^{\alpha-1}\varphi_p(t-s)\Bigg[\sum_{q=1}^n a_{pq}(s) \\ &\times \Big(f_q(x_q(s-\tau_{pq}(s))) - f_q(x_q^*(s-\tau_{pq}(s))\Big) + \sum_{q=1}^n\sum_{l=1}^n b_{pql}(s)\Big(g_q(x_q(s-\sigma_{pql}(s))) \\ &\times g_l(x_l(s-\nu_{pql}(s))) - g_q(x_q^*(s-\sigma_{pql}(s)))g_l(x_l^*(s-\nu_{pql}(s)))\Big)\Bigg]ds, \quad t \ge 0 \end{aligned}$$

and

$$\begin{aligned} |U_p(t)(\psi_p(0) - \psi_p^*(0))|_{\mathcal{A}} \le& \int_0^{+\infty} \varrho_\alpha(\gamma) e^{-a_p^-(t)^\alpha\gamma} d\gamma |\psi_p(0) - \psi_p^*(0)|_{\mathcal{A}} \\ \le& \int_0^{+\infty} \varrho_\alpha(\gamma) d\gamma |\psi_p - \psi_p^*|_{\mathcal{A}} \\ \le& |\psi_p - \psi_p^*|_{\mathcal{A}}. \end{aligned}$$

Denote $\lambda(t) = \max_{1\le p\le n} \sup_{s\in[-\rho,t]} |z_p(s)|_{\mathcal{A}}$, then for $t\in[0,T]$ and $p=1,2,\dots,n$, we find

$$\begin{aligned}|z_p(t)|_{\mathcal{A}} \le &\|\psi-\psi^*\|_\rho + \int_0^t (t-s)^{\alpha-1}\varphi_p(t-s)\Bigg[\sum_{q=1}^n \hat{a}_{pq}L_q^f \\ &+\sum_{q=1}^n\sum_{l=1}^n \hat{b}_{pql}\Big(M_l^g L_q^g + M_q^g L_l^g\Big)\Bigg]\lambda(s)ds \\ \le&\|\psi-\psi^*\|_\rho + M\int_0^t (t-s)^{\alpha-1}\alpha\int_0^{+\infty}\gamma\varrho_\alpha(\gamma)e^{-a_p(t-s)^\alpha\gamma}d\gamma\lambda(s)ds \\ \le&\|\psi-\psi^*\|_\rho + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\lambda(s)ds \\ \le&\|\psi-\psi^*\|_\rho + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(s)ds,\end{aligned}$$

which combined with the fact that $|z_p(s)|_{\mathcal{A}} \le \|\psi-\psi^*\|_\rho$ for $s\in[-\rho,0]$ yields

$$\lambda(t) \le \|\psi-\psi^*\|_\rho + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(s)ds,$$

for $t\in[0,T]$ and $p=1,2,\dots,n$.

Further, invoking Corollary 2 in [43], we infer that

$$\lambda(t) \le \|\psi-\psi^*\|_\rho E_\alpha(Mt^\alpha),\ t\in[0,T].$$

Since the function $E_\alpha(\theta)$ is nondecreasing for $\theta\in\mathbb{R}^+$, we conclude that

$$|z_p(t)|_{\mathcal{A}} \le \lambda(t) \le \|\psi-\psi^*\|_\rho E_\alpha(MT^\alpha), \quad t\in[0,T].$$

By (3.4), for all $t\in[0,T]$, one has

$$\|x(t)-x^*(t)\| < \varepsilon,$$

which gives the conclusion of the theorem. The proof is complete. ∎

4 An example

Our example is as follows.

Example 4.1. *In system* (2.1)*, take $n=m=2$, and for $p=1,2$, let*

$$x_p(t) = e_0x_p^0(t) + e_1x_p^1(t) + e_2x_p^2(t) + e_{12}x_p^{12}(t) \in \mathcal{A},\ a_1 = 12,\ a_2 = 8,$$

$$a_{11}(t) = a_{12}(t) = e_0 \frac{1}{80} \cos t + e_2 \frac{1}{80} \sin t,$$
$$a_{21}(t) = a_{22}(t) = e_0 \frac{3}{400} \sin t + e_1 \frac{1}{80} \cos t + e_2 \frac{1}{80} \sin t + e_{12} \frac{3}{400} \cos t,$$
$$b_{111}(t) = b_{112}(t) = b_{121}(t) = b_{122}(t) = e_0 \frac{1}{100} + e_2 \frac{7}{400} \sin t + e_{12} \frac{7}{400} \cos t,$$
$$b_{211}(t) = b_{212}(t) = b_{221}(t) = b_{222}(t) = e_0 \frac{1}{80} + e_1 \frac{3}{200} \cos \sqrt{2}t + e_2 \frac{3}{200} \sin \sqrt{2}t,$$
$$f_p(x) = e_0 0.5 \sin x^0 + e_1 0.5 \sin x^1 + e_1 0.5 \sin x^2 + e_{12} 0.5 \sin x^{12},$$
$$g_p(x) = e_0 \frac{1}{3} |\sin x^0| + e_1 \frac{1}{4} |\sin x^1| + e_2 \frac{1}{5} |\sin x^2| + e_{12} \frac{1}{6} |\sin x^{12}|,$$
$$\tau_{pq}(t) = 0.1 \sin 3t + 0.3,\ \sigma_{pql}(t) = 0.1 \sin^2 t,\ \nu_{pql}(t) = 0.3 \sin 3t + 0.5,$$
$$I_p(t) = e_0 \sqrt{20.01} \cos \sqrt{5}t + e_{12} \sqrt{20.01} \sin \sqrt{5}t + \frac{1}{1+t^2}.$$

It is easy to get that

$$L_q^f = 0.5,\ L_q^g = \frac{1}{3},\ M_q^f = 1,\ M_q^g = 0.5,\ \hat{a}_{11} = \hat{a}_{12} = 0.0125,\ \hat{a}_{21} = \hat{a}_{22} = 0.0146,$$

$$\hat{b}_{111} = \hat{b}_{112} = \hat{b}_{121} = \hat{b}_{122} = 0.0216,\ \hat{b}_{211} = \hat{b}_{212} = \hat{b}_{221} = \hat{b}_{222} = 0.0195,\ \hat{I}_1 = \hat{I}_2 = 4.5837.$$

Let $r = 10,\ \delta = 0.1,\ \varepsilon = 0.5, t_0 = 0$. *Choose* $\alpha = 0.3, 0.5$ *and* $0.7, T = 10$, *one has*

$$\max_{1\le p\le 2} \left\{ \frac{C_p}{a_p} + \frac{\hat{I}_p}{a_p} \right\} = 0.6075 < 10 = r,\ \xi = \max_{1\le p\le 2} \left\{ \frac{D_p}{a_p} \right\} = 0.0051 < 1,\ M = \max_{1\le p\le 2} D_p = 0.0406.$$

Using the MATLAB program, to compute the Mittag-Leffler function, we obtain

$$E_{0.3}(MT^{0.3}) = E_{0.3}(0.081) \approx 1.0982,$$
$$E_{0.5}(MT^{0.5}) = E_{0.5}(0.1284) \approx 1.1631,$$
$$E_{0.7}(MT^{0.7}) = E_{0.7}(0.2035) \approx 1.2615,$$

then

$$\delta E_{0.3}(MT^{0.3}) = 0.10982 < \varepsilon, \delta E_{0.5}(MT^{0.5}) = 0.11631 < \varepsilon, \delta E_{0.7}(MT^{0.7}) = 0.12615 < \varepsilon.$$

Therefore, according to Theorem 3.2, system (2.1) *possesses a unique pseudo almost periodic mild solution, which is finite-time stable (see Figures 1-9).*

Remark 4.1. *Figures 1-3 show that when* $\alpha = 0.3, 0.5$ *and* 0.7, *each state variable of system* (2.1) *exhibits almost periodic oscillations over time. Figures 4-9 show that when* $\alpha = 0.3, 0.5$ *and* 0.7, *the same state variable with different initial values of system* (2.1) *exhibits finite time stability over time.*

Remark 4.2. *The results of Example 4.1 can not be obtained by Refs. [24, 25] and any other known results.*

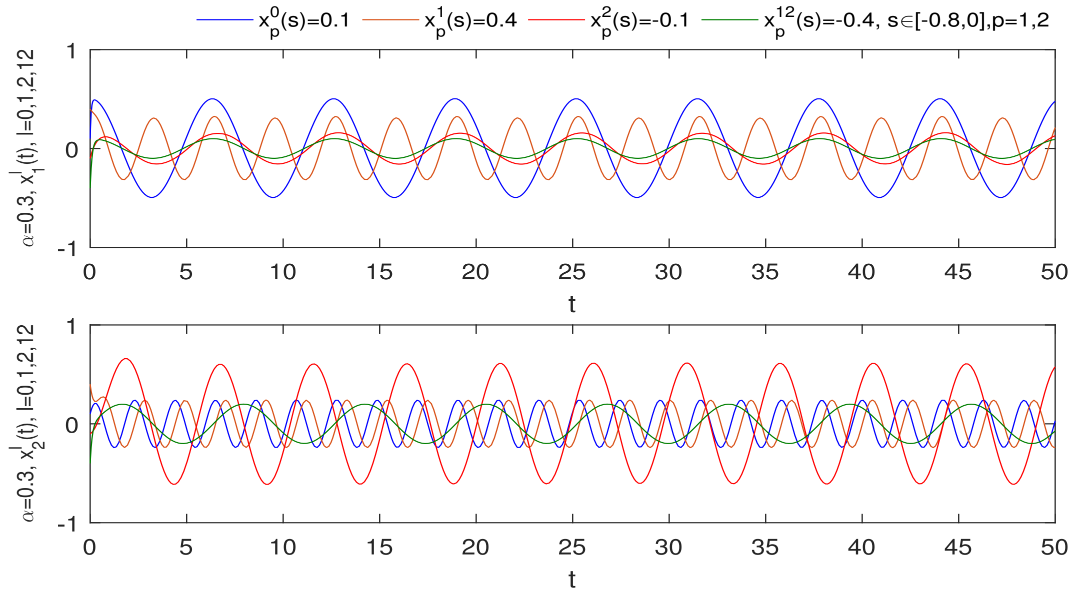

Figure 1: Curves of $x_1^l(t)$ and $x_2^l(t)$ of system (2.1) for $\alpha = 0.3, l = 0, 1, 2, 12$.

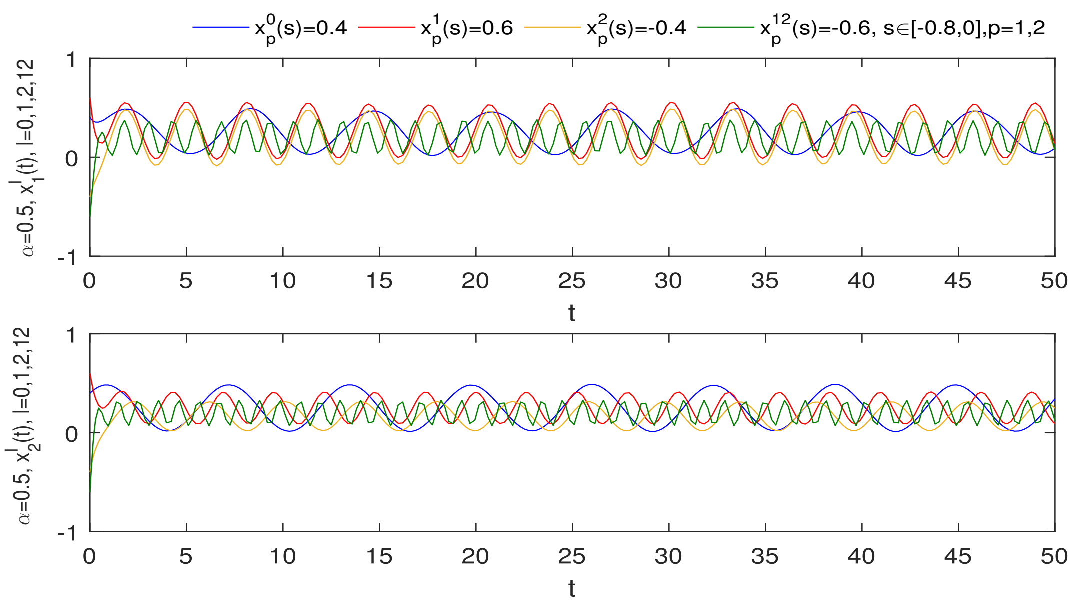

Figure 2: Curves of $x_1^l(t)$ and $x_2^l(t)$ of system (2.1) for $\alpha = 0.5, l = 0, 1, 2, 12$.

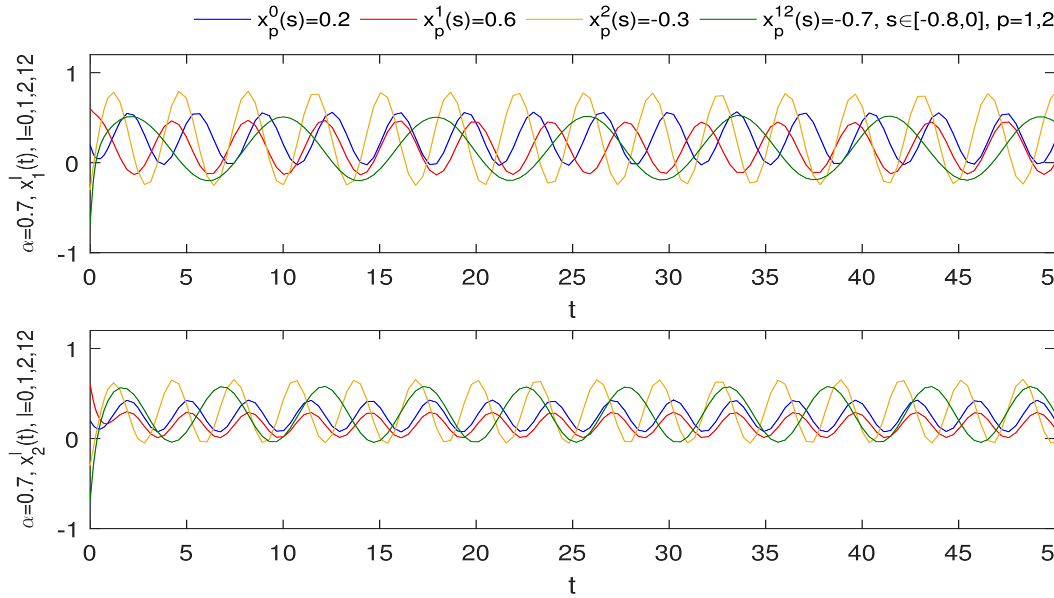

Figure 3: Curves of $x_1^l(t)$ and $x_2^l(t)$ of system (2.1) for $\alpha = 0.7, l = 0, 1, 2, 12$.

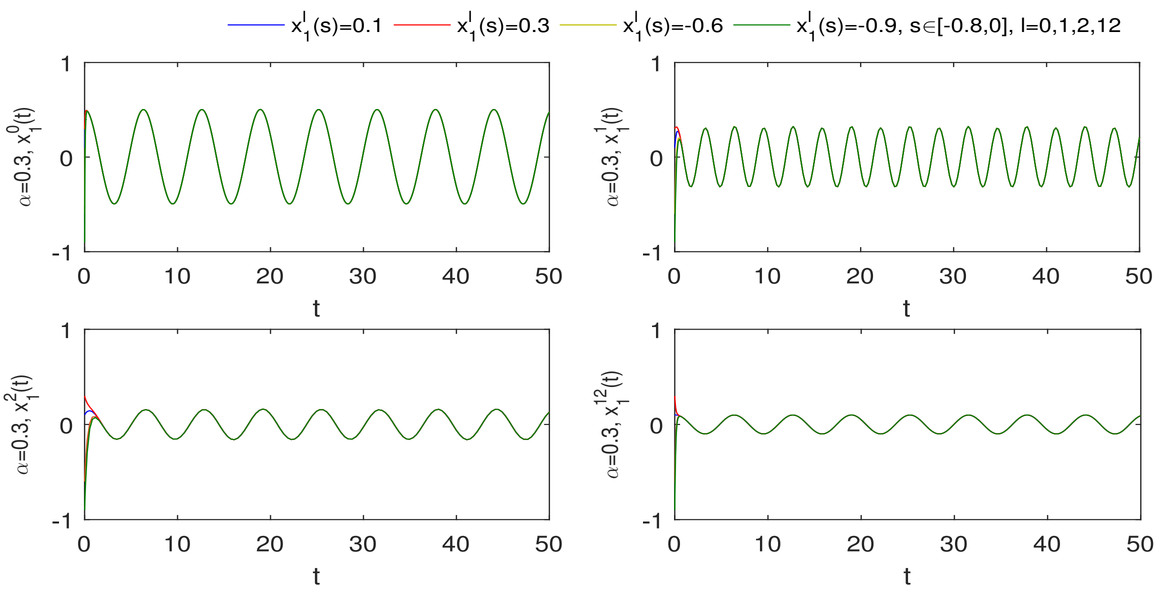

Figure 4: Finite-time stability of $x_1^0(t)$, $x_1^1(t)$, $x_1^2(t)$ and $x_1^{12}(t)$ of system (2.1) for $\alpha = 0.3$.

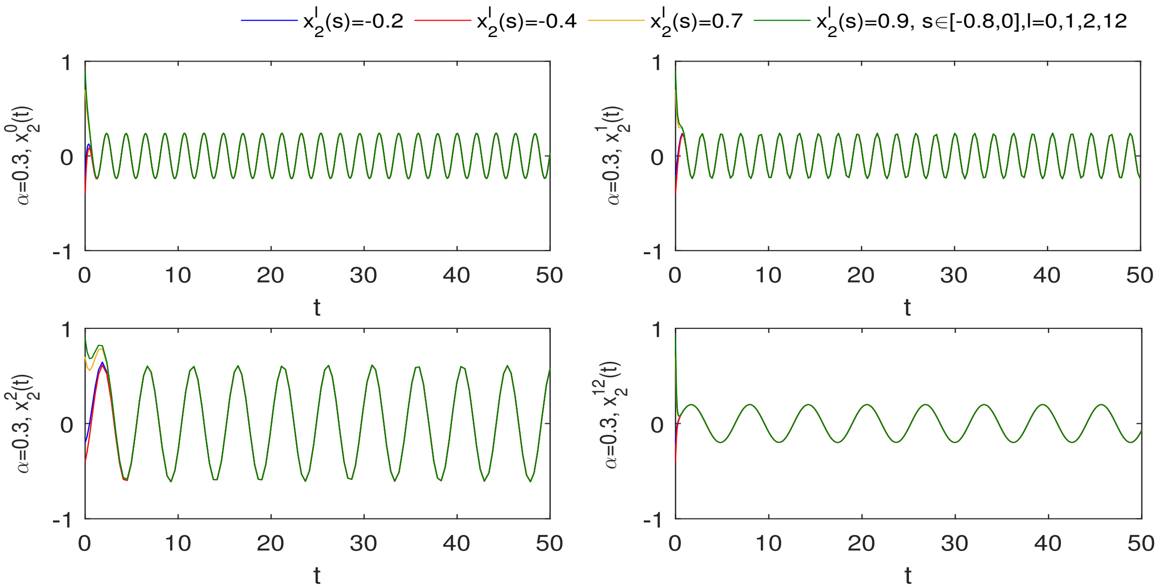

Figure 5: Finite-time stability of $x_2^0(t)$, $x_2^1(t)$, $x_2^2(t)$ and $x_2^{12}(t)$ of system (2.1) for $\alpha = 0.3$.

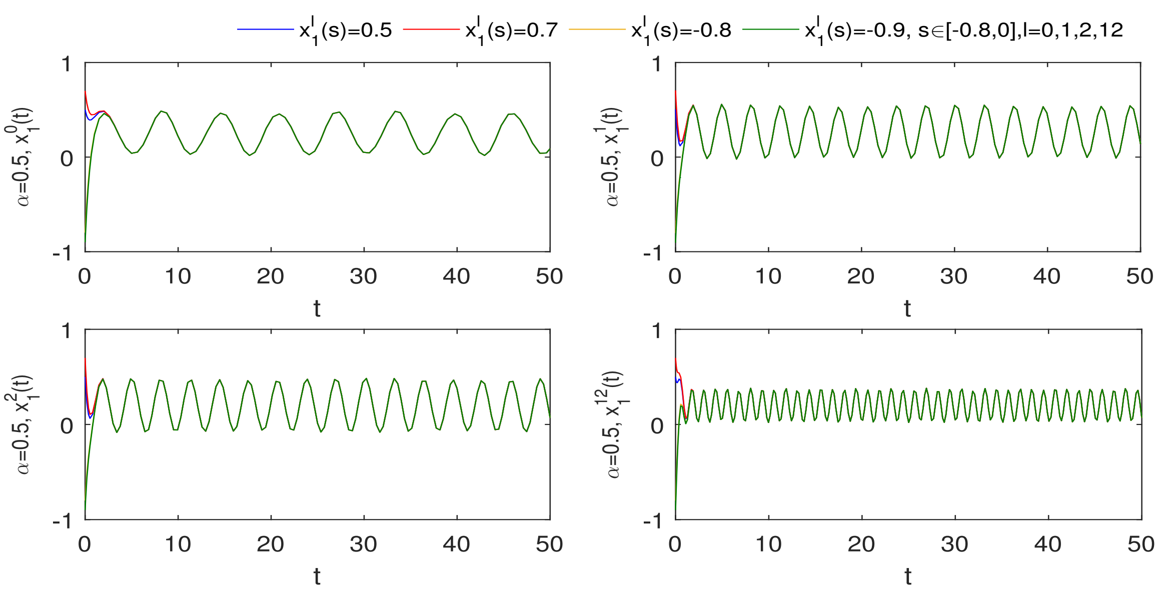

Figure 6: Finite-time stability of $x_1^0(t)$, $x_1^1(t)$, $x_1^2(t)$ and $x_1^{12}(t)$ of system (2.1) for $\alpha = 0.5$.

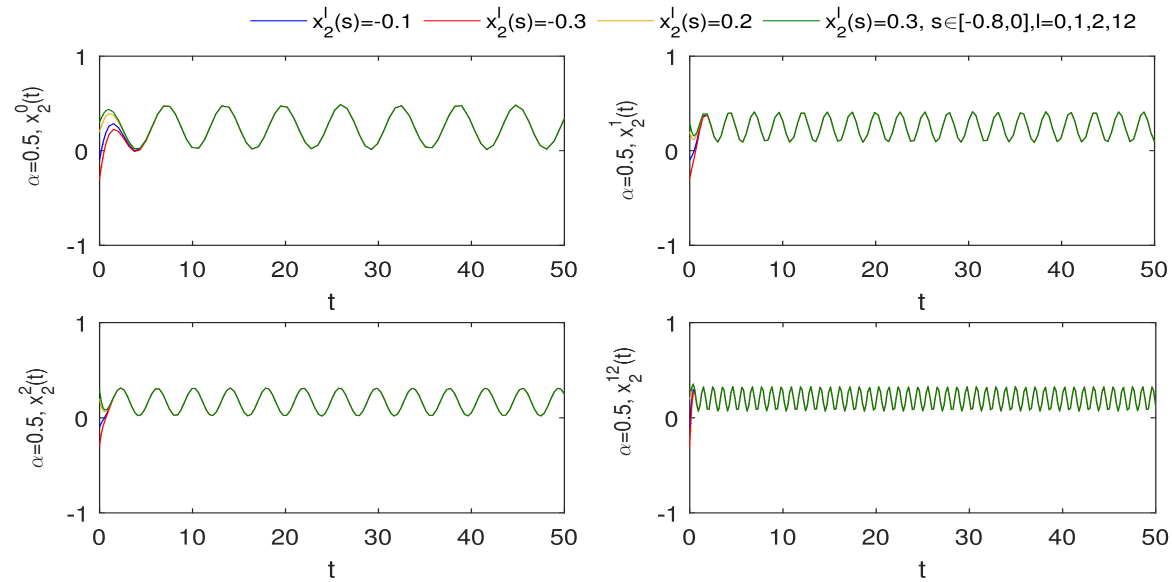

Figure 7: Finite-time stability of $x_2^0(t)$, $x_2^1(t)$, $x_2^2(t)$ and $x_2^{12}(t)$ of system (2.1) for $\alpha = 0.5$.

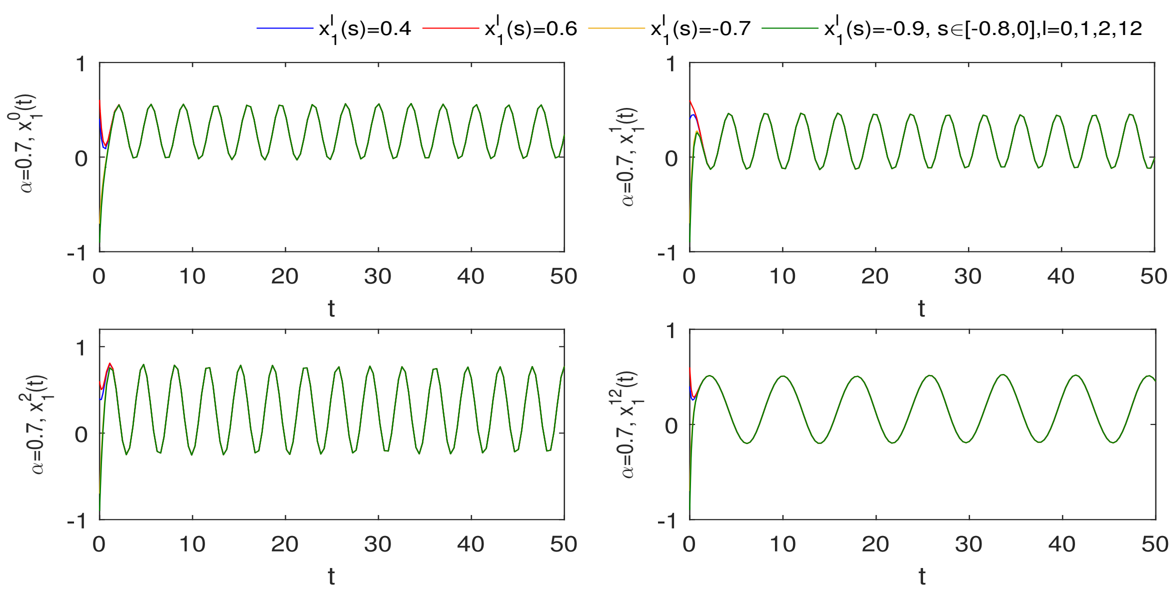

Figure 8: Finite-time stability of $x_1^0(t)$, $x_1^1(t)$, $x_1^2(t)$ and $x_1^{12}(t)$ of system (2.1) for $\alpha = 0.7$.

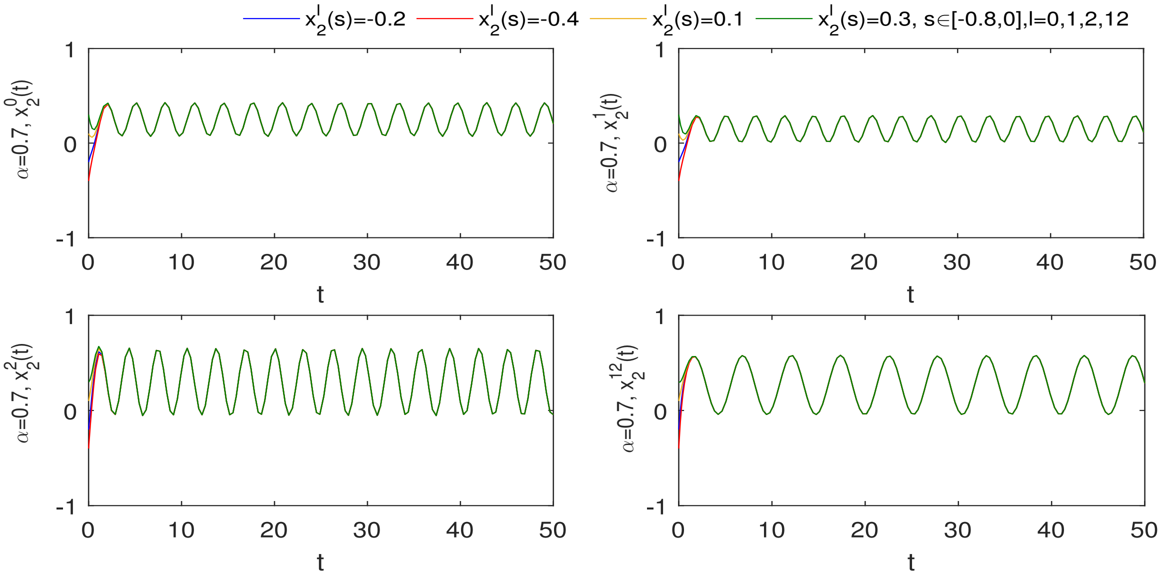

Figure 9: Finite-time stability of $x_2^0(t)$, $x_2^1(t)$, $x_2^2(t)$ and $x_2^{12}(t)$ of system (2.1) for $\alpha = 0.7$.

5 Conclusions

In this paper, we have obtained the existence and finite-time stability of the pseudo almost periodic mild solutions of FCHHNN (2.1). This is the first article to investigate the almost periodic mild solutions of fractional-order Clifford-valued differential equations via direct approach. The results and methods of this paper are new. And the methods used in this paper can be applied to study the existence of almost periodic or almost automorphic mild solutions to other types of fractional-order neural networks. The research on almost periodic synchronization and almost automorphic synchronization of fractional-order complex neural network systems is our future direction.

Abbreviations

FCHHNN	fractional-order Clifford-valued high-order Hopfield neural network
CVNN	Clifford valued neural network
HHNN	high-order Hopfield neural network

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