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EXACT PEAKON SOLUTIONS GIVEN BY THE GENERALIZED HYPERBOLIC FUNCTIONS FOR SOME NONLINEAR WAVE EQUATIONS*

Jibin Li[1,2] and Maoan Han[2,1,†]

Abstract In 1993, Camassa and Holm drived a shallow water equation and found that this equation has a peakon solution with the form $\phi(\xi) = ce^{-|\xi|}$. In this paper, we show that three nonlinear wave systems have peakon solutions which needs to be represented as generalized hyperbolic functions. For the existence of these solutions, some constraint parameter conditions are derived.

Keywords Peakon, traveling wave solution, Arai q-deformed hyperbolic function, multicomponent Korteweg-de Vries equation with dispersion, nonlinear Schrödinger equation, rotation-two-component Camassa-Holm system.

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1. Introduction

In 1993, Camassa and Holm used Hamiltonian methods to derive a new completely integrable dispersive shallow water equation (see [3, 4]):

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

where u is the fluid velocity in the $x-$direction (or equivalently the height of the water's free surface above a flat bottom), κ is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. Considering traveling wave solutions with the form $u = \phi(x - ct) = \phi(\xi)$ of equation (1.1), we have the corresponding traveling wave system [8]:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2}y^2 + \frac{3}{2}\phi^2 + (\kappa - c)\phi + g)}{\phi - c}, \tag{1.2}$$

where g is an integral constant. System (1.7) has the first integral

$$H(\phi, y) = (\phi - c)y^2 - (\kappa - c)\phi^2 - \phi^3 = h. \tag{1.3}$$

We notice that when $\phi = c$, the right hand of the second equation of system (1.2) is not well-definition. In the $(\phi, y)-$phase plan, $\phi = c$ is call a singular straight line.

†Corresponding author.
E-mails: mahan@shnu.edu.cn(M. Han), lijb@zjnu.cn(J. Li)
[1]Department of Mathematics, Shanghai Normal University, Shanghai,200234, China
[2]Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejing 321004, China
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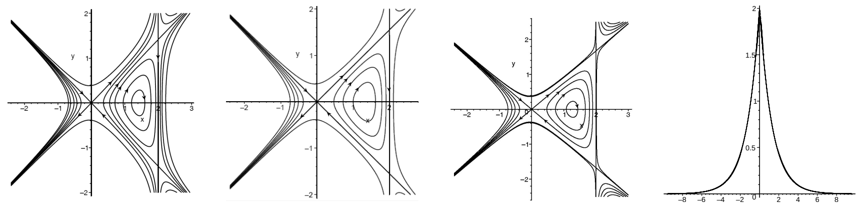


(a) $\kappa < 0$ (b) $\kappa = 0$ (c) $\kappa > 0$ (d) Peakon

Figure 1. The bifurcations of phase portraits of system (1.2) when $g = 0$.

Taking $g = 0$ in system (1.2), for a fixed $c > 0$, near the parameter value $\kappa = 0$, we have the bifurcations of phase portraits of system (1.2) as shown in Fig.1.

Corresponding to the curve triangle in Fig.1 (b) defined by $H(\phi, y) = 0$, by using (1.3) and the first equation of (1.2) , it follows the exact solution

$$\phi(\xi) = ce^{-|\xi|}. \tag{1.4}$$

The profile (i.e., the graph of $\phi(\xi)$) defined by (1.4) is called a peakon (see Fig.1 (d)).

It is well known that the classical Camassa-Holm equation (1.1) has been studied extensively in the last twenty years because of its many remarkable properties: infinity of conservation laws and complete integrability, existence of peakon and multipeakons and compacton solutions (i.e., breaking waves, and meaning solutions that remain bounded, while its slope becomes unbounded in finite time).

For some nonlinear equations, the following relationships are already known for a wave profile of $\phi(\xi)$ with some phase orbits of the corresponding planar dynamical systems (see Li Jibin, et,al., [9–13]).

(1) For a homoclinic orbit (see Fig.1 (c)), if there exists a segment which completely lies in a left (or right) small strip neighborhood of a singular straight line, then this homoclinic orbit defines a pseudo-peakon solution of the system.

(2) For a family of periodic orbits (see Fig.1 (b)), if there exists a segment of every orbit which completely lies in a left (or right) small strip neighborhood of a singular straight line, then these periodic orbits determine a family of periodic peakon solutions of the system. Periodic peakons are two-time-scale smooth classical solutions. Cusp wave parts are locally smooth. Periodic peakons are not weak solutions in any reasonable sense.

(3) If there exists a curve triangle (see Fig.1 (b)) connecting saddle points and surrounding a periodic annulus of a center of the corresponding traveling wave system in the neighborhood of a singular straight line, for which a segment is an edge of the triangle, then as a limiting curve of a family of periodic orbits this curve triangle gives rise to a peakon solution of the system.

In fact, peakon is a limiting solution in the following sense: (i) Under fixed parameter conditions, peakon (or solitary cusp wave solution) is a limiting solution of a family of periodic peakon solutions; (ii) with changeable parameters, peakon is a limiting solution of a family of pseudo-peakons. It should be emphasized that a peakon solution is a C^0-function, i.e., it should be a continuous solution. It is not a weak solution in the sense of distribution.

In [11], we shown that under different parameter conditions, one nonlinear wave equation can have different exact one-peakon solutions and different nonlinear wave equations can have different explicit exact one-peakon solutions. Namely, there are various exact explicit one-peakon solutions, which are different from the one-peakon solution $\phi(x,t) = ce^{-|x-ct|}$.

In this paper, for three nonlinear wave equations, we derive some new exact peakon solutions which need to be given by generalized hyperbolic functions, i.e., so called the Arai q-deformed function (see [1, 2]) defined by $\sinh_q(\xi) = \frac{1}{2}(e^{\xi} - qe^{-\xi}), \cosh_q(\xi) = \frac{1}{2}(e^{\xi} + qe^{-\xi}), \tanh_q(\xi) = \frac{\sinh_q(\xi)}{\cosh_q(\xi)}, \cdots, 0 < q < \infty$. Notice that now we have

$$\cosh_q^2(\xi) - \sinh_q^2(\xi) = q, \quad \frac{d}{d\xi}\cosh_q(\xi) = \sinh_q(\xi), \quad \frac{d}{d\xi}\sinh_q(\xi) = \cosh_q(\xi).$$

In next three sections, we consider the following three nonlinear wave equations, respectively.

(i) The multicomponent Korteweg-de Vries equation with dispersion posed by Kupershmidt in 1985 [7]:

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x + 2v^T v_x + C^T v_{xx}, \\ v_t &= (2uv)_x - u_{xx}C, \end{aligned} \tag{1.5}$$

where $v = (v_1, v_2, \cdots, v_n)^T$ and $C = (c_1, c_2, \cdots, c_n)^T$ is a constant (column) vector. System (1.5) is a bi-Hamiltonian system with an infinite number of conservation laws.

(ii) The nonlinear Schrödinger equation with fourth-order dispersion and cubic-quintic nonlinearity as follows:

$$iE_z - \frac{\beta_2}{2}E_{tt} + \gamma_1|E|^2E = i\frac{\beta_3}{6}E_{ttt} + \frac{\beta_4}{24}E_{tttt} - \gamma_2|E|^4E + i\alpha_1(|E|^2E)_t + i\alpha_2E(|E|^2)_t. \tag{1.6}$$

This equation governs wave dynamics of optical fiber system (see [14]).

(iii) The rotation-two-component Camassa-Holm system (see [5, 6]):

$$\begin{aligned} &u_t - u_{xxt} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) + \mu u_{xxx} + (1-2\Omega A)\rho\rho_x - 2\Omega\rho(u\rho)_x = 0, \\ &\rho_t + (\rho u)_x = 0. \end{aligned} \tag{1.7}$$

We need to use the following conclusion.

Proposition 1.1. *Let $X(\phi) = A + B\phi + C\phi^2$. Assume that $A > 0, \Delta = B^2 - 4AC > 0$. Considering the integral $\xi = \int_{\phi_M}^{\phi} \frac{d\phi}{\phi\sqrt{X(\phi)}}$, i.e., the solutions of the differential equation $\frac{d\phi}{d\xi} = \phi\sqrt{X(\phi)}$, we have*

(1) When $X(\phi_M) = 0$,

$$\begin{aligned} &\phi(\xi) = \frac{2A}{\sqrt{\Delta}\cosh(\sqrt{A}\xi) - B}, \quad \text{if } \phi(0) = -\frac{B+\sqrt{\Delta}}{2C}, \\ &\phi(\xi) = -\frac{2A}{\sqrt{\Delta}\cosh(\sqrt{A}\xi) + B}, \quad \text{if } \phi(0) = \frac{-B+\sqrt{\Delta}}{2C}. \end{aligned} \tag{1.8}$$

(2) When $X(\phi_M) \neq 0$,

$$\phi(\xi) = \frac{2A}{P\cosh_q(\sqrt{A}\xi) - B}, \tag{1.9}$$

where $P = \frac{1}{\phi_M}\left(2\sqrt{AX(\phi_M)} + B\phi_M + 2A\right), q = \frac{\Delta}{P^2}$.

The main result of this paper is the following theorem.

Theorem 1.1. *(i) Considering the traveling wave solutions defined by* (2.1)*, the multicomponent Korteweg-de Vries equation* (1.5) *has peakon solution given by* (2.6)*.*

(ii) Considering the traveling wave solutions defined by (3.1)*, the nonlinear Schrödinger equation* (1.6) *has the peakon and anti-peakon solutions given by* (3.8) *and* (3.9)*.*

(iii) Considering the traveling wave solutions defined by (*), the rotation-two-component Camassa-Holm system (1.7) *has the peakon and anti-peakon solutions given by* (4.6) *and* (4.8)*.*

The proof of the conclusions of this theorem can be seen in sections 2,3 and 4.

2. The exact peakon solutions of the multicomponent Korteweg-de Vries equation (1.5)

To investigate the traveling wave solutions of system (1.4), let

$$u(x,t) = \phi(x-ct) = \phi(\xi), \quad v_i(x,t) = v_i(x-ct) = v_i(\xi), \quad \xi = x - ct. \tag{2.1}$$

Substituting (2.1) into the second equation of system (1.4) and integrating the obtained equation once, we have

$$v_i = \frac{c_i\phi_\xi}{c+2\phi}, \quad (v_i)_\xi = \frac{c_i\phi_{\xi\xi}(c+2\phi) - 2c_i\phi_\xi^2}{(c+2\phi)^2}, \quad i = 1, 2, \cdots, n, \tag{2.2}$$

where we take the integral constant as zero.

Substituting (2.2) into the first equation of system (1.5) and integrating the obtained result onece, we obtain the planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-a^2y^2 + (c+2\phi)^2(3\phi^2 + c\phi + g)}{(c+2\phi)(c+2\phi-a^2)}, \tag{2.3}$$

which has the first integral

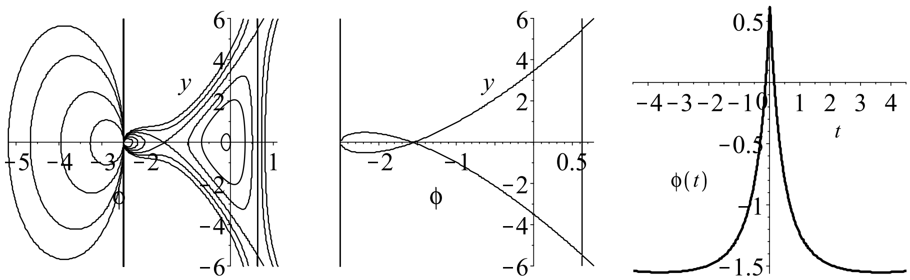
$$H_1(\phi, y) = \frac{(c+2\phi-a^2)y^2}{c+2\phi} - (2g\phi + c\phi^2 + 2\phi^3) = h, \tag{2.4}$$

where h is a constant.

Consider the associated regular system of system (2.3) as follows:

$$\frac{d\phi}{d\zeta} = y(c+2\phi)(c+2\phi-a^2), \quad \frac{dy}{d\zeta} = -a^2y^2 + (c+2\phi)^2(3\phi^2 + c\phi + g), \tag{2.5}$$

where $d\xi = (c+2\phi)(c+2\phi-a^2)d\zeta$, for $\phi \neq \phi_{s1} = -\frac{1}{2}c$ and $\phi \neq \phi_{s2} = \frac{a^2-c}{2}$. Systems (2.3) and (2.5) have the same first integral, but in the phase plane, two systems define different vector fields in the two sides of the singular straight lines.



(a) Phase portrait when $g = g_s$. (b) Level curves defined by $H_1(\phi, y) = h_s$. (c) Peakon.

Figure 2. The curve triangle defined by $H_1(\phi, y) = h_s$ of system (2.2).

Obviously, system (2.5) has the equilibrium points $E_1(\phi_1, 0), E_2(\phi_2, 0)$ and $E_s(\phi_{s1}, 0)$, where $\phi_{1,2} = \frac{1}{6}(-c \mp \sqrt{\Delta})$, when $\Delta = c^2 - 12g > 0$. When $F_s = 3a^4 - 4ca^2 + c^2 + 4g > 0$, in the singular line $\phi = \phi_{s2} = \frac{a^2-c}{2}$, there exist two equilibrium points $S_{\mp}(\phi_{s2}, \mp y_s)$, where $y_s = \frac{a}{2}\sqrt{F_s}$. Clearly, when $\Delta < 0$, we have $F_s > 0$. The point $E_s\left(-\frac{1}{2}c, 0\right)$ is a double equilibrium point of system (2.5).

For the first integral given by (2.4), we write that

$$h_1 = H_1(\phi_1, 0) = \frac{1}{54}(-\Delta^{\frac{3}{2}} - c + 18cg), \quad h_2 = H_1(\phi_2, 0) = \frac{1}{54}(\Delta^{\frac{3}{2}} - c + 18cg)$$

and

$$h_s = H_1(\phi_{s2}, \mp y_s) = -\frac{1}{4}(a^2 - c)(a^4 - ca^2 + 4g).$$

For a fixed pair (a^2, c) with $c > 0$, when $g = g_s = \frac{1}{4}ca^2 - \frac{3}{16}a^4$, we have $h_2 = h_s$. Taking $g = g_s$, we obtain the phase portrait of system (2.3) shown in Fig.2 (a). The level curves defined by $H_1(\phi, y) = h_s$ are shown in Fig.2 (b).

We know from (2.4) that $y^2 = \frac{(h+2g\phi+c\phi^2+2\phi^3)(c+2\phi)}{(c+2\phi-a^2)}$. By using the first equation of (2.3), we have

$$\sqrt{2}\xi = \int_{\phi_0}^{\phi} \frac{(\phi_{s2} - \phi)d\phi}{\sqrt{\left(\frac{1}{2}h + g\phi + \frac{1}{2}c\phi^2 + \phi^3\right)(\phi_{s2} - \phi)(\phi - \phi_{s1})}}.$$

When $h = h_s$, for the curve triangle, this integral becomes $\sqrt{2}\xi = \int_{\phi}^{\phi_{s2}} \frac{d\phi}{(\phi_1 - \phi)\sqrt{\phi - \phi_{s1}}}$. Thus, we obtan the following peakon solution (see Fig.2 (c)) with the parametric representation:

$$\begin{aligned} \phi(\xi) &= \phi_{s1} + (\phi_1 - \phi_{s1})\left(\frac{e^{\omega_1|\xi|} + q_0 e^{-\omega_1|\xi|}}{e^{\omega_1|\xi|} - q_0 e^{-\omega_1|\xi|}}\right)^2 \\ &\equiv \phi_{s1} + \frac{\phi_1 - \phi_{s1}}{\tanh^2_{q_0}(\omega_1|\xi|)} = \phi_{s1} + (\phi_1 - \phi_{s1})\mathrm{ctnh}^2_{q_0}(\omega_1|\xi|), \end{aligned} \tag{2.6}$$

where $q_0 = \frac{\sqrt{\phi_{s2}-\phi_{s1}} - \sqrt{\phi_1 - \phi_{s1}}}{\sqrt{\phi_{s2}-\phi_{s1}} + \sqrt{\phi_1 - \phi_{s1}})}, \omega_1 = \sqrt{\frac{\phi_1 - \phi_{s1}}{2}}$, and $\tanh_{q_0}(\xi), \mathrm{ctnh}_{q_0}(\xi)$ are the Arai q-deformed functions.

3. The exact peakon solution of the nonlinear Schrödinger equation (1.6)

Let

$$E(z,t) = \phi(\xi)\exp(i\theta), \quad \xi = pz - t, \ \theta = kz - ct. \tag{3.1}$$

Substituting (3.1) into equation (1.6) and separating the real and imaginary parts, reducing the fourth order derivative term $\phi^{(4)}$, we obtain from [3] that

$$\begin{aligned}
&[\beta_4(6p - 6\beta_2 c - 3\beta_3 c^2 + \beta_4 c^3) - (\beta_3 - \beta_4 c)(12\beta_2 + 12\beta_3 c - 6\beta_4 c^2)]\phi'' \\
&+\beta_4(18\alpha_1 + 12\alpha_2)[2\phi(\phi')^2 + \phi^2\phi''] \\
&-(\beta_3 - \beta_4 c)[(24k - 12\beta_2 c^2 - 4\beta_3 c^3 + \beta_4 c^4)\phi - 24(\gamma_1 - \alpha_1 c)\phi^3 - 24\gamma_2\phi^5] = 0.
\end{aligned} \tag{3.2}$$

Assume that $A = \beta_4(18\alpha_1 + 12\alpha_2) \neq 0$. We write

$$\begin{aligned}
a &= -\frac{1}{A}\left[\beta_4(6p - 6\beta_2 c - 3\beta_3 c^2 + \beta_4 c^3) - (\beta_3 - \beta_4 c)(12\beta_2 + 12\beta_3 c - 6\beta_4 c^2)\right], \\
r &= \frac{1}{A}(\beta_3 - \beta_4 c)(24k - 12\beta_2 c^2 - 4\beta_3 c^3 + \beta_4 c^4), \\
q &= -\frac{24}{A}(\beta_3 - \beta_4 c)(\gamma_1 - \alpha_1 c), \quad p = -\frac{24\gamma_2}{A}(\beta_3 - \beta_4 c).
\end{aligned}$$

Equation (3.2) is equivalent to the integrable system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{2\phi y^2 - r\phi - q\phi^3 - p\phi^5}{a - \phi^2} \tag{3.3}$$

with the first integral

$$H_2(\phi, y) = (a - \phi^2)^2 y^2 + ar\phi^2 + \frac{1}{2}(aq - r)\phi^4 + \frac{1}{3}(ap - q)\phi^6 - \frac{1}{4}p\phi^8 = h. \tag{3.4}$$

When $a > 0$, system (3.3) is a singular traveling wave system with the singular straight line $\phi_{s_\pm} = \pm\sqrt{a}$ (see [1, 2, 4, 7–11, 13, 14]).

We consider the following associated regular system of equation (3.3):

$$\frac{d\phi}{d\zeta} = (a - \phi^2)y, \quad \frac{dy}{d\zeta} = 2\phi y^2 - r\phi - q\phi^3 - p\phi^5 \equiv 2\phi y^2 - \phi f(\phi), \tag{3.5}$$

where $d\xi = (a - \phi^2)d\zeta$, $f(\phi) = r + q\phi^2 + p\phi^4$. System (3.5) has the same level curves as system (3.3), but two systems define different vector fields in the two sides of the singular straight lines $\phi_{s\mp} = \mp\sqrt{a}$.

Obviously, system (3.5) has the equilibrium point $O(0,0)$. In addition, if $E_j(\phi_j, 0)$ is another equilibrium point of system (3.5), then, we have $f(\phi_j) = 0$, i.e., ϕ_j is a zero point of $f(\phi)$.

When $pq < 0, pr > 0$ and $\Delta = q^2 - 4pr > 0$, in the $\phi-$axis, system (3.5) has five equilibrium points $O(0,0)$, $E_{1\mp}(\mp\phi_1, 0)$ and $E_{2\mp}(\mp\phi_2, 0)$, where $\phi_{1,2} = \left(\frac{-q\mp\sqrt{q^2-4pr}}{2p}\right)^{\frac{1}{2}}$. When $pr < 0$ (or $r = 0$, $pq < 0$), system (3.5) has three equilibrium points.

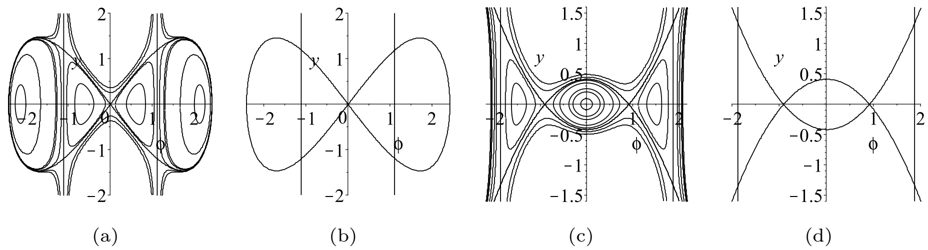


Figure 3. The curve triangle defined by $H_2(\phi, y) = h_s$ of system (3.3). (a) Phase portrait when $\frac{-q+\sqrt{\Delta}}{2p} < a < \frac{-q-\sqrt{\Delta}}{2p}$. (b) Level curves defined by $H_2(\phi, y) = 0 = h_s$. (c) Phase portrait when $a > \frac{-q+\sqrt{\Delta}}{2p}$. (d) Level curves defined by $H_2(\phi, y) = h_2 = h_s$.

When $Y_s = \frac{1}{2}f(\sqrt{a}) > 0$, system (3.5) has two equilibrium points $S_{a\mp}(\phi_{s-}, \mp\sqrt{Y_s})$ and $S_{b\mp}(\phi_{s+}, \mp\sqrt{Y_s})$ on two singular straight lines $\phi = \phi_{s\mp}$.

For the first integral $H_2(\phi, y) = h$ defined by (3.4), we write that

$$h_0 = H_2(0,0) = 0, \quad h_s = H_2(\phi_s, \sqrt{Y_s}) = \frac{a^2}{12}(pa^2 + 2aq + 6r),$$
$$h_1 = H_2(\phi_1, 0) = \frac{\phi_1^2}{24p^2}\left[-(16arp^2 - 2apq^2 + 5pqr - q^3) + (2apq - 3pr + q^2)\sqrt{\Delta}\right],$$
$$h_2 = H_2(\phi_2, 0) = \frac{\phi_2^2}{24p^2}\left[(16arp^2 - 2apq^2 + 5pqr - q^3) + (2apq - 3pr + q^2)\sqrt{\Delta}\right].$$

It is easy to see from the right hand of h_s that if $\Delta_1 = q^2 - 6pr > 0$, then, $h_s = 0$, when $a = \frac{1}{p}(-q \mp \sqrt{\Delta_1})$.

We next assume that $a > 0, pr > 0, \Delta_1 > 0$ such that system (3.5) has five equilibrium points in the $\phi-$axis, for a fixed parameter group (p, q, r). Then, when $p < 0, q > 0, r < 0, \Delta_1 > 0$, and $\frac{-q+\sqrt{\Delta}}{2p} < a < \frac{-q-\sqrt{\Delta}}{2p}, h_s = 0$, we have the phase portrait of system (3.3) as Fig.3 (a). The level curves defined by $H_2(\phi, y) = 0$ are shown in Fig.3 (b).

When $p > 0, q < 0, r > 0, \Delta_1 > 0$, and $a > \frac{-q+\sqrt{\Delta}}{2p}, h_2 = h_s = 0$, we have the phase portrait of system (3.3) as in Fig.3 (c). The level curves defined by $H_2(\phi, y) = h_2 = h_s$ are shown in Fig.3 (d).

To calculate the exact parametric representations of the orbits defined by $H_2(\phi, y) = h$ of system (3.3), we see from (3.4) that

$$y^2 = \frac{p\phi^8 - \frac{4}{3}(pa - q)\phi^6 - 2(aq - r)\phi^4 - 4ar\phi^2 + 4h}{4(a - \phi^2)^2} \equiv \frac{pG(\phi)}{4(a - \phi^2)^2}.$$

By using the first equation of (3.3), we have

$$\xi = \int_{\phi_0}^{\phi} \frac{2|a - \phi^2|d\phi}{\sqrt{p\phi^8 - \frac{4}{3}(pa - q)\phi^6 - 2(aq - r)\phi^4 - 4ar\phi^2 + 4h}} \equiv \int_{\phi_0}^{\phi} \frac{2|a - \phi^2|d\phi}{\sqrt{pG(\phi)}}. \tag{3.6}$$

Making the transformation $\psi=\phi^2$, (3.6) becomes

$$\xi=\int_{\psi_0}^{\psi}\frac{|a-\psi|d\psi}{\sqrt{\psi\left[p\psi^4-\frac{4}{3}(pa-q)\psi^3-2(aq-r)\psi^2-4ar\psi+4h\right]}}\equiv\int_{\psi_0}^{\psi}\frac{2|a-\psi|d\psi}{\sqrt{p\psi\tilde{G}(\psi)}}. \tag{3.7}$$

(i) Corresponding to the level curves defined by $H(\phi,y)=h_s$ in Fig.3 (b), there exist two homoclinic orbits to the origin $O(0,0)$, which passes through two singular straight lines $\phi=\mp\sqrt{a}$ and encloses the equilibrium points $(\mp\phi_1,0)$ and $(\mp\phi_2,0)$, respectively. In addition, there exist two curve triangles enclosing the equilibrium points $(\mp\phi_1,0)$, respectively. In this case, $G(\phi)=(\phi_M^2-\phi)(\phi^2-a)^2\phi^2$. For the right curve triangle, (3.7) becomes $\sqrt{|p|}\xi=\int_{\psi}^{\sqrt{a}}\frac{d\psi}{\psi\sqrt{\psi_M-\psi}}$. Hence, it gives rise to a peakon and an anti-peakon solutions of system (3.3) having the parametric representations:

$$\phi(\xi)=\pm\phi_M\left(1-\left(\frac{e^{\omega_1|\xi|}+q_1e^{-\omega_1|\xi|}}{e^{\omega_1|\xi|}-q_1e^{-\omega_1|\xi|}}\right)^2\right)^{\frac{1}{2}}\equiv\pm\phi_M\sqrt{|q_1|}\mathrm{csch}_{q_1}(\omega_1|\xi|), \tag{3.8}$$

where $q_1=\frac{\sqrt{\phi_M^2-a}-\phi_M}{\sqrt{\phi_M^2-a}+\phi_M},\omega_1=\frac{1}{2}\sqrt{|p|}\phi_M,\mathrm{csch}_{q_1}(\omega_1|\xi|)$ is a generalized hyperbolic function.

(ii) Corresponding to the level curves defined by $H(\phi,y)=h_s$ in Fig.3 (d), there exist two heteroclinic orbits connecting the equilibrium points $(-\phi_1,0)$ and $(\phi_1,0)$, and two curve triangles enclosing the equilibrium points $(-\phi_2,0)$ and $(\phi_2,0)$, respectively. Now, we have $G(\phi)=(a-\phi^2)^2(\phi_1^2-\phi^2)^2$.

For their boundary curves of the two curve triangles, (3.7) becomes that $\sqrt{p}\xi=\int_{\psi}^{a}\frac{d\psi}{(\psi-\psi_1)\sqrt{\psi}}$. We obtain the following peakon and unti-peakon solutions of system (3.3):

$$\phi(\xi)=\pm\phi_1\left(\frac{e^{\omega_2|\xi|}+q_2e^{-\omega_2|\xi|}}{e^{\omega_2|\xi|}-q_2e^{-\omega_2|\xi|}}\right)\equiv\pm\phi_1\mathrm{ctnh}_{q_2}(\omega_2|\xi|), \tag{3.9}$$

where $\omega_2=\frac{1}{2}\sqrt{p}\phi_1,q_2=\frac{\sqrt{a}-\phi_1}{\sqrt{a}+\phi_1}$, and $\mathrm{ctnh}_{q_2}(\omega_2|\xi|)$ is a generalized hyperbolic function.

4. The exact peakon solutions of the rotation-two-component Camassa-Holm system (1.7)

To investigate the traveling wave solutions of system (1.7), by letting

$$u(x,t)=\phi(x-ct)=\phi(\xi),\ \rho(x,t)=v(x-ct)=v(\xi), \tag{$*$}$$

where c is the wave speed, we see from second equation of system (1.7) that $v(\xi)=\frac{\beta}{\phi-c}$, where β is an integration constant and $\beta\neq0$. By the first equation of system (1.7), we have

$$(\sigma\phi-c-\mu)\phi''=-\frac{1}{2}\sigma(\phi')^2-(A+c)\phi+\frac{3}{2}\phi^2+\frac{(1-2A\Omega+2c\Omega)\beta^2}{2(\phi-c)^2}-\frac{1}{2}g,$$

where $\frac{1}{2}g$ is the second integration constant. Write $\alpha=1-2A\Omega+2c\Omega$. Then, the above equation is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi}=y,\quad\frac{dy}{d\xi}=\frac{-\sigma y^2(\phi-c)^2+(\phi-c)^2[3\phi^2-2(A+c)\phi-g]+\alpha\beta^2}{2(\phi-c)^2(\sigma\phi-c-\mu)}, \tag{4.1}$$

which has the following first integral:

$$H_3(\phi, y) = y^2(\sigma\phi - c - \mu) - \phi^3 + (A+c)\phi^2 + g\phi + \frac{\alpha\beta^2}{(\phi - c)} = h. \tag{4.2}$$

Assume that $A > 0$. Imposing the transformation $d\xi = (\phi - c)^2(\sigma\phi - c - \mu)d\zeta$ for $\phi \neq c$, $\frac{c+\mu}{\sigma}$ on system (4.1) leads to the following regular system:

$$\frac{d\phi}{d\zeta} = y(\phi - c)^2(\sigma\phi - c - \mu), \quad \frac{dy}{d\zeta} = -\frac{1}{2}\sigma y^2(\phi - c)^2 + \frac{1}{2}\left[(\phi - c)^2(3\phi^2 - 2(A+c)\phi - g) + \alpha\beta^2\right]. \tag{4.3}$$

Apparently, two singular lines $\phi = c$ and $\phi = \frac{c+\mu}{\sigma}$ are two invariant constant solutions of system (4.3). Near these two straight lines, the variable "ζ" is a fast variable while the variable "ξ" is a slow variable in the sense of the geometric singular perturbation theory.

To see the equilibrium points of system (4.3), write $f(\phi) = (\phi - c)^2(3\phi^2 - 2(A+c)\phi - g) + \alpha\beta^2$, $f'(\phi) = 2(\phi - c)[6\phi^2 - 3(A+2c)\phi + c(A+c) - g]$, $f''(\phi) = 2(18\phi^2 - 6(A+4c)\phi + c(4A+7c) - 2g$. Apparently, $f'(\phi)$ has one zero at $\phi = \phi_{s1} = c$. When $\Delta = 9A^2 + 12Ac + 12c^2 + 24g > 0$, $f'(\phi)$ has two real zeros, at $\phi = \tilde{\phi}_{1,2} = \frac{1}{12}[3(A+2c) \mp \sqrt{\Delta}]$. Thus, $f(c) = \alpha\beta^2, f'(c) = 0$ and $f''(c) = 2(c^2 - 2cA - g)$, $f(0) = \alpha\beta^2 - gc^2$.

In the ϕ-axis, the equilibrium points $E_j(\phi_j, 0)$ of system (4.3) satisfy $f(\phi_j) = 0$. Obviously, system (4.3) has at most 4 equilibrium points at $E_j(\phi_j, 0)$, $j = 1, 2, 3, 4$. On the straight line $\phi = c$, there is no equilibrium point of system (4.3) because $\beta \neq 0$. On the straight line $\phi = \frac{c+\mu}{\sigma}$, there exist two equilibrium points, $S_{\mp}\left(\frac{c+\mu}{\sigma}, \mp Y_s\right)$ of system (4.3), with $Y_s = \sqrt{\frac{f(\frac{c+\mu}{\sigma})}{\sigma(\frac{c+\mu}{\sigma} - c)^2}}$, if $\sigma f(\frac{c+\mu}{\sigma}) > 0$.

Next, we assume that $\alpha > 0$. In this case, $f(c) = \alpha\beta^2 > 0$.

(i) Case of $g \geq 0$. In this case, one always has $\Delta > 0$. It can be easily shown that, when $0 < c < A + \sqrt{A^2 + g}$, one has $f''(c) < 0$, and when $A + \sqrt{A^2 + g} < c$, one has $f''(c) > 0$. The condition $f''(c) < 0$ implies $\tilde{\phi}_1 < c < \tilde{\phi}_2$. The condition $f''(c) > 0$ implies $\tilde{\phi}_1 < \tilde{\phi}_2 < c$.

When $0 < c < A + \sqrt{A^2 + g}$, if $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$, system (4.3) has four simple equilibrium points, $E_j(\phi_j, 0)$, satisfying $\phi_1 < \tilde{\phi}_1 < \phi_2 < c < \phi_3 < \tilde{\phi}_2 < \phi_4$.

(ii) Case of $g < 0$ and $\Delta > 0$. In this case, the requirement of $\Delta > 0$ is either $A^2 + 4g > 0$ or $A^2 + 4g < 0$, $c > \frac{1}{2}(\sqrt{2(4|g| - A^2)} - A) > 0$.

When $A^2 + 4g > 0$ and $A - \sqrt{A^2 + g} < c < A + \sqrt{A^2 + g}$, one has $f''(c) < 0$ and $\tilde{\phi}_1 < c < \tilde{\phi}_2$.

If $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$, then system (4.3) has four simple equilibrium points, $E_j(\phi_j, 0)$, $j = 1, 2, 3, 4$, satisfying $\phi_1 < \tilde{\phi}_1 < \phi_2 < c < \phi_3 < \tilde{\phi}_2 < \phi_4$.

Now, for a given wave speed $c + \mu > 0$, assume that one of the following two conditions holds:

(a_1) $g > 0, c < A + \sqrt{A^2 + g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.

(a_2) $g < 0$, $A^2 + 4g > 0$, $A - \sqrt{A^2 + g} < c < A + \sqrt{A^2 + g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.

Then, system (4.3) has four simple equilibrium points, $E_j(\phi_j, 0)$, $j = 1, 2, 3, 4$, satisfying $\phi_1 < \tilde{\phi}_1 < \phi_2 < c < \phi_3 < \tilde{\phi}_2 < \phi_4$.

Notice that for every $j = 1, 2, 3, 4$, ϕ_j does not depend on the parameter σ.

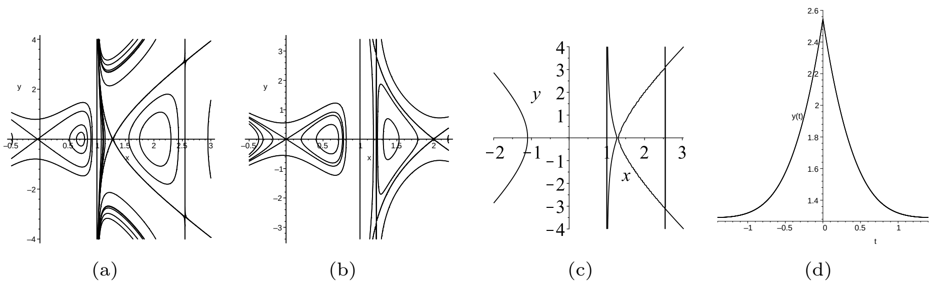


Figure 4. Phase portraits and curve triangle defined by $H_3(\phi, y) = h_s$ of system (4.1). (a) $h_1 < h_2 < h_s = h_3 < h_4, \phi_4 < \frac{c+\mu}{\sigma}, \sigma < 1$. (b) $h_1 < h_2 < h_3 < h_4 = h_s, c < \frac{c+\mu}{\sigma} < \phi_3, \sigma < 1$. (c) In Fig.4 (a), the level curves defined by $H_3(\phi, y) = h_s$. (d) Peakon solution of system (4.1) corresponding to the curve triangle in Fig.4 (c).

Now, let $h_i = H_3(\phi_i, 0)$ and $h_s = H_3\left(\frac{c+\mu}{\sigma}, \mp Y_s\right)$, where H_3 is given by (4.2).

In the case of $\alpha > 0$, suppose that $\sigma \neq 0$. For a fixed $c+\mu$, by increasing σ from $\sigma < 1$ to $\sigma \geq 1$, i.e., letting the singular line $\phi = \frac{c+\mu}{\sigma}$ move from right to left in the (ϕ, y)-phase plane, one can obtain different topological phase portraits of system (4.3). Especially, we have two phase portraits to appear curve triangle shown in Fig.4 (a) and (b) where the corresponding values of $H_3(\phi_j, 0)$ and parameter conditions are given.

We discuss the exact parametric representations of peakon and anti-peakon solutions of system (4.1) corresponding to two curve triangles in Fig.4 (a) and (b). As can be seen from (4.2), for a fixed integral constant h, one has

$$
\begin{aligned}
y^2 &= \frac{\phi^4 - (A+2c)\phi^3 + (c^2 + Ac - g)\phi^2 + (h + cg)\phi - (ch + \alpha\beta^2)}{(\phi - c)(\sigma\phi - c - \mu)} \\
&\equiv \frac{G(\phi)}{(\phi - c)(\sigma\phi - c - \mu)}.
\end{aligned}
$$

By using the first equation of system (4.1) and taking integration along a branch of the level curve defined by $H_3(\phi, y) = h$ with initial value $\phi(\xi_0) = \phi_0$, one obtains

$$
\xi - \xi_0 = \int_{\phi_0}^{\phi} \sqrt{\frac{(\phi - c)(\sigma\phi - c - \mu)}{G(\phi)}} d\phi. \tag{4.4}
$$

(i) Corresponding to the orbit triangle (see Fig.4 (c)) connecting the equilibrium points $E_3(\phi_3, 0)$ and $S_\mp$ of system (4.3) and enclosing the center $E_4(\phi_4, 0)$ in Fig.4 (c), which is the level curve defined by $H_3(\phi, y) = h_s = h_3$, one has $G(\phi) = (\frac{c+\mu}{\sigma} - \phi)(\phi - \phi_3)^2(\phi - \phi_l)$. Hence, taking integrals along the curves E_3S_+ and S_-E_3, it yields from (4.4) that

$$
\pm \frac{\xi}{\sqrt{\sigma}} = \int_{\phi_4}^{\phi} \frac{d\phi}{\sqrt{(\phi - c)(\phi - \phi_l)}} + (\phi_3 - c) \int_{\phi_4}^{\phi} \frac{d\phi}{(\phi - \phi_3)\sqrt{(\phi - c)(\phi - \phi_l)}}. \tag{4.5}
$$

Thus, by Poposition 1.1, one obtains the following exact peakon solution of system

(1.7) (see Fig.4 (d)):

$$\phi(\chi)=\phi_3+\frac{2(\phi_3-c)(\phi_3-\phi_l)}{\tilde{P}_1\cosh_{\tilde{q}_a}\left(\sqrt{(\phi_3-c)(\phi_3-\phi_l)}\chi\right)-(2\phi_3-c-\phi_l)},\quad \chi\in(-\infty,-\chi_{03}),\ \text{and}\ \ (\chi_{03},\infty),$$
$$\xi(\chi)=\mp\sqrt{\sigma}\left[(\phi_3-c)\chi+\ln\left(\sqrt{(\phi(\chi)-c)(\phi(\chi)-\phi_l)}+\phi(\chi)-\tfrac{1}{2}(c+\phi_l)\right)-\xi_{01}\right], \tag{4.6}$$

where

$$\chi_{03}=\frac{1}{\sqrt{(\phi_3-c)(\phi_3-\phi_l)}}\cosh^{-1}_{\tilde{q}_a}\left(\frac{1}{\tilde{P}_1}\left(\frac{2(\phi_3-c)(\phi_3-\phi_l)}{\frac{c+\mu}{\sigma}-\phi_3}+(2\phi_3-c-\phi_l)\right)\right),$$
$$\tilde{P}_1=\frac{1}{\phi_4}\left[2\sqrt{(\phi_3-c)(\phi_3-\phi_l)(\phi_4-c)(\phi_4-\phi_l)}+(2\phi_3-c-\phi_3)\phi_4+2(\phi_3-c)(\phi_3-\phi_l)\right],$$
$$\tilde{q}_a=\frac{(c-\phi_l)^2}{(\tilde{P}_1)^2},\xi_{01}=\ln\left(\sqrt{(\phi_4-c)(\phi_4-\phi_l)}+\phi_4-\frac{1}{2}(c+\phi_l)\right).$$

(ii) Corresponding to the orbit triangle (see Fig.4 (b)) connecting the equilibrium points $E_4(\phi_4,0)$ and $S_\mp$ of system (4.3) and enclosing the center $E_3(\phi_3,0)$ in Fig.4 (b), which is the level curve defined by $H(\phi,y)=h_s=h_4$, one has $G(\phi)=(\phi_4-\phi)^2(\phi-\frac{c+\mu}{\sigma})(\phi-\phi_l)$.

Then, taking integrals along the curves S_+E_4 and E_4S_-, it yields from (4.4) that

$$\pm\frac{\xi}{\sqrt{\sigma}}=-\int_{\phi_3}^{\phi}\frac{d\phi}{\sqrt{(\phi-c)(\phi-\phi_l)}}+(\phi_4-c)\int_{\phi_3}^{\phi}\frac{d\phi}{(\phi_4-\phi)\sqrt{(\phi-c)(\phi-\phi_l)}}. \tag{4.7}$$

Thus, by Poposition 1.1, one obtains the following exact anti-peakon solution of system (1.7):

$$\phi(\chi)=\phi_4-\frac{2(\phi_4-c)(\phi_4-\phi_l)}{\tilde{P}_2\cosh_{\tilde{q}_b}\left(\sqrt{(\phi_4-c)(\phi_4-\phi_l)}\chi\right)+(2\phi_4-c-\phi_l)},\quad \chi\in(-\infty,-\chi_{04}),\ \text{and}\ (\chi_{04},\infty),$$
$$\xi(\chi)=\mp\sqrt{\sigma}\left[(\phi_3-c)\chi-\ln\left(\sqrt{(\phi(\chi)-c)(\phi(\chi)-\phi_l)}+\phi(\chi)-\tfrac{1}{2}(c+\phi_l)\right)+\xi_{02}\right], \tag{4.8}$$

where

$$\chi_{04}=\frac{1}{\sqrt{(\phi_4-c)(\phi_4-\phi_l)}}\cosh^{-1}_{\tilde{q}_b}\left(\frac{1}{\tilde{P}_2}\left(\frac{2(\phi_4-c)(\phi_4-\phi_l)}{\phi_4-\frac{c+\mu}{\sigma}}-(2\phi_4-c-\phi_l)\right)\right),$$
$$\tilde{P}_2=\frac{1}{\phi_4}\left[2\sqrt{(\phi_4-c)(\phi_4-\phi_l)(\phi_3-c)(\phi_3-\phi_l)}-(2\phi_4-c-\phi_3)\phi_3+2(\phi_4-c)(\phi_4-\phi_l)\right],$$
$$\tilde{q}_b=\frac{(c-\phi_l)^2}{(\tilde{P}_2)^2},\ \xi_{02}=\ln\left(\sqrt{(\phi_3-c)(\phi_3-\phi_l)}+\phi_3-\frac{1}{2}(c+\phi_l)\right).$$

References

[1] A. Arab, *Exactly solvable supersymmetric quantum mechanics,* J. Math. Anal. Appl., 1991, 158, 63–79.

[2] A. Arab, *Exact solutions of multi-component nonlinear Schrödinger and Klein-Gordon equations in two-dimensional space-time,* J. Phys. A: Math. Gen., 2001, 34, 4281.

[3] R. Cammasa and D. D. Holm, *An integrable shallow water equation with peaked solution,* Phys. Rev. Lett., 1993, 71, 1161–1164.

[4] R. Cammasa, D. D. Holm and J. M. Hyman, *A new integrable shallow water equation,* Adv. Appl. Mech., 1994, 31, 1–33.

[5] E. Fan and M. Yuen, *Peakon weak solutions for the rotation-two-component Camassa–Holm system,* Appl. Math. Lett, 2019, 97, 53–59.

[6] L. Fan, H. Gao and Y. Liu, *On the rotation-two-component Camassa–Holm system modelling the equatorial water waves,* Adv. Math., 2016, 291, 59–89.

[7] B. A. Kupershmidt, *A coupled Korteweg-de Vries equation with dispersion,* J. Phys A., 1985, 18, L571–573.

[8] J. Li, *Singular Nonlinear Traveling Wave Equations: Bifurcations and Exact Solutions,* Science Press, Beijing, 2013.

[9] J. Li and G. Chen, *On a class of singular nonlinear traveling wave equations,* Int. J. Bifur. Chaos, 2007, 17, 4049–4065.

[10] J. Li and Z. Qiao, *Bifurcations and exact travelling wave solutions of the generalized two-component Camassa-Holm equation,* Int. J. Bifur. Chaos, 2012, 22, 1250305, 1–13.

[11] J. Li, *Variform exact one-peakon solutions for some singular nonlinear traveling wave equations of the first kind,* Int. J. of Bifurcation and Chaos, 2014, 24, 1450160, 1–15.

[12] J. Li, W. Zhu, and G. Chen, *Understanding peakons, periodic peakons and compactons via a shallow water wave equation,* Int. J. Bifur. Chaos, 2016, 26, 1650207.

[13] J. Liang, and J. Li, *Bifurcations and exact solutions of nonlinear Schrödinger equation with an anti-cubic nonlinearity,* Journal of Applied Analysis and Computation, 2018, 8, 1194–1210.

[14] J. Zhang and C. Dai, *Bright and dark optical solitons in the nonlinear Schrödinger equation with fourth-order dispersion and cubic-quintic nonlinearity*, Chinese Optics Letters, 2005, 3, 295–298.