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INVARIANT MEASURE AND STATISTICAL SOLUTIONS FOR NON-AUTONOMOUS DISCRETE KLEIN-GORDON-SCHRÖDINGER-TYPE EQUATIONS*

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Abstract In this article, we first prove the existence of the pullback attractor for no-autonomous discrete Klein-Gordon-Schrödinger-type equations. Then we construct the invariant measure and statistical solutions for this discrete equations via the generalized Banach limit.

Keywords Discrete Klein-Gordon-Schrödinger-type equations, pullback attractor, invariant measures, statistical solutions.

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1. Introduction

In this article, we consider the following non-autonomous discrete Klein-Gordon-Schrödinger-type equations:

$$\begin{cases} i\dot{z}_m - \kappa(Az)_m + i\alpha z_m = z_m u_m + f_m(t),\ m \in \mathbb{Z},\ t > 0, \\ \ddot{u}_m + (Au)_m + u_m + \lambda \dot{u}_m = -\mathbf{Re}(Bz)_m + g_m(t),\ m \in \mathbb{Z},\ t > 0, \end{cases} \tag{1.1}$$

with initial conditions:

$$u_m(\tau) = u_{m,\tau}, \quad \dot{u}_m(\tau) = u_{1m,\tau}, \quad z_m(\tau) = z_{m,\tau}, \quad m \in \mathbb{Z}, \quad \tau \in \mathbb{R}. \tag{1.2}$$

where $z_m(t) \in \mathbb{C}, u_m(t) \in \mathbb{R}$, κ, α and λ are positive constants, A and B are linear operators defined as

$$\begin{aligned} (Au)_m &= 2u_m - u_{m+1} - u_{m-1}, \quad \forall u = (u_m)_{m\in\mathbb{Z}}, \\ (Bu)_m &= u_{m+1} - u_m, \quad \forall u = (u_m)_{m\in\mathbb{Z}}. \end{aligned}$$

Equations (1.1) can be regarded as a discrete version of the following continuous Klein-Gordon-Schrödinger-type equations on $\mathbb{C} \times \mathbb{R}$,

$$\begin{cases} iz_t - \kappa z_{xx} + i\alpha z = zu + f(t),\ x \in \mathbb{R},\ t > 0, \\ u_{tt} - u_{xx} + u + \lambda u_t = -\mathbf{Re} z_x + g(t),\ x \in \mathbb{R},\ t > 0. \end{cases} \tag{1.3}$$

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Equations (1.3) describe the nonlinear interaction between high-frequency electron waves and low-frequency ion plasma waves, where $z = z(x,t) \in \mathbb{C}$ represents the dimensionless low frequency electron field, and $u = u(x,t) \in \mathbb{R}$ represents the dimensionless low frequency density, $i\alpha z$ and λu_t denote the dissipative mechanism of the system. The term $\mathbf{Re} z_x$ reflects the contribution of the effect caused by polarization drift to the system of equations. The well-posedness of the continuous version of Klein-Gordon-Schrödinger-type equations (1.3) has been studied extensively, we refer the readers to [1, 6, 12–14] and the references cited therein.

The invariant measures and statistical solutions are very useful concepts to understand the complexity of dynamical systems. For example, in order to analysis the turbulence in the case of Navier-Stokes equations, the measurements of several aspects of turbulent flows are actually measurements of time-average quantities, see [?] for details. Statistical solutions were used to formalize the notion of ensemble average in the conventional statistical theory of turbulence. Foias and Prodi [4] introduced the so-called Foias-Prodi statistical solution, which are associated to some invariant measures defined on the phase space (independent of time t) of the addressed system. Vishik and Furshikov [15] developed the so-called Vishik-Furshikov statistical solution, which are associated to some invariant measures defined on the trajectory space (dependent of time t), see [3] for details. Wang [16] talked about invariant Borel probability measures for discrete long-wave-short-wave resonance equations. Zhao [19] constructed trajectory statistical solutions possess an invariant property and satisfy a Liouville type equation. There are a series of papers concerning the invariant measures for well-posed dissipative systems, see [7–9, 11, 17, 18].

For the autonomous discrete Klein-Gordon-Schrödinger-type equations, Li, Hsu, Lin and Zhao established the existence of global attractor and obtained an upper bound of fractal dimension for the global attractor in [10]. Recently, Zhao, Xue and Łukaszewicz [20] studied the existence of the pullback attractor and invariant measure for a kind of lattice Klein-Gordon-Schrödinger equations, but they do not study the statistical solution for these equations.

In the current article, we first prove the existence of the pullback attractor for no-autonomous discrete Klein-Gordon-Schrödinger-type equations. Then we construct the invariant measure and statistical solutions for this discrete equations via the generalized Banach limit. To the best of the authors' knowledge, this is the first reference investigating the statistical solutions for lattice dynamical systems.

The rest of this article is organized as follows. Section 2 are preliminaries. In Section 3, we prove the existence of the pullback attractor. In section 4, we construct the invariant measure and statistical solutions via the generalized Banach limit.

2. Preliminaries

Set

$$\ell^2 = \left\{ u = (u_m)_{m\in\mathbb{Z}} \left| u_m \in \mathbb{R} \text{ and } \sum_{m\in\mathbb{Z}} |u_m|^2 < +\infty \right. \right\},$$

$$L^2 = \left\{ u = (u_m)_{m\in\mathbb{Z}} \left| u_m \in \mathbb{C} \text{ and } \sum_{m\in\mathbb{Z}} |u_m|^2 < +\infty \right. \right\}.$$

We use X to denote ℓ^2 or L^2 for brevity, and equip X with the inner product and norm by

$$(u,v)=\sum_{m\in\mathbb{Z}}u_m\bar{v}_m,\ \|u\|^2=(u,u),\ u=(u_m)_{m\in\mathbb{Z}},\ v=(v_m)_{m\in\mathbb{Z}}\in X,$$

where $\bar{v}_m$ denote the conjugate of v_m. Define a bilinear form $(\cdot,\cdot)_1$ and the adjoint operator B^* of B respectively as

$$\begin{aligned}&(u,v)_1=(Bu,Bv)+(u,v),\quad u,v\in X,\\&(B^*u,v)=(Bu,v),\quad u,v\in X.\end{aligned}$$

Obviously, the bilinear form $(\cdot,\cdot)_1$ is also an inner product in X, and the norm $\|u\|_1:=(u,u)_1$. Then we have:

$$\begin{aligned}&(Au,v)=(B^*Bu,v)=(Bu,Bv),\quad (Bu,v)=(u,B^*v)\,,\quad \forall u,v\in X,\\&\|Au\|^2\leqslant 16\|u\|^2,\ \|Bu\|^2=\|B^*u\|^2\leqslant 4\|u\|^2,\quad \forall u\in X,\\&\|u\|^2\leqslant\|u\|^2+\|Bu\|^2=\|u\|_1^2\leqslant 5\|u\|^2,\quad \forall u\in X.\end{aligned}$$

Thus, the norm $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|$.

We next denote

$$E=L^2\times\ell_1^2\times\ell^2,$$

and equip it with the inner products

$$\begin{aligned}\left(\psi^{(1)},\psi^{(2)}\right)_E&=\left(z_m^{(1)}\bar{z}_m^{(2)}+u^{(1)},u^{(2)}\right)_1+\left(v^{(1)},v^{(2)}\right)+\left(z^{(1)},z^{(2)}\right)\\&=\sum_{m\in\mathbb{Z}}\left(\left(Bu^{(1)}\right)_m\left(Bu^{(2)}\right)_m+u_m^{(1)}u_m^{(2)}+v_m^{(1)}v_m^{(2)}\right),\\&\qquad\forall\psi^{(k)}=\left(z^{(k)},u^{(k)},v^{(k)}\right)^T\in E,\quad k=1,2,\end{aligned}$$

and the corresponding norm

$$\|\psi\|_E=\sqrt{(\psi,\psi)_E},\quad \psi\in E.$$

Now we set

$$v=\dot{u}+\delta u,\quad\text{where}\quad \delta=\frac{\lambda}{\lambda^2+4}>0.$$

Then equations (1.2)-(1.4) can be rewritten as following systems

$$\begin{cases}\dot{\psi}+\Theta\psi=G(\psi,t),\quad t>\tau,\\ \psi(\tau)=\psi_\tau=(z_\tau,u_\tau,v_\tau)^T=(z_\tau,u_\tau,\dot{u}_\tau+\delta u_\tau),\quad \tau\in\mathbb{R},\end{cases}\tag{2.1}$$

where

$$\begin{aligned}&\psi=(z,u,v)^T,\quad v=\dot{u}+\delta u,\\&G(\psi,t)=(-izu-if,0,-\mathbf{Re}(Bz)+g)^T,\\&\Theta=\begin{pmatrix}i\kappa A+\alpha I&0&0\\0&\delta I&-I\\0&A+I+\delta(\delta-\lambda)I&(\lambda-\delta)I\end{pmatrix}.\end{aligned}$$

For the locally existence of solution to problem (2.1), we have

Lemma 2.1 ([20]). *Let $f \in L^2$ and $g \in l^2$. For any initial data $\psi_0 = (z_0, u_0, v_0)^T \in E$, there exist a unique local solution $\psi(t) = (z(t), u(t), v(t)) \in E$ of* (2.1) *such that $\psi(\cdot) \in \mathcal{C}([0,T), E) \cap \mathcal{C}^1((0,T), E)$ for some $T > 0$. Moreover, if $T < +\infty$,then $lim_{t\to T^-}\|\psi(t)\|_E = +\infty$.*

Lemma 2.2. *For any $\psi = (z, u, v)^{\mathrm{T}} \in E$, there holds*

$$\boldsymbol{Re}(\Theta\psi, \psi)_E \geq \beta\left(\|u\|_1^2 + \|v\|^2\right) + \frac{\lambda}{2}\|v\|^2 + \alpha\|z\|^2, \quad \forall t \geqslant 0, \tag{2.2}$$

where

$$\beta := \lambda\left[\sqrt{\lambda^2+4}\left(\sqrt{\lambda^2+4}+\lambda\right)\right]^{-1} > 0. \tag{2.3}$$

We next verify the boundness of the solutions.

Lemma 2.3. *Let $f \in L^2$, $g \in l^2$ and $\psi(t) = (z(t), u(t), v(t)) \in E$ be the solution of problem* (2.1) *with initial data $\psi_\tau = (z_\tau, u_\tau, v_\tau)^T \in E$. Then*

$$\|z(t)\|^2 \leqslant \|z_\tau\|^2 e^{-\alpha(t-\tau)} + \frac{e^{-\alpha t}}{\alpha}\int_\tau^t e^{\alpha s}\|f(s)\|^2 ds, \quad \forall t \geqslant \tau. \tag{2.4}$$

Proof. Taking the imaginary part of the inner product $(L^2, (\cdot,\cdot))$ of (1.1) with z, we obtain

$$\frac{d}{dt}\|z(t)\|^2 + \alpha\|z(t)\|^2 \leqslant \frac{1}{\alpha}\|f(t)\|^2.$$

Using the Gronwall inequality, we obtain the estimation (2.4) and the proof is completed. □

Lemma 2.4. *Let $f(t) = (f_m(t))_{m\in Z} \in \mathcal{C}\left(\mathbb{R}, L^2\right)$ and $g(t) = (g_m(t))_{m\in Z} \in \mathcal{C}\left(\mathbb{R}, \ell^2\right)$ Then the solution $\psi(t) = (z(t), u(t), v(t))^T \in E$ of problem* (2.1) *corresponding to initial condition $\psi_\tau = (z_\tau, u_\tau, v_\tau)^T \in E$ satisfies*

$$\begin{aligned}\|\psi(t)\|_E^2 \leqslant \|\psi_\tau\|_E^2 e^{-\sigma(t-\tau)} &+ c_1 e^{-\sigma t}\int_\tau^t e^{\sigma s}\left(\|f(s)\|^2 + \|g(s)\|^2\right) ds \\ &+ c_1 e^{-\sigma t}\int_\tau^t e^{\sigma s}\|z(s)\|^2 ds, \quad \forall t \geqslant \tau.\end{aligned} \tag{2.5}$$

where $c_1 = max\{1/\alpha, 4/\lambda\}, \sigma = min\{2\beta, \alpha\}$, and β is defined in Lemma 2.2.

Proof. Taking the imaginary part of the inner product $(\cdot,\cdot)_E$ of (2.1) with $\psi(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_E^2 + \mathbf{Re}(\Theta\psi, \psi)_E = \mathbf{Re}(G(\psi, t), \psi)_E. \tag{2.6}$$

Now, we estimate the term $\mathbf{Re}(G(\psi, t), \psi)$:

$$\mathbf{Re}(G(\psi,t), \psi)_E = -(\mathbf{Re}(Bz), v) + (g, v) + \mathbf{Im}(f, z). \tag{2.7}$$

By Lemma 2.3, we obtain that

$$\begin{aligned}-(\mathbf{Re}(Bz), v) &\leqslant \frac{\lambda}{4}\|v\|^2 + \frac{4}{\lambda}\|z\|^2, \\ (g, v) &\leqslant \frac{\lambda}{4}\|v\|^2 + \frac{1}{\lambda}\|g\|^2,\end{aligned}$$

$$\mathbf{Im}(f, z) \leqslant \frac{\alpha}{2}\|z\|^2 + \frac{1}{2\alpha}\|f\|^2.$$

Combining the above three terms, we have

$$\mathbf{Re}(G(\psi, t), \psi)_E \leqslant \frac{\lambda}{2}\|v\|^2 + \frac{\alpha}{2}\|z\|^2 + \frac{1}{2\alpha}\|f\|^2 + \frac{1}{\lambda}\|g\|^2 + \frac{4}{\lambda}\|z\|^2, \quad \forall t \geqslant 0. \tag{2.8}$$

It follows from (2.6), (2.7) and (2.8) that

$$\frac{d}{dt}\|\psi\|_E^2 + \sigma\|\psi\|_E^2 \leqslant \frac{1}{\alpha}\|f\|^2 + \frac{2}{\lambda}\|g\|^2 + \frac{4}{\lambda}\|z\|^2, \quad \forall t \geqslant 0,$$

where $\sigma = \min\{2\beta, \alpha\}$. Apply Gronwall inequality to (2), we get the desired result. □

By (2.4) and (2.5), we see that for any $\psi_\tau = (z_\tau, u_\tau, v_\tau)^T \in E$ the corresponding solution $\psi(t) = (z(t), u(t), v(t))^T \in E$ of problem (2.1) is bounded for all $t \in [\tau, +\infty)$, hence the solution $\psi(t)$ exists globally on $[\tau, +\infty)$. Moreover, from Lemma 2.1 we see that

$$\psi(\cdot) \in \mathcal{C}([\tau, +\infty), E) \cap \mathcal{C}^1((\tau, +\infty), E).$$

Thus, Lemma 2.1 implies that the maps of solutions operators

$$U(t, \tau) : \psi_\tau = (z_\tau, u_\tau, v_\tau)^T \in E \longmapsto \psi(t) = \big(z(t), u(t), v(t)\big)^T \in E, \quad t \geqslant \tau,$$

generate a continuous process $\{U(t, \tau)\}_{t \geqslant \tau}$ in E.

3. Existence of the pullback attractor

We first prove some estimates of solutions to problem (2.1). To guarantee the existence of pullback absorbing set, we need some assumption on the functions $f(t)$ and $g(t)$.

Assumption **(H)**: Let $f(t) \in \mathcal{C}(\mathbb{R}; L^2)$, $g(t) \in \mathcal{C}(\mathbb{R}; \ell^2)$ and

$$\int_{-\infty}^{s} e^{\sigma\eta}\|f(\eta)\|^2 \mathrm{d}\eta < e^{(\frac{\sigma}{2}+\rho)s}K(s), \tag{3.1}$$

for some continuous function $K(\cdot)$ on the real line, bounded on intervals of the form $(-\infty, t)$, with $0 < \rho < \frac{\sigma}{2}$, and let

$$\int_{-\infty}^{s} e^{\sigma\eta}\|g(\eta)\|^2 \mathrm{d}\eta < +\infty, \quad \text{for each } s \in \mathbb{R}. \tag{3.2}$$

Next, we denote by $\mathcal{P}(E)$ the family of all nonempty subsets of E and by $\mathcal{D}_\sigma$ the class of families of nonempty subsets $\widehat{D} = \{D(s) | s \in \mathbb{R}\} \subseteq \mathcal{P}(E)$ satisfying

$$\lim_{s \to -\infty} \Big(e^{\frac{\sigma s}{2}} \sup_{\psi \in D(s)} \|\psi\|_E^2 \Big) = 0, \tag{3.3}$$

hereinafter, the constant σ comes from Lemma 2.4. The class $\mathcal{D}_\sigma$ will be called a universe in $\mathcal{P}(E)$. Clearly, all fixed bounded subsets of E lie in $\mathcal{D}_\sigma$.

We next prove the existence of the pullback-$\mathcal{D}_\sigma$ absorbing set for $\{U(t, \tau)\}_{t \geqslant \tau}$ in E.

Lemma 3.1. *Let assumption* **(H)** *hold. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *possesses a bounded pullback-*$\mathcal{D}_\sigma$ *absorbing set* $\widehat{\mathcal{B}}_0 = \{\overline{\mathcal{B}}_0(s)|s\in\mathbb{R}\}\subseteq\mathcal{P}(E)$ *in the sense that for any* $t\in\mathbb{R}$ *and any* $\widehat{D}=\{D(s)|\, s\in\mathbb{R}\}\in\mathcal{D}_\sigma$*, there exists a* $\tau_0(t,\widehat{D})\leqslant t$ *such that* $U(t,\tau)D(\tau)\subseteq\overline{\mathcal{B}}_0(t)$ *for all* $\tau\leqslant\tau_0(t,\widehat{D})$*, where* $\overline{\mathcal{B}_0}(s)=\overline{\mathcal{B}}(0,R_\sigma(s))\subset E$ *is the closed ball of radius* $R_\sigma(s)$ *and centered at zero.*

Proof. Set

$$R_\sigma^2(t)\triangleq 1+c_1e^{-\sigma t}\int_{-\infty}^{t}e^{\sigma s}\Big(\|f(s)\|^2+\|g(s)\|^2\Big)\mathrm{d}s,\quad t\in\mathbb{R}. \tag{3.4}$$

Using assumption **(H)** and (2.6), we can establish that for each t

$$\lim_{\tau\to-\infty}e^{-\sigma t}\int_{\tau}^{t}e^{\sigma s}\|z(s)\|^2ds=0.$$

Then from (2.5) and (3.3) we conclude that the family $\widehat{\mathcal{B}}_0=\{\overline{\mathcal{B}}(0,R_\sigma^2(s))\,|s\in\mathbb{R}\}$ is the desired bounded pullback-$\mathcal{D}_\sigma$ absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$ in E. The proof is completed. $\square$

The following lemma reveals the pullback-$\mathcal{D}_\sigma$ asymptotic nullness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in E.

Lemma 3.2. *Let assumption (H) hold. Then for any given* $t\in\mathbb{R},\epsilon>0$ *and* $\hat{D}=\{D(s)|s\in\mathbb{R}\}$*, there exist* $M_*=M_*(t,\epsilon,\hat{D})\in\mathbb{N}$ *and* $\tau_*=\tau_*(t,\epsilon,\hat{D})\leqslant t$*, such that*

$$\sup_{\psi_\tau\in D(\tau)}\sum_{|m|\geqslant M_*}\left|(U(t,\tau)\psi_\tau)_m\right|_E^2\leqslant\epsilon^2,\quad\forall\tau\leqslant\tau_*,$$

where

$$|\psi_m|_E^2\triangleq|(Bu)_m|^2+u_m^2+v_m^2+|z_m|^2\,,\psi_m=(u_m,v_m,z_m)^T.$$

Proof. First, for any $M\in\mathbb{Z}_+$, we set

$$p:=(p_m)_{m\in\mathbb{Z}}\,,\ q:=(q_m)_{m\in\mathbb{Z}}\,,\ w:=(w_m)_{m\in\mathbb{Z}}\,,\ \phi:=(\phi_m)_{m\in\mathbb{Z}}\,,\ p_m:=\chi\left(\tfrac{|m|}{M}\right)u_m$$
$$q_m:=\chi\left(\tfrac{|m|}{M}\right)v_m,\ w_m:=\chi\left(\tfrac{|m|}{M}\right)z_m,\ \phi_m:=(w_m,p_m,q_m)^T.$$

where $\chi(x)\in\mathcal{C}^1(\mathbb{R}_+,[0,1])$ as

$$\chi(x):=\begin{cases}0,\ 0\leqslant x\leqslant 1;\\ 1,\ x\geqslant 2,\end{cases}\ \text{and}\ |\chi'(x)|\leqslant\chi_0,\quad\forall x\in\mathbb{R}_+\,. \tag{3.5}$$

Take inner product with (2.1) by ϕ and take the real part, we have:

$$\mathbf{Re}(\dot{\psi},\phi)_E+\mathbf{Re}(\Theta\psi,\phi)_E=\mathbf{Re}(G(\psi,t),\phi)_E.$$

Direct calculation implies that

$$\begin{aligned}\mathbf{Re}(\dot{\psi},\phi)_E&=(\dot{u},p)_1+(\dot{v},q)+\mathbf{Re}(\dot{z},w)\\&=(B\dot{u},Bp)+(\dot{u},p)+(\dot{v},q)+\mathbf{Re}(\dot{z},w)\\&=\frac{1}{2}\frac{d}{dt}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m|_E^2\\&\quad+\sum_{m\in\mathbb{Z}}\left(\chi\left(\frac{|m+1|}{M}\right)-\chi\left(\frac{|m|}{M}\right)\right)(\dot{u}_{m+1}-\dot{u}_m)\,u_{m+1}.\end{aligned}$$

Then, using the definition of $\chi(x)$ and Lemma 3.1, we obtain that

$$\begin{aligned}
&\mathbf{Re}(\dot{\psi},\phi)_E-\frac{1}{2}\frac{d}{dt}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m|_E^2\\
=&\sum_{m\in\mathbb{Z}}\left(x\left(\frac{|m+1|}{M}\right)-x\left(\frac{|m|}{M}\right)\right)(\dot{u}_{m+1}-\dot{u}_m)\,u_{m+1}\\
=&\sum_{m\in\mathbb{Z}}\chi'\left(\frac{\tilde{m}}{M}\right)\frac{\operatorname{sgn} m}{M}\,(v_{m+1}-\delta u_{m+1}-v_m+\delta u_m)\,u_{m+1}\\
\geqslant&-\frac{2\chi_0R_0^2}{M}(\delta+2),\quad \forall t\geqslant t_0,
\end{aligned}\tag{3.6}$$

where

$$\operatorname{sgn} x:=\left\{\begin{array}{l}1,\ \ x\geqslant 0\\ -1,\ x<0\end{array}\right\}.$$

Next, we estimate $\mathbf{Re}(\Theta\psi,\phi)$. It is easy to see that

$$\begin{aligned}
\mathbf{Re}(\Theta\psi,\phi)_E=&\delta(Bu,Bp)+\delta(u,p)-(Bv,Bp)-(v,p)+(Au,q)+(u,q)\\
&+\delta(\delta-\lambda)(u,q)+(\lambda-\delta)(v,q)-\kappa\operatorname{Im}(Az,w)+\alpha(z,w).
\end{aligned}$$

By elementary computations, we have

$$\begin{aligned}
&(Bu,Bp)\geqslant\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|(Bu)_m|^2-\frac{2\chi_0R_0^2}{M},\quad\forall t\geqslant t_0.\\
&(u,p)=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)u_m^2,\quad (v,q)=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)v_m^2.\\
&(u,q)=(v,p)=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)u_mv_m,\quad (z,w)=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2.\\
&(Au,q)-(Bv,Bp)=\sum_{m\in Z}\left(x\left(\frac{|m+1|}{M}\right)-x\left(\frac{|m|}{M}\right)\right)(u_{m+1}v_m-u_mv_{m+1})\\
&\qquad\qquad\geqslant-\frac{4\chi_0R_0^2}{M},\quad\forall t\geqslant t_0.\\
&-\operatorname{Im}(Az,w)\geqslant-\sum_{m\in\mathcal{Z}}\left|x\left(\frac{|m+1|}{M}\right)-x\left(\frac{|m|}{M}\right)\|z_{m+1}\|\,z_m\right|\geqslant-\frac{\chi_0R_0^2}{M},\ \forall t\geqslant t_0.
\end{aligned}$$

Combining these inequality, when $t>t_0$, we get

$$\begin{aligned}
\mathbf{Re}(\Theta\psi,\phi)_E\geqslant&\delta\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\left(|(Bu)_m|^2+u_m^2\right)+(\lambda-\delta)\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)v_m^2\\
&+\alpha\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2+\delta(\delta-\lambda)\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)u_mv_m\\
&-\frac{\chi_0R_0^2}{M}(2\delta+\kappa+4).
\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{Re}(\Theta\psi,\phi)_E-\textstyle\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\left[\beta\left(|(Bu)_m|^2+u_m^2+v_m^2\right)+\frac{\lambda}{2}v_m^2+\alpha\,|z_m|^2\right]\\ \geqslant&\textstyle\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\left[(\delta-\beta)\left(|(Bu)_m|^2+u_m^2\right)+\left(\frac{\lambda}{2}-\delta-\beta\right)v_m^2\right]\\ &+\delta(\delta-\lambda)\textstyle\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)u_mv_m-\frac{\chi_0R_0^2}{M}(2\delta+4+\kappa),\quad\forall t\geqslant t_0.\end{aligned}\tag{3.7}$$

Since $4(\delta-\beta)(\lambda/2-\delta-\beta)=\delta^2\lambda^2$, for any $m\in\mathbb{Z}$, there holds

$$\begin{aligned}&(\delta-\beta)\left(|(Bu)_m|^2+u_m^2\right)+\left(\tfrac{\lambda}{2}-\delta-\beta\right)v_m^2+\delta(\delta-\lambda)u_mv_m\\ \geqslant&(\delta-\beta)u_m^2+\left(\tfrac{\lambda}{2}-\delta-\beta\right)v_m^2-\delta\lambda\,|u_mv_m|\geqslant0.\end{aligned}$$

Therefore, we obtain

$$\sum_{m\in Z}\chi\left(\frac{|m|}{M}\right)\left[(\delta-\beta)\left(|(Bu)_m|^2+u_m^2\right)+\left(\frac{\lambda}{2}-\delta-\beta\right)v_m^2+\delta(\delta-\lambda)u_mv_m\right]\geqslant0.$$

We derive from (3.7) that

$$\begin{aligned}\mathbf{Re}(\Theta\psi,\phi)_E\geqslant&\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\left[\beta\left(|(Bu)_m|^2+u_m^2+v_m^2\right)+\frac{\lambda}{2}v_m^2+\alpha\,|z_m|^2\right]\\ &-\frac{\chi_0R_0^2}{M}(2\delta+\kappa+4),\quad\forall t\geqslant t_0.\end{aligned}\tag{3.8}$$

Finally, we estimate $\mathbf{Re}(G(\psi,t),\phi)_E$. Direct computations show that

$$\begin{aligned}\mathbf{Re}(G(\psi,t),\phi)_E=&-\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\mathbf{Re}(Bz)_mv_m+\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)g_mv_m\\ &+\mathrm{Im}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)f_m\overline{z}_m\end{aligned}$$

and

$$\begin{aligned}-\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\mathbf{Re}(Bz)_mv_m&\leqslant\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)(|z_{m+1}v_m|+|z_mv_m|)\\ &\leqslant\frac{\lambda}{4}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)v_m^2+\frac{2}{\lambda}\left(I_1+I_2\right),\\ \sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)g_mv_m&\leqslant\frac{\lambda}{4}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)v_m^2+\frac{1}{\lambda}\sum_{|m|\geqslant M}g_m^2,\\ \mathrm{Im}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)f_m\overline{z}_m&\leqslant\frac{\alpha}{2}\sum_{m\in\overline{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2+\frac{1}{2\alpha}\sum_{|m|\geqslant M}|f_m|^2,\end{aligned}$$

where

$$I_1:=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2\quad\text{and}\quad I_2:=\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_{m+1}|^2.$$

Now, we estimate terms I_1 and I_2. Taking the imaginary part of the inner product of equation (2.7) with $w=(w_m)_{m\in\mathbb{Z}}$ and using (3.10), we obtain

$$\frac{d}{dt}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2+\alpha\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m|^2\leqslant\frac{1}{\alpha}\sum_{|m|\geqslant M}|f_m|^2+\frac{2\kappa\chi_0R_0^2}{M}.$$

Then, applying Gronwall inequality ,we have

$$I_1\leqslant\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|z_m(\tau)|^2e^{-\alpha(t-\tau)}+\int_\tau^t\left(\frac{1}{\alpha}\sum_{|m|\geqslant M}|f_m(s)|^2+\frac{2\kappa\chi_0R_0^2(s)}{M}\right)e^{-\alpha(t-s)}ds,$$

and I_2 is similar to I_1.

It follows that

$$\begin{aligned}&\mathbf{Re}(G(\psi,t),\phi(t))_E\\&\leqslant\frac{\lambda}{2}\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)v_m^2(t)+\frac{\alpha}{2}\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)|z_m(t)|^2+\frac{1}{\lambda}\sum_{|m|\geqslant M}g_m^2(t)\\&\quad+\frac{1}{2\alpha}\sum_{|m|\geqslant M}|f_m(t)|^2+\frac{2}{\lambda}\int_\tau^t\left(\sum_{|m|\geqslant M}|f_m(s)|^2+\frac{2\chi_0R_\sigma(s)}{\alpha M}\right)e^{-\alpha(t-s)}ds\\&\quad+\frac{2}{\lambda}e^{-\alpha(t-\tau)}\left(\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)|z_m(\tau)|^2\right),\ \forall\tau\leqslant\tau(t,\widehat{D})<t.\end{aligned}\tag{3.9}$$

Combing (3.6) ,(3.8) and (3.9) to get the following estimate:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m|_E^2+\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)\left[\beta\left(|(Bu)_m|^2+u_m^2+v_m^2\right)+\frac{\alpha}{2}|z_m|^2\right]\\&\leqslant\frac{1}{\lambda}\sum_{|m|\geqslant M}g_m^2+\frac{1}{2\alpha}\sum_{|m|\geqslant M}|f_m|^2+\frac{2}{\lambda}\int_\tau^t\left(\sum_{|m|\geqslant M}|f_m(s)|^2+\frac{2\chi_0R_\sigma(s)}{\alpha M}\right)e^{-\alpha(t-s)}ds\\&\quad+\frac{2}{\lambda}e^{-\alpha(t-\tau)}\left(\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)|z_m(\tau)|^2\right)+\frac{\chi_0R_0^2(t)(4\delta+\kappa+8)}{M}.\end{aligned}\tag{3.10}$$

Then, we can see that for any $\varepsilon>0$, there exist $M_1(\varepsilon,\mathcal{B}_0)\in\mathbb{Z}_+$ such that

$$\frac{\chi_0R_0^2(t)(4\delta+\kappa+8)}{M}\leqslant\frac{\varepsilon^2}{12},\qquad\text{for }M\geqslant M_1(\varepsilon,\mathcal{B}_0).\tag{3.11}$$

Note that

$$\frac{2}{\lambda}\int_\tau^te^{-\alpha(t-s)}\sum_{|m|\geqslant M}|f_m(s)|^2ds=\frac{2}{\lambda}e^{-\alpha t}\int_\tau^te^{\alpha s}\sum_{|m|\geqslant M}|f_m(s)|^2ds.$$

Due to the $\int_{-\infty}^te^{\alpha\theta}\|f(s)\|^2ds<+\infty$ for each $t\in\mathbb{R}$, there exist $M_2(\varepsilon,\mathcal{B}_0)\in\mathbb{Z}_+$ such that

$$\frac{2}{\lambda}\int_\tau^te^{-\alpha(t-s)}\sum_{|m|\geqslant M}|f_m(s)|^2ds\leqslant\epsilon^2/12.\tag{3.12}$$

Note that

$$
\begin{aligned}
&\int_\tau^t \frac{2\chi_0 R_\sigma(s)}{\alpha M} e^{-\alpha(t-s)} ds = \frac{2\chi_0}{\alpha M}\int_\tau^t R_\sigma(s) e^{-\alpha(t-s)} ds\\
\leqslant & \frac{2\chi_0}{\alpha M}\int_\tau^t e^{-\alpha(t-s)} ds + \frac{2c_1\chi_0 e^{-\alpha t}}{\alpha M}\int_\tau^t e^{(\alpha-\sigma)s}\int_{-\infty}^t e^{\sigma\theta}\left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta ds\\
\leqslant & \frac{2\chi_0}{\alpha^2 M} + \frac{2c_1\chi_0 e^{-\alpha t}}{\alpha M}\int_\tau^t e^{(\alpha-\sigma)s} ds \int_{-\infty}^t e^{\sigma\theta}\left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta\\
\leqslant & \frac{2\chi_0}{\alpha^2 M} + \frac{2c_1\chi_0 e^{-\sigma t}}{\alpha(\alpha-\sigma) M}\int_{-\infty}^t e^{\sigma\theta}\left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta.
\end{aligned}
$$

Hence, there exists some $M_3 = M_3(t,\epsilon) \in \mathbb{N}$ such that

$$
\frac{2}{\lambda}\int_\tau^t \frac{2\chi_0 R_\sigma(s)}{\alpha M} e^{-\alpha(t-s)} ds \leqslant \sigma\epsilon^2/12, \quad \forall M > M_3. \tag{3.13}
$$

It is obvious that

$$
\frac{2}{\lambda} e^{-\alpha(t-\tau)}\left(\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)|z_m(\tau)|^2\right) \leqslant \frac{2}{\lambda} e^{-\alpha(t-\tau)}\|z_\tau\|^2.
$$

Hence, we can get from the discussion of theorem 3.1 that there exists some $\tau_1 = \tau_1(t,\epsilon) \in \mathbb{N}$ such that

$$
\frac{2}{\lambda} e^{-\alpha(t-\tau)}\left(\sum_{|m|\geqslant M}\chi\left(\frac{|m|}{M}\right)|z_m(\tau)|^2\right) \leqslant \sigma\epsilon^2/12, \quad \forall \tau \leqslant \tau_1 \leqslant \tau(t,\widehat{D}) < t. \tag{3.14}
$$

Then combing (3.10) to (3.14), for $\forall\tau\leqslant\tau_1\leqslant\tau(t,\widehat{D}) < t$ and $M > \max\{M_1, M_2, M_3\}$, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m|_E^2 + \sigma\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m|_E^2\\
\leqslant & \frac{2}{\lambda}\sum_{|m|\geqslant M} g_m^2(t) + \frac{1}{\alpha}\sum_{|m|\geqslant M}|f_m(t)|^2 + \sigma\epsilon^2/3,
\end{aligned} \tag{3.15}
$$

where $\xi = min\{\beta, \alpha/2\}$. Apply Gronwall inequality to (3.15), we get

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}}\chi\left(\frac{|m|}{M}\right)|\psi_m(t)\|_E^2 \leqslant & \| \psi_\tau|_E^2\, e^{-\sigma(t-\tau)} + \frac{2e^{-\sigma t}}{\lambda}\int_\tau^t e^{\sigma\theta}\sum_{|m|\geqslant M} g_m^2(\theta) d\theta\\
& + \epsilon^2/3 + \frac{e^{-\sigma t}}{\alpha}\int_\tau^t e^{\sigma\theta}\sum_{|m|\geqslant M}|f_m(\theta)|^2 d\theta.
\end{aligned} \tag{3.16}
$$

We derive from (3.1) and (3.2) that there is some $M_4 = M_4(t,\epsilon) \in \mathbb{N}$ such that

$$
\begin{aligned}
&\frac{2e^{-\sigma t}}{\lambda}\int_\tau^t e^{\sigma\theta}\sum_{|m|\geqslant M} g_m^2(\theta) d\theta + \frac{e^{-\sigma t}}{\alpha}\int_\tau^t e^{\sigma\theta}\sum_{|m|\geqslant M}|f_m(\theta)|^2 d\theta\\
\leqslant & \frac{2e^{-\sigma t}}{\lambda}\sum_{|m|\geqslant M}\int_{-\infty}^t e^{\sigma\theta} g_m^2(\theta) d\theta + \frac{e^{-\sigma t}}{\alpha}\sum_{|m|\geqslant M}\int_{-\infty}^t e^{\sigma\theta}|f_m(\theta)|^2 d\theta\\
\leqslant & \epsilon^2/3, \forall M > M_4.
\end{aligned} \tag{3.17}
$$

Now, by (3.3), there exists some $\tau_2 = \tau_2(t, \epsilon, \widehat{D})$ such that

$$e^{-\sigma t} \cdot e^{\sigma\tau} \sup_{\psi(\tau)\in D(\tau)} \|\psi(\tau)\|_E^2 \leqslant \frac{\epsilon^2}{3}, \quad \forall \tau \leqslant \tau_2. \tag{3.18}$$

Choosing

$$M_* = \max\{M_1, M_2, M_3, M_4\}, \quad \tau_* = \min\left\{\tau(t, \widehat{D}), \tau_1, \tau_2\right\},$$

due to (3.11) to (3.18), we have :

$$\sup_{\psi_\tau\in D(\tau)} \sum_{|m|\geqslant 2M_*} |(U(t,\tau)\psi_\tau)_m|_E^2 = \sup_{\psi_\tau\in D(\tau)} \sum_{|m|\geqslant 2M_*} |\psi_m(t)|_E^2 \leqslant \epsilon^2, \forall \tau \leqslant \tau_*.$$

we obtain that the process $\mathcal{U}(t,\tau)_{t>\tau}$ have pullback-$\mathcal{D}$ asymptotical nullness. The proof of Lemma is completed. □

Now, we can state the following existence of pullback attractors of the system (2.1).

Theorem 3.1. *Let assumption (H) hold. Then there exists a pullback-D_σ attractor $\hat{\mathcal{A}}_{\mathcal{D}_\sigma} = \left\{\mathcal{A}_{\mathcal{D}_\sigma(t)} | t \in \mathbb{R}\right\}$ for the process $\mathcal{U}(t,\tau)_{t\geq\tau}$ in E associated to the solution operators of equations* (2.1).

4. Invariant measures and statistical solutions

In this section, we will first construct the invariant measure for $\{U(t,\tau)\}_{t\geqslant\tau}$. Then we prove that this invariant measure is a statistical solution of problem (2.1). In the beginning, we recall the definition of generalized Banach limit.

Definition 4.1 ([5]). A generalized Banach limit is any linear functional, which we denote by $LIM_{T\to\infty}$, defined on the space of all bounded real-valued functions on $[0,\infty)$ that satisfies

(i) $\mathrm{LIM}_{T\to\infty}\phi(T) \geqslant 0$ for nonnegative functions $\phi(\cdot)$,

(ii) $\mathrm{LIM}_{T\to\infty}\phi(T) = \lim_{T\to\infty}\phi(T)$ is the usual limit $\lim_{T\to\infty}\phi(T)$ exists.

Definition 4.2 ([5]). A process $\{\mathcal{U}(t,\tau)\}_{t\geqslant\tau}$ is said to be τ-continuous on a metric space X if for every $x_0 \in X$ and every $t \in \mathbb{R}$, the X-valued function $\tau \to \mathcal{U}(t,\tau)x_0$ is continuous and bounded on $(-\infty, t]$.

Theorem 4.1 ([7]). *Let $\{\mathcal{U}(t,\tau)\}_{t\geqslant\tau}$ be a τ-continuous evolutionary process in a complete metric space X that has a pullback-$\mathcal{D}$ attractor $\mathcal{A}(\cdot)$. Fix a generalized Banach limit $LIM_{T\to\infty}$ and let $\gamma : \mathbb{R} \to X$ be a continuous map such that $\gamma(\cdot) \in \mathcal{D}$. Then there exists a unique family of Borel probability measure $\{\mu_t\}_{t\in\mathbb{R}}$ in X such that the support of the measure μ_t is contained in $\mathcal{A}_t$ and*

$$LIM_{\tau\to-\infty} \frac{1}{t-\tau} \int_\tau^t \phi(\mathcal{U}(t,s)\gamma(s))ds = \int_{\mathcal{A}(t)} \phi(\nu)d\mu_t(\nu)$$

for any real-valued continuous functional ϕ on X. In addition, μ_t is invariant in the sense that

$$\int_{\mathcal{A}(t)} \phi(\nu)d\mu_t(\nu) = \int_{\mathcal{A}(\tau)} \phi(\mathcal{U}(t,\tau)\nu)d\mu_t(\nu), \quad t \geqslant \tau.$$

Lemma 4.1. *Let $\psi^{(1)}(t)$ and $\psi^{(2)}(t)$ be two solutions of problem* (2.1) *corresponding to the initial conditions $\psi_{\tau}^{(1)}$ and $\psi_{2\tau}^{(2)} \in E$, respectively. Then*

$$\left\|\psi^{(1)}(t)-\psi^{(2)}(t)\right\|_{E} \leqslant e^{C(t-\tau)}\left\|\psi^{1}(\tau)-\psi^{2}(\tau)\right\|_{E}. \tag{4.1}$$

Proof. Let $\psi^1(t), \psi^2(t) \in E$ be the solutions of problem (2.1) with the initial condition $\psi^1(\tau), \psi^2(\tau) \in E$. Denote that

$$\begin{cases} \tilde{u}(t)=u^{(1)}(t)-u^{(2)}(t), \\ \tilde{v}(t)=v^{(1)}(t)-v^{(2)}(t), \\ \tilde{z}(t)=z^{(1)}(t)-z^{(2)}(t), \\ \tilde{\psi}(t)=\psi^{(1)}(t)-\psi^{(2)}(t). \end{cases}$$

Then $\tilde{\psi}$ satisfies

$$\begin{cases} \dfrac{d}{dt}\tilde{\psi}(t)+\Theta\tilde{\psi}(t)=G\left(\psi^{(1)}(t),t\right)-G\left(\psi^{(2)}(t),t\right), \quad \forall t>\tau, \\ \tilde{\psi}\Big|_{t=\tau}=\tilde{\psi}(\tau)=\psi_{\tau}^{(1)}-\psi_{2\tau}^{(2)}. \end{cases} \tag{4.2}$$

According to lemma [20, Lemma 3.3], we have

$$\mathbf{Re}(\Theta\tilde{\psi},\tilde{\psi})_E \geqslant \beta\left(\|\tilde{u}\|_1^2+\|\tilde{v}\|^2\right)+\frac{\lambda}{2}\|\tilde{v}\|^2+\alpha\|\tilde{z}\|^2, \quad \forall t \geqslant 0. \tag{4.3}$$

For the right side

$$\begin{aligned} &\left\|G\left(\psi^{(1)},t\right)-G\left(\psi^{(2)},t\right)\right\|_E^2 \\ =&\left\|\left(iz^{(2)}u^{(2)}-iz^{(1)}u^{(1)},0,\mathbf{Re}\left(Bz^{(2)}\right)-\mathbf{Re}\left(Bz^{(1)}\right)\right)^T\right\|_E^2 \\ \leqslant&\left\|Bz^{(2)}-Bz^{(1)}\right\|^2+\left\|z^{(2)}\left(u^{(2)}-u^{(1)}\right)+u^{(1)}\left(z^{(2)}-z^{(1)}\right)\right\|^2 \\ \leqslant&(4+4L(\mathcal{B}))\left\|\psi^{(1)}-\psi^{(2)}\right\|_E^2, \end{aligned} \tag{4.4}$$

where $L(\mathcal{B})$) is a positive constant depending on B. Take the real part of the inner product of (4.2) with $\tilde{\psi}$:

$$\frac{1}{2}\frac{d}{dt}\left\|\tilde{\psi}\right\|_E^2+\mathbf{Re}\left(\Theta\tilde{\psi},\tilde{\psi}\right)_E=\mathbf{Re}\left(G\left(\psi^{(1)},t\right)-G\left(\psi^{(2)},t\right),\tilde{\psi}\right), \quad \forall t \geqslant 0. \tag{4.5}$$

Coombing (4.3) to (4.5), we have

$$\frac{d}{ds}\|\tilde{\psi}(s)\|_E^2+\sigma\|\tilde{\psi}(s)\|_E^2 \leqslant \frac{2}{\sigma}\left\|G\left(\psi^{(1)},s\right)-G\left(\psi^{(2)},s\right)\right\|_E^2 \leqslant \frac{8(1+L(\mathcal{B}))}{\sigma}\|\tilde{\psi}(s)\|_E^2,$$

for all $s>\tau$, where σ is defined as Lemma 2.4. Using the Gronwall inequality, we can get

$$\left\|\psi^{(1)}(t)-\psi^{(2)}(t)\right\|_{E} \leqslant e^{C(t-\tau)}\left\|\psi^{1}(\tau)-\psi^{2}(\tau)\right\|_{E}.$$

Here C is a constant depending on σ and $L(\mathcal{B})$. □

Lemma 4.2. *Let $f(t)$ and $g(t)$ satisfy the conditions of (H). Then for every $\psi_* \in E$ and every $t \in \mathbb{R}$, the $E-$valued function $\tau \to U(t,\tau)\psi_*$ is continuous and bounded on $(-\infty, t]$.*

Proof. Consider any $\psi_* = (u_*, v_*, z_*)^T \in E$ and $t \in \mathbb{R}$. We shall prove that for any $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$, such that if $r < t, s < t$ and $|r - s| < \delta$ then $\|U(t,r)\psi_* - U(t,s)\psi_*\|_E < \epsilon$. We assume $r < s$ without loss of generality. Set

$$\begin{cases} U(\cdot,s)U(s,r)\psi_* = \left(u_*^{(1)}(\cdot), v_*^{(1)}(\cdot), z_*^{(1)}(\cdot)\right)^T, \\ U(\cdot,s)U(r,r)\psi_* = \left(u_*^{(2)}(\cdot), v_*^{(2)}(\cdot), z_*^{(2)}(\cdot)\right)^T. \end{cases}$$

Employing Lemma 4.1 and the continuity property of the process, we have

$$\begin{aligned} &\|U(t,r)\psi_* - U(t,s)\psi_*\|_E^2 \\ =&\|U(t,s)U(s,r)\psi_* - U(t,s)U(r,r)\psi_*\|_E^2 \\ \leqslant&\|U(s,r)\psi_* - U(r,r)\psi_*\|_E^2 e^{C(t-s)}. \end{aligned}$$

Now, we have shown that solutions of problem (4.2) belong to the space $\mathcal{C}([\tau, +\infty), E)$, hence for any $s \in \mathbb{R}$ with $s \leqslant t$. Therefore, the E-valued function $\tau \longmapsto U(t,\tau)\psi_*$ is continuous with respect to $\tau \in (-\infty, t]$ in the space E . Finally, for above $\psi_* \in E$ and $t \in \mathbb{R}$, we see from lemma 2.4 and theorem 3.1 that

$$\begin{aligned} &\lim_{\tau \to -\infty} \|U(t,\tau)\psi_*\|_E^2 \\ \leqslant&\lim_{\tau \to -\infty} \|\psi_\tau\|_E^2 e^{-\sigma(t-\tau)} + \lim_{\tau \to -\infty} c_1 e^{-\sigma t} \int_\tau^t e^{\sigma\theta} \left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta \\ &+ \lim_{\tau \to -\infty} c_1 e^{-\sigma t} \int_\tau^t e^{\sigma\theta} \|z(\theta)\|^2 d\theta \\ =&c_1 e^{-\sigma t} \int_{-\infty}^t e^{\sigma\theta} \left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta < +\infty. \end{aligned}$$

where $c_1 e^{-\sigma t} \int_{-\infty}^t e^{\sigma\theta} \left(\|f(\theta)\|^2 + \|g(\theta)\|^2\right) d\theta$ is independent of τ. Notice that the E -valued function $\tau \longmapsto U(t,\tau)\psi_*$ is continuous with respect to $\tau \in (-\infty, t]$ in the space E. Thus the E -valued function $\tau \longmapsto U(t,\tau)\psi_*$ is bounded on $(-\infty, t]$. The proof is complete. □

Combing theorem 4.1 lemma 4.1 and lemma 4.2, we have the existence of invariant measures:

Theorem 4.2. *Let $f(t)$ and $g(t)$ satisfy assumption (H). Let $\{U(t,\tau)\}_{t\geqslant\tau}$be the process associated to the solution operators of problem* (2.1) *and $\mathcal{A}_{\mathcal{D}_\sigma} = \{\mathcal{A}_{\mathcal{D}_\sigma}(t) | t \in \mathbb{R}\}$be the pullback $D_\sigma-$attractor obtained in Theorem 3.1. Fix a generalized Banach limit $LIM_{T\to\infty}$ and let :$\psi : \mathbb{R} \to E$ be a continuous map such that $\psi \in D_\sigma$. Then there exists a unique family of Borel probability measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ in the space E such that the support of the measure $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ is contained in $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ and*

$$\mathrm{LIM}_{\tau\to-\infty} \frac{1}{t-\tau} \int_\tau^t \lambda(U(t,s)\psi(s))ds = \int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \lambda(\phi) dm_t(\phi) \tag{4.6}$$

for any real-valued continuous functional λ *on* E. *In addition,* $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ *is invariant in the sense that*

$$\int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \lambda(\phi) dm_t(\phi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma(\tau)}} \lambda(U(t,\tau)\phi) d\mathfrak{m}_\tau(\phi), \quad t \geqslant \tau, \tag{4.7}$$

$$\mathrm{LIM}_{\tau\to-\infty} \frac{1}{t-\tau} \int_\tau^t \int_{\ell^2} \Phi(U(t,s)u) d\mathfrak{m}_s(u) ds = \int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)} \Phi(u) d\mathfrak{m}_t(u). \tag{4.8}$$

We next prove that the obtained invariant measures $\{m_t\}_{t\in\mathbb{R}}$ is a statistical solutions for problem (2.1). We introduce some notations.

Definition 4.3 (cf. [5])**.** We define the class $\mathcal{T}_E$ of test functions to be the set of real-valued functionals $\Upsilon = \Upsilon(\phi)$ on E that are bounded on bounded subset of E and satisfy

(a) for any $\phi \in E$, the Fréchet derivative $\Upsilon'(\phi)$ exists: for each $\phi \in E$ there exists an element $\Upsilon'(\phi)$ such that

$$\frac{|\Upsilon(\phi+\varphi) - \Upsilon(\phi) - \langle \Upsilon'(\phi), \varphi\rangle|}{\|\varphi\|_E} \longrightarrow 0 \ \ as \ \ \|\varphi\|_E \to 0, \ \ \varphi \in E,$$

where $\langle\cdot,\cdot\rangle$ is the dual product between E^* and E;

(b) $\Upsilon'(\phi) \in E^*$ for all $\phi \in E$, and the mapping $\phi \longmapsto \Upsilon'(\phi)$ is continuous and bounded as a function from E to E^*.

We write equation (2.1) as

$$\psi_t = F(t,\psi) = -\Theta\psi + G(\psi,t). \tag{4.9}$$

The conditions (a) and (b) in Definition 4.3 are sufficient to ensure that if $\psi(t)$ solves equation (2.1) then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(\psi(t)) = \langle \Upsilon'(\psi(t)), F(\psi(t))\rangle.$$

Definition 4.4. Let $\hat{\mathcal{A}}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}_\sigma}(t)\}_{t\in\mathbb{R}}$ be the pullback attractors obtained in Theorem 4.2. We say a family of Borel probability measures ρ_t is a statistical solutions for (2.1), if the ρ_t satisfies

(i) The function

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \Phi(\psi) d\rho_t(\psi)$$

is continuous on $[\tau, +\infty)$ for every $\Phi \in \mathcal{T}_E$.

(ii) For almost every $t \in [\tau, +\infty)$, and the function $\psi \mapsto \langle F(t,\psi), v\rangle_{E^*,E}$ is ρ_t-integral, for every $v \in E$. Moreover, the map

$$t \mapsto \int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \langle F(t,\psi), v\rangle_{E^*,E} d\rho_t(\psi)$$

belongs to $L^1_{loc}([\tau, +\infty))$, for every $v \in E$,

(iii) For any cylindrical test function Ψ in $\mathcal{T}_E$, it follows that, for all $t \geqslant \tau$

$$\int_{\mathcal{A}_{\mathcal{D}_\lambda(t)}} \Phi(u) d\rho_t(u) - \int_{\mathcal{A}_{\mathcal{D}_\lambda(\tau)}} \Phi(u) d\rho_\tau(u) = \int_\tau^t \int_{\mathcal{A}_{\mathcal{D}_\lambda(\eta)}} (\Phi'(u), F(u,\eta))\, d\rho_\eta(u) d\eta.$$

Theorem 4.3. *The family of invariant measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ obtained in Theorem 4.1 are statistical solutions of equation* (2.1).

Proof. Let $f \in L^2$ and $g \in \ell^2$. There exists a unique local solution $\psi(t) = (z(t), \psi(t), v(t))^T \in E$ of equations (2.1) such that for any initial data $\psi_0 = (z_0, \psi_0, v_0)^T \in E$, $\psi(\cdot) \in \mathcal{C}([0,T), E) \cap \mathcal{C}^1((0,T), E)$ for some $T > 0$. So we have

$$t \to \int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \Phi(\phi) dm_t(\phi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma(\tau)}} \Phi(U(t,\tau)\phi) d\mathfrak{m}_\tau(\phi), \quad t \geqslant \tau.$$

Note that the invariant property of $\mathfrak{m}_t$ and the continuity of $\mathcal{U}(t,\tau)$, the function $t \to \int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \Phi(\phi) dm_t(\phi)$ is continuous. It follows from condition (2) that $F(t,\psi) = \Theta\psi + G(t,\psi)$. Using the property of $\mathfrak{m}_t$ in Theorem 4.2, we get

$$\int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \langle F(t,\phi), v\rangle_{E',E} d\mathfrak{m}_t(\psi) = \int_{\mathcal{A}_{\mathcal{D}_\sigma(\tau)}} \langle F(t, \mathcal{U}(t,\tau)\phi), v\rangle_{E',E} d\mathfrak{m}_\tau(\phi). \tag{4.10}$$

Combining with the estimates in Lemma 2.4, we derive that

$$\int_{\mathcal{A}_{\mathcal{D}_\sigma(t)}} \langle F(t,\phi), v\rangle_{E',E} d\mathfrak{m}_t(\psi) \leq \frac{1}{\alpha}\|f\|^2 + \frac{2}{\lambda}\|g\|^2 + \frac{4}{\lambda}\|z\|^2 \leqslant C(r_\sigma(t), f(t), g(t)) < \infty. \tag{4.11}$$

According to the $\mathfrak{m}_t$ are a unique family of Borel probability measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ in the space E such that the support of the measure $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ is contained in $\mathcal{A}_{\mathcal{D}_\sigma}(t)$, the condition (2) satisfies.

To verify condition (3), note that Φ is Frechét differential, then we have

$$\Phi(\psi(t+\Delta t)) - \Phi(\psi(t)) = \langle \Phi'(\psi(t)), F(\psi(t), t)\rangle + o(\Delta t).$$

Integrating (4.9) over $[\tau, t]$, we have

$$\Phi(\psi(t)) - \Phi(\psi(\tau)) = \int_\tau^t (\Phi'(\psi(\theta)), F(\psi(\theta), \theta)) d\theta.$$

For any $s < \tau$ and $\psi(\eta) = U(\eta, s)\psi_0$, we can obtain

$$\Phi(U(t,s)\psi_0) - \Phi(U(\tau,s)\psi_0) = \int_\tau^t (\Phi(U(\eta,s)\psi_0), F(U(\eta,s)\psi_0, \eta))\, d\eta. \tag{4.12}$$

Using the property of $\mathfrak{m}_t$, we have

$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\lambda(t)}} \Phi(\psi) d\mathfrak{m}_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\lambda}(\tau)} \Phi(\psi) d\mathfrak{m}_\tau(\psi) \\
=& LIM_{M\to-\infty} \frac{1}{t-M} \int_M^t \int_E \Phi(U(t,s)\psi_0) d\mathfrak{m}_s(\psi_0)\, ds
\end{aligned}$$

$$
\begin{aligned}
&- LIM_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E \Phi(U(\tau,s)\psi_0)d\mathfrak{m}_s(\psi_0)\,ds\\
=&LIM_{M\to-\infty}\frac{1}{\tau-(M-t+\tau)}\int_M^\tau\int_E \Phi(U(t,s)\psi_0)d\mathfrak{m}_s(\psi_0)\,ds\\
&+\mathrm{LIM}_{M\to-\infty}\frac{1}{t-M}\int_\tau^t\int_E \Phi(U(t,s)\psi_0)d\mathfrak{m}_s(\psi_0)\,ds\\
&- LIM_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E \Phi(U(\tau,s)\psi_0)d\mathfrak{m}_s(\psi_0)\,ds. \qquad (4.13)
\end{aligned}
$$

It follows from (4.6) and (4.7) that

$$
\int_E \Phi(U(t,s)\psi_0)d\mathfrak{m}_s(\psi_0)=\int_{\mathcal{A}_{\mathcal{D}_\lambda(s)}}\Phi(U(t,s)\psi_0)d\mathfrak{m}_s(\psi_0)=\int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)}\Phi(\psi_0)d\mathfrak{m}_t(\psi_0)
$$

is irrelevant to s, and $\mathcal{A}_{\mathcal{D}_\lambda}(t)$ is compact sets. Hence

$$
\int_E \|U(t,s)\psi_0\|^2\,d\mathfrak{m}_s(\psi_0)=\int_{\mathcal{A}_{\mathcal{D}_\lambda(t)}}\|\psi_0\|^2\,d\mathfrak{m}_t(\psi_0)<+\infty.
$$

Thus, we have

$$
\begin{aligned}
&\mathrm{LIM}_{M\to-\infty}\frac{1}{t-M}\int_\tau^t\int_E \Phi(U(t,s)\psi_0)d\mathfrak{m}_s(\psi_0)\,ds\\
&=\lim_{M\to-\infty}\frac{1}{t-M}\int_\tau^t\int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)}\Phi(\psi_0)d\mathfrak{m}_t(\psi_0)\,ds=0.
\end{aligned}
$$

Meanwhile, we can derive that

$$
\begin{aligned}
&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-(M-t+\tau)}\int_M^\tau\int_\ell \|U(t,s)\psi_0\|^2\,d\mathfrak{m}_s(\psi_0)\,ds\\
&= LIM_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E \|U(t,s)\psi_0\|^2\,d\mathfrak{m}_s(\psi_0)\,ds.
\end{aligned}
$$

Using the Fubini theorem to (4.13) gives

$$
\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\lambda(t)}(t)}\Phi(\psi)d\mathfrak{m}_t(\psi)-\int_{\mathcal{A}_{\mathcal{D}_\lambda}(\tau)}\Phi(\psi)d\mathfrak{m}_\tau(\psi)\\
&=\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E \left(\Phi(U(t,s)\psi_0)-\Phi(U(\tau,s)\psi_0)\right)d\mathfrak{m}_s(\psi_0)\,ds\\
&=\lim_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E \left(\Phi(U(t,s)\psi_0)-\Phi(U(\tau,s)\psi_0)\right)d\mathfrak{m}_s(\psi_0)\,ds\\
&=\lim_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_E\int_\tau^t \left(\Phi'(U(\eta,s)\psi_0),F\left(U(\eta,s)\psi_0,\eta\right)\right)d\eta d\mathfrak{m}_s(\psi_0)\,ds\\
&=\lim_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_\tau^t\int_E \left(\Phi'(U(\eta,s)\psi_0),F\left(U(\eta,s)\psi_0,\eta\right)\right)d\mathfrak{m}_s(\psi_0)\,d\eta ds. \qquad (4.14)
\end{aligned}
$$

It follows that

$$\begin{aligned}
&\int_E \left(\Phi'(U(\eta,s)\psi_0), F\left(U(\eta,s)\psi_0,\eta\right)\right) d\mathfrak{m}_s\left(\psi_0\right)\\
=&\int_E \left(\Phi\left(U(\eta,\theta)U(\theta,s)\psi_0\right), F\left(U(\eta,\theta)U(\theta,s)\psi_0,\eta\right)\right) d\mathfrak{m}_s\left(\psi_0\right)\\
=&\int_E \left(\Phi'(U(\eta,\theta)\psi_0), F\left(U(\eta,\theta)\psi_0,\eta\right)\right) d\mathfrak{m}_\theta\left(\psi_0\right),
\end{aligned}$$

which is independent of s. Note that the support set of $\mathfrak{m}_t$ is contained by $\mathcal{A}_{\mathcal{D}_\lambda}(t)$, then we get

$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{D}_\lambda}(t)} \Phi(\psi) d\mathfrak{m}_t(\psi) - \int_{\mathcal{A}_{\mathcal{D}_\lambda}(\tau)} \Phi(\psi) d\mathfrak{m}_\tau(\psi)\\
=&\int_\tau^t \int_E \left(\Phi'(U(\eta,\theta)\psi_0), F\left(U(\eta,\theta)\psi_0,\eta\right)\right) d\mathfrak{m}_\theta\left(\psi_0\right)\\
=&\int_\tau^t \int_E (\Phi'(\psi), F(\psi,\eta)) d\mathfrak{m}_\eta(\psi) d\eta\\
=&\int_\tau^t \int_{\mathcal{A}_{\mathcal{D}_\lambda}(\eta)} (\Phi'(\psi), F(\psi,\eta)) d\mathfrak{m}_\eta(\psi) d\eta, \forall t \geqslant \tau.
\end{aligned}$$

Therefore, the existence of statistical solutions is proved. □

References

[1] M. Abounouh, O. Goubet and A. Hakim, *Regularity of the attractor for a coupled Klein-Gordon-Schrödinger system*, Differ. Integral Equ., 2003, 16(5), 573–581.

[2] A. Bronzi, C. F. Mondaini and R. Rosa, *Abstract framework for the theory of statistical solutions*, J. Differ. Equ., 2016, 260(12), 8428–8484.

[3] T. Caraballo, G. Łukaszewicz and J. Real, *Invariant measures and statistical solutions of the globally modified Navier Stokes equations*, Discrete Contin. Dyn. Syst. Ser. B, 2008, 10(4), 761–781.

[4] C. Foias and G. Prodi, *Sur les solutions statistiques equations de Navier-Stokes*, Ann. Mat. Pura Appl., 1976, 111(1), 307–330.

[5] C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

[6] B. Guo and Y. Li, *Attractor for dissipative Klein-Gordon-Schroödinger equations in R^3*, J. Differ. Equ., 1997, 136(2), 356–377.

[7] G. Kaszewicz, *Pullback attractors and statistical solutions for 2D Navier Stokes equations*, Discrete Contin. Dyn. Syst. Ser. B , 2008, 9(3/4,May), 643–659.

[8] G. Kaszewicz, J. Real and J. C. Robinson, *Invariant measures for dissipative dynamical systems and generalised Banach limits*, J. Dynam. Differential Equations, 2011, 23(2), 225–250.

[9] G. Kaszewicz and J. C. Robinson, *Invariant measures for non-autonomous dissipative dynamical systems*, Discrete Contin. Dyn. Syst. Ser. A, 2014, 34(10), 211–4222.

[10] C. Li, C. Hsu, J. Lin and C. Zhao, *Global attractors for the discrete Klein-Gordon-Schrödinger type equations*, J. Differ. Equ. Appl., 2014, 20(10), 1404–1426.

[11] X. Li, W. Shen and C. Sun, *Invariant measures for complex-valued dissipative dynamical systems and applications*, Discrete Contin. Dyn. Syst. Ser. B, 2017, 22(6), 2427–2446.

[12] K. Lu and B. Wang, *Global attractors for the Klein-Gordon-Schroödinger equation in unbounded domains* , J. Differ. Equ., 2001, 170(2), 281–316.

[13] K. Lu and B. Wang, *Upper semicontinuity of attractors for Klein-Gordon-Schroödinger equation*, Int. J. Bifur. Chaos., 2005, 15(1), 157–168.

[14] M. Poulou and N. Stavrakakis, *Global attractor for a system of Klein-Gordon-Schrödinger type in all R* , Nonlinear Anal., 2011, 74(7), 2548–2562

[15] M. Vishik and A. Fursikov, *Translationally homogeneousstatistical solutions and individual solutions with infinite energy of a system of Navier-Stokes equations*, Sib. Math. J., 1978, 19(5), 710–729.

[16] C. Wang, G. Xue and C. Zhao, *Invariant Borel probability measures for discrete long-wave-short-wave resonance equations*, Appl. Math. Comput., 2018, 339(1), 853–865.

[17] C. Zhao and S. Zhou, *Upper semicontinuity of attractors for lattice systems under singular perturbations*, Non-Linear Analysis., 2015, 72(5), 2149–2158.

[18] C. Zhao and L Yang, *Pullback attractor and invariant measures for three dimensional globally modified Navier-Stokes equations* , Comm. Math. Sci., 2017(6), 1565–1580.

[19] C. Zhao, Z. Song and T. Caraballo, *Strong trajectory statistical solutions and Liouville type equations for dissipative Euler equations*, Appl. Math. Lett., 2020. DOI: 10.1016/2020/105981.

[20] C. Zhao, G. Xue and G. Łukaszewicz, *Pullback attractors and invariant measures for discrete Klein-Gordon-Schrödinger equations* . Discrete Contin. Dyn. Syst. Ser. B, 2018, 23(9), 4021–4044.

[21] C. Zhao and T. Caraballo. *Asymptotic regularity of trajectory attractor and trajectory statistical solution for the 3D globally modified Navier-Stokes equations* , J. Differ. Equ., 2019, 266(11), 7205–7229.

[22] S. Zhou and W. Shi. *Attractors and dimension of dissipative lattice systems*, J. Differ. Equ., 2006, 24(1), 172–204.