Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 1107–1117
Website:http://jaac-online.com/
DOI:10.11948/20190206

UNITARILY INVARIANT NORM AND Q-NORM ESTIMATIONS FOR THE MOORE–PENROSE INVERSE OF MULTIPLICATIVE PERTURBATIONS OF MATRICES

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Abstract Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by $B = D_1^* A D_2$, where $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$ are both nonsingular. New upper bounds for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$ are derived, where $A^\dagger, B^\dagger$ are the Moore-Penrose inverses of A and B, and $\|\cdot\|_U, \|\cdot\|_Q$ are any unitarily invariant norm and Q-norm, respectively. Numerical examples are provided to illustrate the sharpness of the obtained upper bounds.

Keywords Moore-Penorse inverse, multiplicative perturbation, unitarily invariant norm, Q-norm, norm upper bound.

MSC(2010) 15A09, 15A60.

1. Introduction

Throughout this paper, $\mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}, \mathbb{C}^n$ and $\mathbb{C}^{m\times n}$ are the sets of positive integers, real numbers, nonnegative real numbers, complex numbers, column vector of n-dimensions and $m \times n$ complex matrices, respectively. Let $0_{m\times n}$ be the zero matrix of $\mathbb{C}^{m\times n}$. When $m = n$, let I_m denote the identity matrix of $\mathbb{C}^{m\times m}$. An element $P \in \mathbb{C}^{m\times m}$ is said to be an orthogonal projection if $P^2 = P$ and $P^* = P$.

For any $A \in \mathbb{C}^{m\times n}$, let $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^T, A^*, $\|A\|_F$, $\|A\|_2, \|A\|_U$ and $\|A\|_Q$ denote the range, the null space, the transpose, the conjugate transpose, the Frobenius norm, the 2-norm, any unitarily invariant norm and any Q-norm of A, respectively. The Moore-Penrose inverse of A [6], written $A^\dagger$, is the unique element of $\mathbb{C}^{n\times m}$ which satisfies

$$AA^\dagger A = A,\ A^\dagger A A^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger \text{ and } (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse has various applications. One research field of the Moore-Penrose inverse is its perturbation theory. In this paper, we deal with norm estimations for the Moore-Penrose inverse associated with rank-preserving perturbations of matrices. Let $A \in \mathbb{C}^{m\times n}$ be given and $B \in \mathbb{C}^{m\times n}$ be a perturbation of A. Clearly, $rank(B) = rank(A)$ if and only if there exist $D_1 \in \mathbb{C}^{m\times m}$ and $D_2 \in \mathbb{C}^{n\times n}$ such that

$$B = D_1^* A D_2, \text{ where } D_1 \text{ and } D_2 \text{ are both nonsingular.} \tag{1.1}$$

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The matrix B given by (1.1) is usually called a multiplicative perturbation of A, and norm estimations for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$ are studied in the literatures [2, 5, 8], where $\|\cdot\|_U$ and $\|\cdot\|_Q$ denote any unitarily invariant norm and Q-norm [1], respectively. Upper bounds for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$ are figured out in [2, Theorems 4.1 and 4.2] firstly. The results obtained in [2] are improved in [8, Theorems 3.1, 3.3 and 3.5], which are improved further in [5, Theorems 3.1 and 3.2]. Note that the Frobenius norm and the 2-norm are two special kinds of Q-norms. In a recent paper [4], new upper bounds for $\|B^\dagger - A^\dagger\|_F$ and $\|B^\dagger - A^\dagger\|_2$ are derived without using the Singular Value Decomposition (SVD), which serves however as the main tool in [2, 5, 8]. Based on the new method employed in [4], improvements of [5, Theorems 2.1, 3.1 and 3.2] are made in the special cases of the Frobenius norm and the 2-norm.

The purpose of this paper is to generalize the main results of [4] from the Frobenius norm and the 2-norm to the general unitarily invariant norm and Q-norm. Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). In this paper, we focus on the study of norm estimations for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$, and have managed to derive new upper bounds along the line initiated in [4, Theorems 2.2 and 3.3]. Thus, the main results of [5] are improved in the cases of the unitarily invariant norm and the Q-norm; see the comparison of (3.30) with (3.37), and (4.3) with (4.4).

The rest of this paper is organized as follows. In Section 2, we put forward some basic knowledge about the unitarily invariant norm, especially a norm equality of $P - PQ$ and $Q - QP$ is provided in the case that $\dim \mathcal{R}(P) = \dim \mathcal{R}(Q)$, where P and Q are orthogonal projections acting on the same finite-dimensional Hilbert space. Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1), and $\|\cdot\|_U$ and $\|\cdot\|_Q$ be any unitarily invariant norm and Q-norm. In Sections 3 and 4, we focus on the study of upper bounds for $\|B^\dagger - A^\dagger\|_U$ and $\|B^\dagger - A^\dagger\|_Q$, respectively. Finally in Section 5, we provide three numerical examples to illustrate the sharpness of the upper bounds (3.30) and (4.3).

2. Some properties of the unitarily invariant norm

The term of the unitarily invariant norm can be found in [1, Sec. IV.2], which is originally defined on $\mathbb{C}^{n\times n}$ for some $n \in \mathbb{N}$. We extend such a term in two steps. An extension to $\mathbb{C}^{m\times n}$, called the (m,n)-unitarily invariant norm, is given in Definition 2.1. A further extension to $\bigcup\limits_{m,n=1}^{\infty} \mathbb{C}^{m\times n}$ is given in Definition 2.2, which is the exact meaning of the unitarily invariant norm adopted in this paper. The purpose of this section is to put forward some basic knowledge about this new kind of unitarily invariant norm.

Now, we give the definitions as follows:

Definition 2.1. Let $m, n \in \mathbb{N}$ be given. An (m,n)-unitarily invariant norm is a norm $\|\cdot\|$ defined on the linear space $\mathbb{C}^{m\times n}$ such that $\|A\| = \|UAV\|$, for any $A \in \mathbb{C}^{m\times n}$ and any unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$.

Definition 2.2. A unitarily invariant norm $\|\cdot\|_U$ is a mapping defined on $\bigcup\limits_{m,n=1}^{\infty} \mathbb{C}^{m\times n}$ such that for each $m, n \in \mathbb{N}$, the restriction of $\|\cdot\|_U$ to $\mathbb{C}^{m\times n}$ is an (m,n)-unitarily invariant norm, and for any $A \in \mathbb{C}^{m\times n}$, any $k, l \in \mathbb{N}$, it holds that

$$\|A\|_U = \left\| \begin{pmatrix} A & 0_{m\times k} \\ 0_{l\times n} & 0_{l\times k} \end{pmatrix} \right\|_U .$$

Trivial as it is, Lemma 2.1 below is stated for the sake of completeness.

Lemma 2.1. *Let $\|\cdot\|_U$ be any unitarily invariant norm. Then for any $A \in \mathbb{C}^{m\times n}$, it holds that $\|A\|_U = \|A^*\|_U$.*

Proof. Let $A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*$ be the SVD of A, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices. Then $A^* = V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^*$, so

$$\|A\|_U = \left\| \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \right\|_U = \|\Sigma\|_U = \left\| V \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^* \right\|_U = \|A^*\|_U .$$

□

Recall that any (n, n)-unitarily invariant norm is a symmetric norm, which can be stated as follows:

Lemma 2.2 ([1, Proposition IV.2.4])**.** *Let $\|\cdot\|$ be an (n, n)-unitarily invariant norm. Then for any $A, B, C \in \mathbb{C}^{n\times n}$, it holds that*

$$\|ABC\| \le \|A\|_2 \cdot \|B\| \cdot \|C\|_2 .$$

An extension of the preceding lemma is as follows:

Corollary 2.1. *Let $\|\cdot\|_U$ be any unitarily invariant norm. Then for any $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{k\times l}$, it holds that*

$$\|ABC\|_U \le \|A\|_2 \cdot \|B\|_U \cdot \|C\|_2 . \tag{2.1}$$

Proof. Let $\widetilde{A}$ be the inclusion of A into $\mathbb{C}^{(m+n+k+l)\times(m+n+k+l)}$ defined by

$$\widetilde{A} = \begin{pmatrix} A & 0_{m\times(m+k+l)} \\ 0_{(n+k+l)\times n} & 0_{(n+k+l)\times(m+k+l)} \end{pmatrix} .$$

Similarly, define $\widetilde{B}$ and $\widetilde{C}$. Then

$$\widetilde{A}\widetilde{B}\widetilde{C} = \begin{pmatrix} ABC & 0_{m\times(m+n+k)} \\ 0_{(n+k+l)\times l} & 0_{(n+k+l)\times(m+n+k)} \end{pmatrix} ,$$

hence by Lemma 2.2 we have

$$\|ABC\|_U = \|\widetilde{A}\widetilde{B}\widetilde{C}\|_U \le \|\widetilde{A}\|_2 \cdot \|\widetilde{B}\|_U \cdot \|\widetilde{C}\|_2 = \|A\|_2 \cdot \|B\|_U \cdot \|C\|_2 .$$ □

For any Hilbert spaces H and K, let $\mathbb{B}(H, K)$ be the set of bounded linear operators from H to K. If $H = K$, then $\mathbb{B}(H, H)$ is simplified to be $\mathbb{B}(H)$. Let I_H denote the identity operator on H.

At the end of this section, we state a result of [7] as follows.

Theorem 2.1. [7, Theorem 7.1] *Let H be a finite-dimensional Hilbert space, P, $Q \in \mathbb{B}(H)$ be orthogonal projections such that $dim\mathcal{R}(P)=dim\mathcal{R}(Q)$. Then for any unitarily invariant norm $\|\cdot\|_U$, it holds that $\|Q(I_H - P)\|_U = \|P(I_H - Q)\|_U$.*

3. Unitarily invariant norm estimations for the Moore–Penrose inverse of multiplicative perturbations of matrices

Throughout this section, $\|\cdot\|_U$ is any unitarily invariant norm. Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). In this section, we study estimations for $\|B^\dagger - A^\dagger\|_U$ and get a new upper bound established in Theorem 3.1, which leads to the generalization of the main technique result in [5, Sec. 3]; see Corollary 3.2 below.

Lemma 3.1. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then*

$$\|BB^\dagger(I_m - AA^\dagger)\|_U = \|AA^\dagger(I_m - BB^\dagger)\|_U,$$
$$\|B^\dagger B(I_n - A^\dagger A)\|_U = \|A^\dagger A(I_n - B^\dagger B)\|_U.$$

Proof. Since both D_1 and D_2 are nonsingular, we have

$$rank(AA^\dagger) = rank(A) = rank(B) = rank(BB^\dagger),$$
$$rank(A^\dagger A) = rank(A^*) = rank(B^*) = rank(B^\dagger B).$$

The conclusion follows immediately from Theorem 2.1. □

Theorem 3.1. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $s_1, s_2, s_3, s_4, s_5 \in \mathbb{C}$, we have*

$$\|B^\dagger - A^\dagger\|_U \le \xi_1(s_1) + \xi_2(s_2, s_3) + \xi_3(s_4, s_5),$$

where

$$\Lambda_1 = AA^\dagger(I_m - \bar{s_1}\, D_1), \Lambda_2 = (I_n - s_1\, D_2^{-1})A^\dagger A, \tag{3.1}$$
$$\Lambda_3 = AA^\dagger(I_m - s_2\, D_1), \Lambda_4 = (I_m - s_3\, D_1^{-1})(I_m - AA^\dagger), \tag{3.2}$$
$$\Lambda_5 = A^\dagger A(I_n - s_4\, D_2), \Lambda_6 = (I_n - s_5\, D_2^{-1})(I_n - A^\dagger A), \tag{3.3}$$
$$\xi_1(s_1) = \|B^\dagger\|_2 \cdot \|\Lambda_1\|_U + \|A^\dagger\|_2 \cdot \|\Lambda_2\|_U, \tag{3.4}$$
$$\xi_2(s_2, s_3) = \|B^\dagger\|_2 \cdot \min\{\|\Lambda_3\|_U, \|\Lambda_4\|_U\},$$
$$\xi_3(s_4, s_5) = \|A^\dagger\|_2 \cdot \min\{\|\Lambda_5\|_U, \|\Lambda_6\|_U\}.$$

Proof. Clearly, $B^\dagger - A^\dagger = \Omega_1 + \Omega_2 + \Omega_3$, where

$$\Omega_1 = B^\dagger AA^\dagger - B^\dagger BA^\dagger = B^\dagger B \cdot \Omega_1 \cdot AA^\dagger, \tag{3.5}$$
$$\Omega_2 = B^\dagger(I_m - AA^\dagger) = B^\dagger B \cdot \Omega_2 \cdot (I_m - AA^\dagger), \tag{3.6}$$
$$\Omega_3 = -(I_n - B^\dagger B)A^\dagger = (I_n - B^\dagger B) \cdot \Omega_3. \tag{3.7}$$

Therefore, we have

$$\|B^\dagger - A^\dagger\|_U \le \|\Omega_1\|_U + \|\Omega_2\|_U + \|\Omega_3\|_U. \tag{3.8}$$

First, we derive an upper bound for $\|\Omega_1\|_U$. Since $B = D_1^* A D_2$, we have

$$BD_2^{-1} = D_1^* A \text{ and } (D_1^{-1})^* B = AD_2. \tag{3.9}$$

The first equation above yields

$$B^\dagger B D_2^{-1} A^\dagger = B^\dagger D_1^* A A^\dagger. \tag{3.10}$$

It follows from (3.5) and (3.10) that for any $s_1 \in \mathbb{C}$,

$$\Omega_1 = X - Y, \text{ hence } \|\Omega_1\|_U \le \|X\|_U + \|Y\|_U, \tag{3.11}$$

where

$$X = B^\dagger (I_m - s_1 D_1^*) A A^\dagger \text{ and } Y = B^\dagger B (I_n - s_1 D_2^{-1}) A^\dagger.$$

The equations above, together with Corollary 2.1 and Lemma 2.1 yield

$$\begin{aligned}
\|X\|_U &= \|B^\dagger (I_m - s_1 D_1^*) A A^\dagger\|_U \le \|B^\dagger\|_2 \cdot \|(I_m - s_1 D_1^*) A A^\dagger\|_U \\
&= \|B^\dagger\|_2 \cdot \|A A^\dagger (I_m - \bar{s_1} D_1)\|_U, &(3.12)\\
\|Y\|_U &= \|B^\dagger B (I_n - s_1 D_2^{-1}) A^\dagger A A^\dagger\|_U \\
&\le \|B^\dagger B\|_2 \cdot \|(I_n - s_1 D_2^{-1}) A^\dagger A\|_U \cdot \|A^\dagger\|_2 \\
&\le \|A^\dagger\|_2 \cdot \|(I_n - s_1 D_2^{-1}) A^\dagger A\|_U. &(3.13)
\end{aligned}$$

Next, we derive upper bounds for $\|\Omega_2\|_U$ and $\|\Omega_3\|_U$. By (3.6)–(3.7), Corollary 2.1 and Lemmas 2.1 and 3.1, we have

$$\begin{aligned}
\|\Omega_2\|_U &= \|B^\dagger (I_m - A A^\dagger)\|_U = \|B^\dagger B B^\dagger (I_m - A A^\dagger)\|_U \\
&\le \|B^\dagger\|_2 \cdot \|B B^\dagger (I_m - A A^\dagger)\|_U &(3.14)\\
&= \|B^\dagger\|_2 \cdot \|A A^\dagger (I_m - B B^\dagger)\|_U, &(3.15)\\
\|\Omega_3\|_U &= \|(I_n - B^\dagger B) A^\dagger\|_U = \|(I_n - B^\dagger B) A^\dagger A A^\dagger\|_U \\
&\le \|A^\dagger\|_2 \cdot \|(I_n - B^\dagger B) A^\dagger A\|_U &(3.16)\\
&= \|A^\dagger\|_2 \cdot \|A^\dagger A (I_n - B^\dagger B)\|_U = \|A^\dagger\|_2 \cdot \|B^\dagger B (I_n - A^\dagger A)\|_U \\
&= \|A^\dagger\|_2 \cdot \|(I_n - A^\dagger A) B^\dagger B\|_U. &(3.17)
\end{aligned}$$

It follows from (3.9) that for any $s_2, s_3, s_4, s_5 \in \mathbb{C}$,

$$\begin{aligned}
&(I_m - B B^\dagger) A A^\dagger = (I_m - B B^\dagger)(I_m - s_2 D_1)^* A A^\dagger, \\
&(I_m - A A^\dagger) B B^\dagger = (I_m - A A^\dagger)(I_m - s_3 D_1^{-1})^* B B^\dagger, \\
&A^\dagger A (I_n - B^\dagger B) = A^\dagger A (I_n - s_4 D_2)(I_n - B^\dagger B), \\
&B^\dagger B (I_n - A^\dagger A) = B^\dagger B (I_n - s_5 D_2^{-1})(I_n - A^\dagger A).
\end{aligned}$$

Taking $*$-operation we get

$$\begin{aligned}
&A A^\dagger (I_m - B B^\dagger) = A A^\dagger (I_m - s_2 D_1)(I_m - B B^\dagger), &(3.18)\\
&B B^\dagger (I_m - A A^\dagger) = B B^\dagger (I_m - s_3 D_1^{-1})(I_m - A A^\dagger), &(3.19)\\
&(I_n - B^\dagger B) A^\dagger A = (I_n - B^\dagger B)(I_n - s_4 D_2)^* A^\dagger A, &(3.20)\\
&(I_n - A^\dagger A) B^\dagger B = (I_n - A^\dagger A)(I_n - s_5 D_2^{-1})^* B^\dagger B. &(3.21)
\end{aligned}$$

It follows from (3.15) and (3.18) that

$$\begin{aligned}\|\Omega_2\|_U &\le \|B^\dagger\|_2 \cdot \|AA^\dagger(I_m - s_2D_1)(I_m - BB^\dagger)\|_U \\ &\le \|B^\dagger\|_2 \cdot \|AA^\dagger(I_m - s_2D_1)\|_U. \end{aligned} \tag{3.22}$$

Similarly, by (3.14) and (3.19) we can get

$$\|\Omega_2\|_U \le \|B^\dagger\|_2 \cdot \|(I_m - s_3D_1^{-1})(I_m - AA^\dagger)\|_U. \tag{3.23}$$

We may combine (3.16), (3.17), (3.20) with (3.21) to get

$$\|\Omega_3\|_U \le \|A^\dagger\|_2 \cdot \|A^\dagger A(I_n - s_4D_2)\|_U, \tag{3.24}$$
$$\|\Omega_3\|_U \le \|A^\dagger\|_2 \cdot \|(I_n - s_5D_2^{-1})(I_n - A^\dagger A)\|_U. \tag{3.25}$$

The conclusion then follows from (3.8), (3.11)–(3.13) and (3.22)–(3.25). □

Remark 3.1. Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then $A = \left(D_1^{-1}\right)^* BD_2^{-1}$, which means that A is also a multiplicative perturbation of B. In view of such an observation, by the preceding theorem we get the following corollary:

Corollary 3.1. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $u_1, u_2, u_3, u_4, u_5 \in \mathbb{C}$, we have*

$$\|B^\dagger - A^\dagger\|_U \le \rho_1(u_1) + \rho_2(u_2, u_3) + \rho_3(u_4, u_5),$$

where

$$\Theta_1 = BB^\dagger(I_m - \bar{u_1}D_1^{-1}), \Theta_2 = (I_n - u_1D_2)B^\dagger B, \tag{3.26}$$
$$\Theta_3 = BB^\dagger(I_m - u_2D_1^{-1}), \Theta_4 = (I_m - u_3D_1)(I_m - BB^\dagger), \tag{3.27}$$
$$\Theta_5 = B^\dagger B(I_n - u_4D_2^{-1}), \Theta_6 = (I_n - u_5D_2)(I_n - B^\dagger B), \tag{3.28}$$
$$\rho_1(u_1) = \|A^\dagger\|_2 \cdot \|\Theta_1\|_U + \|B^\dagger\|_2 \cdot \|\Theta_2\|_U, \tag{3.29}$$
$$\rho_2(u_2, u_3) = \|A^\dagger\|_2 \cdot \min\left\{\|\Theta_3\|_U, \|\Theta_4\|_U\right\},$$
$$\rho_3(u_4, u_5) = \|B^\dagger\|_2 \cdot \min\left\{\|\Theta_5\|_U, \|\Theta_6\|_U\right\}.$$

Corollary 3.2. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $s_i, u_i \in \mathbb{C}\,(1 \le i \le 5)$, we have*

$$\|B^\dagger - A^\dagger\|_U \le \min\{L_U(s_1, s_2, s_3, s_4, s_5), R_U(u_1, u_2, u_3, u_4, u_5)\}, \tag{3.30}$$

where $\xi_1(s_1), \xi_2(s_2, s_3), \xi_3(s_4, s_5), \rho_1(u_1), \rho_2(u_2, u_3)$ and $\rho_3(u_4, u_5)$ are given by Theorem 3.1 and Corollary 3.1 respectively, and

$$L_U(s_1, s_2, s_3, s_4, s_5) = \xi_1(s_1) + \xi_2(s_2, s_3) + \xi_3(s_4, s_5),$$
$$R_U(u_1, u_2, u_3, u_4, u_5) = \rho_1(u_1) + \rho_2(u_2, u_3) + \rho_3(u_4, u_5).$$

Remark 3.2. With the notations above, we have

$$\xi_1(1) \le \|B^\dagger\|_2 \cdot \|I_m - D_1\|_U + \|A^\dagger\|_2 \cdot \|I_n - D_2^{-1}\|_U \overset{def}{=} \lambda_1(U), \tag{3.31}$$
$$\xi_2(1,1) \le \|B^\dagger\|_2 \cdot \min\left\{\|I_m - D_1\|_U, \|I_m - D_1^{-1}\|_U\right\} \overset{def}{=} \lambda_2(U), \tag{3.32}$$
$$\xi_3(1,1) \le \|A^\dagger\|_2 \cdot \min\left\{\|I_n - D_2\|_U, \|I_n - D_2^{-1}\|_U\right\} \overset{def}{=} \lambda_3(U), \tag{3.33}$$
$$\rho_1(1) \le \|A^\dagger\|_2 \cdot \|I_m - D_1^{-1}\|_U + \|B^\dagger\|_2 \cdot \|I_n - D_2\|_U \overset{def}{=} \mu_1(U), \tag{3.34}$$
$$\rho_2(1,1) \le \|A^\dagger\|_2 \cdot \min\left\{\|I_m - D_1^{-1}\|_U, \|I_m - D_1\|_U\right\} \overset{def}{=} \mu_2(U), \tag{3.35}$$
$$\rho_3(1,1) \le \|B^\dagger\|_2 \cdot \min\left\{\|I_n - D_2^{-1}\|_U, \|I_n - D_2\|_U\right\} \overset{def}{=} \mu_3(U). \tag{3.36}$$

Thus, we can apply Corollary 3.2 to get the technique result of [5, Sec. 3] as follows:

Corollary 3.3 ([5, Theorem 3.1])**.** *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then*

$$\|B^\dagger - A^\dagger\|_U \le \min\{\Phi_1(D_1, D_2), \Phi_2(D_1, D_2)\}, \tag{3.37}$$

where $\lambda_i(U), \mu_i(U)$ are defined by (3.31)–(3.36) for $i = 1, 2, 3$, and

$$\Phi_1(D_1, D_2) = \sum_{i=1}^{3} \lambda_i(U),\ \ \Phi_2(D_1, D_2) = \sum_{i=1}^{3} \mu_i(U).$$

4. Q-norm estimations for the Moore–Penrose inverse of multiplicative perturbations of matrices

Recall that a unitarily invariant norm is called a Q-norm, written $\|\cdot\|_Q$, if there exists another unitarily invariant norm $\|\cdot\|_U$ such that

$$\|A\|_Q^2 = \|A^*A\|_U, \ \text{for any } m, n \in \mathbb{N} \text{ and } A \in \mathbb{C}^{m\times n}. \tag{4.1}$$

Throughout the rest of this section, $\|\cdot\|_Q$ and $\|\cdot\|_U$ are Q-norm and unitarily invariant norm respectively, both of them are defined on $\bigcup\limits_{m,n=1}^{\infty} \mathbb{C}^{m\times n}$ such that (4.1) is satisfied.

Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). In this section, we study the Q-norm estimations for $B^\dagger - A^\dagger$, and get a new upper bound established in Theorem 4.1, which leads to the generalization of [5, Theorems 3.2]; see Corollary 4.2 for the details.

Theorem 4.1. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $s_1, s_2, s_3, s_4, s_5 \in \mathbb{C}$, we have*

$$\|B^\dagger - A^\dagger\|_Q \le \sqrt{\Upsilon_1(s_1)^2 + \Upsilon_2(s_2, s_3)^2 + \Upsilon_3(s_4, s_5)^2},$$

where Λ_1–Λ_6 are defined by (3.1)–(3.3), and

$$\Upsilon_1(s_1) = \|B^\dagger\|_2 \cdot \|\Lambda_1\|_Q + \|A^\dagger\|_2 \cdot \|\Lambda_2\|_Q,$$
$$\Upsilon_2(s_2, s_3) = \|B^\dagger\|_2 \cdot \min\{\|\Lambda_3\|_Q, \|\Lambda_4\|_Q\},$$
$$\Upsilon_3(s_4, s_5) = \|A^\dagger\|_2 \cdot \min\{\|\Lambda_5\|_Q, \|\Lambda_6\|_Q\}.$$

Proof. By the proof of Theorem 3.1 we know that $B^\dagger - A^\dagger = \Omega_1 + \Omega_2 + \Omega_3$, where Ω_1, Ω_2 and Ω_3 are given by (3.5)–(3.7) such that

$$(\Omega_1 + \Omega_2)^*\Omega_3 = 0 \text{ and } \Omega_1\Omega_2^* = 0,$$

which means that

$$\begin{aligned}
\|B^\dagger - A^\dagger\|_Q^2 &= \|(\Omega_1 + \Omega_2 + \Omega_3)^*(\Omega_1 + \Omega_2 + \Omega_3)\|_U \\
&= \|(\Omega_1 + \Omega_2)^*(\Omega_1 + \Omega_2) + \Omega_3^*\Omega_3\|_U \\
&\le \|(\Omega_1 + \Omega_2)^*(\Omega_1 + \Omega_2)\|_U + \|\Omega_3^*\Omega_3\|_U = \|\Omega_1 + \Omega_2\|_Q^2 + \|\Omega_3\|_Q^2 \\
&= \|(\Omega_1 + \Omega_2)^*\|_Q^2 + \|\Omega_3\|_Q^2 = \|(\Omega_1 + \Omega_2)(\Omega_1 + \Omega_2)^*\|_U + \|\Omega_3\|_Q^2 \\
&= \|\Omega_1\Omega_1^*\|_U + \|\Omega_2\Omega_2^*\|_U + \|\Omega_3\|_Q^2 = \|\Omega_1^*\|_Q^2 + \|\Omega_2^*\|_Q^2 + \|\Omega_3\|_Q^2 \\
&= \|\Omega_1\|_Q^2 + \|\Omega_2\|_Q^2 + \|\Omega_3\|_Q^2.
\end{aligned}$$

Therefore,

$$\|B^\dagger - A^\dagger\|_Q \le \sqrt{\|\Omega_1\|_Q^2 + \|\Omega_2\|_Q^2 + \|\Omega_3\|_Q^2}. \tag{4.2}$$

The conclusion then follows from (4.2), and from (3.11)–(3.13) and (3.22)–(3.25) by replacing $\|\cdot\|_U$ with $\|\cdot\|_Q$ therein. □

In view of Remark 3.1, a corollary can be induced as follows:

Corollary 4.1. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $u_1, u_2, u_3, u_4, u_5 \in \mathbb{C}$, we have*

$$\|B^\dagger - A^\dagger\|_Q \le \sqrt{\sigma_1(u_1)^2 + \sigma_2(u_2, u_3)^2 + \sigma_3(u_4, u_5)^2},$$

where Θ_1–Θ_6 are defined by (3.26)–(3.28), and

$$\begin{aligned}
\sigma_1(u_1) &= \|A^\dagger\|_2 \cdot \|\Theta_1\|_Q + \|B^\dagger\|_2 \cdot \|\Theta_2\|_Q, \\
\sigma_2(u_2, u_3) &= \|A^\dagger\|_2 \cdot \min\{\|\Theta_3\|_Q, \|\Theta_4\|_Q\}, \\
\sigma_3(u_4, u_5) &= \|B^\dagger\|_2 \cdot \min\{\|\Theta_5\|_Q, \|\Theta_6\|_Q\}.
\end{aligned}$$

Corollary 4.2. *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then for any $s_i, u_i \in \mathbb{C}\,(1 \le i \le 5)$, we have*

$$\|B^\dagger - A^\dagger\|_Q \le \min\{L_Q(s_1, s_2, s_3, s_4, s_5), R_Q(u_1, u_2, u_3, u_4, u_5)\}, \tag{4.3}$$

where $\Upsilon_1(s_1), \Upsilon_2(s_2, s_3), \Upsilon_3(s_4, s_5), \sigma_1(u_1), \sigma_2(u_2, u_3)$ and $\sigma_3(u_4, u_5)$ are given by Theorem 4.1 and Corollary 4.1 respectively, and

$$\begin{aligned}
L_Q(s_1, s_2, s_3, s_4, s_5) &= \sqrt{\Upsilon_1(s_1)^2 + \Upsilon_2(s_2, s_3)^2 + \Upsilon_3(s_4, s_5)^2}, \\
R_Q(u_1, u_2, u_3, u_4, u_5) &= \sqrt{\sigma_1(u_1)^2 + \sigma_2(u_2, u_3)^2 + \sigma_3(u_4, u_5)^2}.
\end{aligned}$$

As illustrated by Remark 3.2, a result of [5, Sec. 3] turns out to be the special case of Corollary 4.2, which can be stated as follows:

Corollary 4.3 ([5, Theorem 3.2])**.** *Let B be a multiplicative perturbation of $A \in \mathbb{C}^{m\times n}$ given by (1.1). Then*

$$\|B^\dagger - A^\dagger\|_Q \le \min\{\Psi_1(D_1, D_2), \Psi_2(D_1, D_2)\}, \tag{4.4}$$

where

$$\begin{aligned}
\lambda_1(Q) &= \|B^\dagger\|_2 \cdot \|I_m - D_1\|_Q + \|A^\dagger\|_2 \cdot \|(I_n - D_2^{-1}\|_Q,\\
\lambda_2(Q) &= \|B^\dagger\|_2 \cdot \min\left\{\|I_m - D_1\|_Q, \|I_m - D_1^{-1}\|_Q\right\},\\
\lambda_3(Q) &= \|A^\dagger\|_2 \cdot \min\left\{\|I_n - D_2\|_Q, \|I_n - D_2^{-1}\|_Q\right\},\\
\mu_1(Q) &= \|A^\dagger\|_2 \cdot \|I_m - D_1^{-1}\|_Q + \|B^\dagger\|_2 \cdot \|I_n - D_2\|_Q,\\
\mu_2(Q) &= \|A^\dagger\|_2 \cdot \min\left\{\|I_m - D_1^{-1}\|_Q, \|I_m - D_1\|_Q\right\},\\
\mu_3(Q) &= \|B^\dagger\|_2 \cdot \min\left\{\|I_n - D_2^{-1}\|_Q, \|I_n - D_2\|_Q\right\},\\
\Psi_1(D_1, D_2) &= \left(\sum_{i=1}^{3} \lambda_i(Q)^2\right)^{\frac{1}{2}}, \ \Psi_2(D_1, D_2) = \left(\sum_{i=1}^{3} \mu_i(Q)^2\right)^{\frac{1}{2}}.
\end{aligned}$$

5. Numerical examples

In this section, we provide three numerical examples to illustrate the sharpness of upper bounds (3.30) and (4.3). For any $A \in \mathbb{C}^{m\times n}$, let $s_j(A)$ denote the j-th element in the sequence of singular values of A which are sorted from large to small. The Schatten p-norm of A [1, p92] is defined by

$$\|A\|_p = \left(\sum_{j=1}^{n} \big(s_j(A)\big)^p\right)^{\frac{1}{p}}, \ \text{for } 1 \le p < +\infty.$$

Note that each $\|\cdot\|_p$ is a unitarily invariant norm, which is furthermore a Q-norm if $p \ge 2$ [1, p95].

Example 5.1. We consider the optimality of upper bounds (3.30) and (4.3). Let $a, b \in \mathbb{R}$ be given such that $0 < b < a$. Let B be a multiplicative perturbation of $A \in \mathbb{C}^{3\times 2}$ given by (1.1), where $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and

$$A = \begin{pmatrix} \sin(t) & -\cos(t) \\ \cos(t) & \sin(t) \\ 0 & 0 \end{pmatrix}, D_1 = I_3 \text{ and } D_2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Then $A^\dagger = A^T$, and

$$B = \begin{pmatrix} a\sin(t) & -b\cos(t) \\ a\cos(t) & b\sin(t) \\ 0 & 0 \end{pmatrix}, B^\dagger = \begin{pmatrix} \frac{1}{a}\sin(t) & \frac{1}{a}\cos(t) & 0 \\ -\frac{1}{b}\cos(t) & \frac{1}{b}\sin(t) & 0 \end{pmatrix},$$

so $A^\dagger A = B^\dagger B = I_2, AA^\dagger = BB^\dagger = \operatorname{diag}(I_2, 0), \|A^\dagger\|_2 = 1,\ \|B^\dagger\|_2 = \frac{1}{b}$, and

$$\|B^\dagger - A^\dagger\|_p = \left[\left|1 - \frac{1}{a}\right|^p + \left|1 - \frac{1}{b}\right|^p\right]^{\frac{1}{p}}.$$

For any $s_1, u_1 \in \mathbb{C}$, let $\xi_1(s_1)$ and $\rho_1(u_1)$ be defined by (3.4) and (3.29), respectively. Then

$$\xi_1(s_1) = \frac{1}{b} \cdot 2^{\frac{1}{p}} \cdot |1 - \bar{s_1}| + \left[\left|1 - \frac{s_1}{a}\right|^p + \left|1 - \frac{s_1}{b}\right|^p\right]^{\frac{1}{p}},$$
$$\rho_1(u_1) = 2^{\frac{1}{p}} \cdot |1 - \bar{u_1}| + \frac{1}{b} \cdot \Big[\,|1 - au_1|^p + |1 - bu_1|^p\,\Big]^{\frac{1}{p}}.$$

In particular,

$$\xi_1(1) = \left[\left|1 - \frac{1}{a}\right|^p + \left|1 - \frac{1}{b}\right|^p\right]^{\frac{1}{p}} = \|B^\dagger - A^\dagger\|_p.$$

Now we let $s_1 = 1$, $u_1 \in \mathbb{C}$ be arbitrary, and $s_i = u_i = 1$ for $2 \le i \le 5$. Then for any $1 \le p < +\infty$, by (3.30) we have

$$\|B^\dagger - A^\dagger\|_p \le \min\{\xi_1(1), \rho_1(u_1)\} \le \xi_1(1) = \|B^\dagger - A^\dagger\|_p.$$

Therefore, with the choice of s_i and u_i as above, upper bound (3.30) is optimal for any Schatten p-norm $\|\cdot\|_p$ in the case that $1 \le p < \infty$. The same is true for upper bound (4.3) with respect to any Schatten p-norm $\|\cdot\|_p$ with $2 \le p < \infty$.

It is remarkable that upper bounds (3.37) and (4.4) may both fail to be optimal. More precisely, let $\theta = \Big[\,|1 - a|^p + |1 - b|^p\,\Big]^{\frac{1}{p}}$. Then for $1 \le p < +\infty$, upper bound (3.37) turns out to be

$$\|B^\dagger - A^\dagger\|_p \le \min\left\{\xi_1(1) + \min\{\theta, \xi_1(1)\}, \frac{1}{b}\Big[\theta + \min\{\theta, \xi_1(1)\}\Big]\right\}. \tag{5.1}$$

If we choose $a, b \in \mathbb{R}$ with $0 < b \le 1 < a$ such that $\xi_1(1) \le \theta$ (for instance, $a > 1$ and $b = \frac{1}{a}$ or $b = 1$), then $\|B^\dagger - A^\dagger\|_p = \xi_1(1) \le \theta$, hence the right side of (5.1) is $2\,\|B^\dagger - A^\dagger\|_p$. Furthermore, with the choice of such a and b, upper bound (4.4) emerges as

$$\|B^\dagger - A^\dagger\|_p \le \min\{L, R\} = L = \sqrt{2}\,\|B^\dagger - A^\dagger\|_p,$$

where $2 \le p < +\infty$, and

$$L = \sqrt{\xi_1(1)^2 + \Big(\min\{\theta, \xi_1(1)\}\Big)^2} \text{ and } R = \frac{1}{b}\sqrt{\theta^2 + \Big(\min\{\theta, \xi_1(1)\}\Big)^2}.$$

Example 5.2. Let B be a multiplicative perturbation of A given by (1.1), where A and D_2 are of the same forms as in Example 5.1, whereas D_1 is given by $D_1 = \mathrm{diag}(1, c, 1)$ for some $c > 0$. Taking Schatten 1.5-norm as an example of the unitarily invariant norm, we make a comparison of upper bounds (3.30) and (3.37) by choosing the parameters as follows:

$$t = 0.366527946360803, a = 1.03010885403880, b = 1.03119860898309,$$
$$c = 0.970535221105394, s_1 = 1.0305 - 0.0010i, s_2 = 1.6849 + 1.4915i,$$
$$s_3 = 1.0000, s_4 = 1.2144 - 0.0023i, s_5 = 1.1963 + 0.0035i,$$
$$u_1 = 0.9708 - 0.0002i, u_2 = 1.3014 + 1.2306i, u_3 = 1.0000,$$
$$u_4 = 1.1039 + 0.0006i, u_5 = 1.0905 + 0.0002i.$$

The details are listed in Table 1, where ε_1 and ε_2 are the relative errors of upper bound (3.30) and upper bound (3.37), respectively.

Table 1. Numerical values of Schatten 1.5-norm associated to Example 5.2.

$\|B^\dagger - A^\dagger\|_{1.5}$	upper bound (3.30), ε_1	[5, Theorem 3.1], ε_2
0.03013147685055	0.03039184726925 0.8641%	0.15336733380427 408.9937%

Table 2. Numerical values of Schatten 2.5-norm associated to Example 5.3.

$\|B^\dagger - A^\dagger\|_{2.5}$	upper bound (4.3), Δ_1	[5, Theorem 3.2], Δ_2
0.06565210054381	0.06736956871418 2.6160%	0.13894465418377 111.6378%

Example 5.3. Let B be a multiplicative perturbation of A given by (1.1), where A, D_1 and D_2 are of the same forms as in Example 5.2. Taking Schatten 2.5-norm as an example of the Q-norm, we make a comparison of upper bounds (4.3) with (4.4) by choosing the parameters as follows:

$$
\begin{aligned}
&t = 1.349998626675060, a = 0.940000808375520, b = 0.980000334983814,\\
&c = 1.050000338313970, s_1 = 0.9667, s_2 = 1.0000 + 1.0000i,\\
&s_3 = 1.0000, s_4 = 1.0000 + 1.0000i, s_5 = 1.0000 + 1.0000i,\\
&u_1 = 1.0331, u_2 = 1.0240, u_3 = 1.1192 + 1.0342i,\\
&u_4 = 0.9609 + 1.3259i, u_5 = 1.3285 - 0.0005i.
\end{aligned}
$$

The details are listed in Table 2, where Δ_1 and Δ_2 are the relative errors of upper bound (4.3) and upper bound (4.4), respectively.

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