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ROUGH CONVERGENCE OF DOUBLE SEQUENCES OF FUZZY NUMBERS

Funda Babaarslan [1,†] and A. Nihal Tuncer[2]

Abstract In this paper, we define the concepts of rough convergence and rough Cauchy sequence of double sequences of fuzzy numbers. Then, we investigate some relations between rough limit set and extreme limit points of such sequences.

Keywords Fuzzy numbers, rough convergence, double sequence.

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1. Introduction

The convergence of double sequences was introduced by Pringsheim [10] as follows: A double sequence $x = (x_{nm})$ is said to be convergent in the Pringsheim's sense if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_{nm} - L| < \epsilon$ whenever $n, m \geq N$. In here, L is called the Pringsheim limit of x. Also, a double sequence $x = (x_{nm})$ is said to be Cauchy sequence if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_{kl} - x_{nm}| < \epsilon$ whenever $k \geq n \geq N, l \geq m \geq N$.

Phu [9] introduced the concept of rough convergence in normed linear space as follows: Let $x = (x_n)$ be a sequence in some normed space $(X, \|.\|)$ and r be a non-negative real number. Then, $x = (x_n)$ is said to be rough convergent to $x_* \in X$, if for every $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon$ provided that $\|x_n - x_*\| < r + \epsilon$. In here, $r \geq 0$ is called roughness degree of x. Also, Phu [9] defined $r - limit\ set$ as $LIM^r x := \{L \in \mathbb{R} : x_n \xrightarrow{r} L\}$. If $LIM^r x \neq \emptyset$, then $x = (x_n)$ is said to be r-convergent.

The concepts of $r-$limit inferior, $r-$limit superior and the rough core of a real sequence were studied by Aytar [2]. Then, Aytar [3] introduced rough statistical convergence by using the natural density. Also, he defined the set of rough statistical limit points of a sequence and he showed that this set is closed and convex.

Since double sequences have more application areas in summability theory, Dündar and Çakan [6] extended the convergence in Pringsheim's sense to rough convergence. The concepts of rough statistical convergence and rough statistical Cauchy of a real double sequence were given by Aytar [4]. More recent developments on rough convergence and its statistical analogues can be found in [5, 7, 8, 11].

Moreover, Akçay and Aytar [1] studied the notion of rough convergence in the metric space $(L(R), \bar{d})$, where $L(R)$ denotes the set of all fuzzy numbers and $\bar{d}$ denotes the supremum metric on $L(R)$. This work motivated us to study rough

[†]the corresponding author. Email address:funda.kocabiyik@bozok.edu.tr (F. Babaarslan)
[1]Department of Mathematics, Yozgat Bozok University, Yozgat, 66100, Turkey
[2]Department of Mathematics, Erciyes University, Kayseri, 38039, Turkey

convergence of double sequences of fuzzy numbers. We define rough convergence in Pringsheim's sense, $r - limit\ set$ and rough Cauchy of a double sequence of fuzzy numbers. Also, we give some properties of the $r - limit\ set$ and we examine relation between the set of rough limit and the extreme limit points of such sequences by using the similar tecniques that in [1].

2. Basic notions and some properties

Let A and B be compact and convex subsets of $\mathbb{R}^n$. The Hausdorff distance between them is defined as

$$\delta_\infty(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

A fuzzy number is a fuzzy subset of $\mathbb{R}^n$ which is normal, bounded and convex. Let $L(\mathbb{R}^n)$ denotes the set of all n-dimensional fuzzy numbers which are upper semi continuous and have a compact support. Then, the linear identity of $L(\mathbb{R}^n)$ is defined as follows:

$$[X+Y]^\gamma = [X]^\gamma + [Y]^\gamma,\ (X, Y \in L(\mathbb{R}^n))$$

and

$$[\lambda X]^\gamma = \lambda[X]^\gamma,\ (\lambda \in \mathbb{R}),$$

where γ-level set $X^\gamma := \{x \in \mathbb{R}^n : X(x) \geq \gamma\}$, for $0 < \gamma \leq 1$. Also, the metric d_q is defined as

$$d_q(X,Y) = \left(\int_0^1 \delta_\infty(X^\gamma, Y^\gamma)^q d_q\right)^{\frac{1}{q}},$$

for each $1 \leq q < \infty$. Furthermore, $d_\infty = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. In this paper, d_q will be denoted by d for $1 \leq q \leq \infty$.

Definition 2.1. A double sequence $X = (X_{nm})$ of fuzzy numbers is a function X from $\mathbb{N}\times\mathbb{N}$ into $L(\mathbb{R}^n)$. Here, X_{nm} is the value of the function at a point (n,m). By the convergence of double sequences, the convergence in Pringsheim's sense is understood, i.e. $X = (X_{nm})$ is said to be P-convergent to a finite number L, if X_{nm} tends to L as both n and m tends to ∞, independently each other [12].

Definition 2.2. A double sequence $X = (X_{nm})$ is said to be bounded, if there exists a positive number M such that $d(X_{nm}, 0) < M$ for all $n, m \in \mathbb{N}$ [12].

Throughout the paper, let $X = (X_{nm})$ be a double sequence of fuzzy numbers and let r be a nonnegative real number.

3. Main results

Definition 3.1. The sequence $X = (X_{nm})$ is said to be rough convergent in Pringsheim's sense to a fuzzy number X_*, denoted by $X_{nm} \xrightarrow{r} X_*$, if for every $\epsilon > 0$, there exists an integer i_ϵ such that

$$d(X_{nm}, X_*) < r + \epsilon,$$

whenever $n, m \geq i_\epsilon$.

Here, r is called roughness degree. The concept of rough convergence reduces the classical convergence of double sequences of fuzzy numbers for $r = 0$. In case $r > 0$, r-limit point of (X_{nm}) is usually no more unique, so we have defined so-called r-limit set as

$$LIM^r X_{nm} := \{X_* \in L(\mathbb{R}^n) : X_{nm} \xrightarrow{r} X_*\}.$$

$X = (X_{nm})$ is said to be r-convergent if this r-limit set is nonempty.

A double sequence of fuzzy numbers which is divergent can be convergent in Pringsheim's sense with a certain roughness degree. Now, we give the following example.

Example 3.1. The sequence $X = (X_{nm})$ is defined as follows:

$$X_{nm}(x) = \begin{cases} \ell_1(x)\ , & \text{if } (n+m) \text{ is odd,} \\ \ell_2(x)\ , & \text{if } (n+m) \text{ is even,} \end{cases}$$

where

$$\ell_1(x) = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,2], \\ \frac{-x+4}{2}, & \text{if } x \in [2,4], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\ell_2(x) = \begin{cases} \frac{x-5}{2}, & \text{if } x \in [5,7], \\ \frac{-x+9}{2}, & \text{if } x \in [7,9], \\ 0, & \text{otherwise.} \end{cases}$$

The γ-level set of the sequence is

$$[\ell_1(x)]^\gamma = [2\gamma, 4 - 2\gamma]$$

and

$$[\ell_2(x)]^\gamma = [2\gamma + 5, 9 - 2\gamma].$$

Then, we have

$$LIM^r X_{nm} := \begin{cases} \emptyset, & \text{if } r < \frac{5}{2}, \\ \left[\ell_2 - \acute{r}, \ell_1 + \acute{r}\right], & \text{otherwise,} \end{cases}$$

which $\acute{r}$ is nonnegative real number with $\{X \in L(\mathbb{R}^n) : \ell_2 - \acute{r} \preceq X \preceq \ell_1 + \acute{r}\}$.

Definition 3.2. The sequence $X = (X_{nm})$ is said to be rough Cauchy sequence with roughness degree ρ or ρ-Cauchy if for every $\epsilon > 0$, there exists i_ϵ such that

$$d(X_{nm}, X_{kl}) < \rho + \epsilon$$

whenever $k \geq n \geq i_\epsilon$ and $l \geq m \geq i_\epsilon$. Here, ρ is called Cauchy degree of (X_{nm}).

Let $Y=(Y_{nm})$ double convergent to X_*. Then, (Y_{nm}) often cannot be determined exactly, so we have to do with an approximated sequence (X_{nm}) provided that

$$d(X_{nm},Y_{nm})\leq\Delta$$

for all n,m, where $\Delta>0$ is an upper bound of approximation errors. This sequence (X_{nm}) may not be classical convergent, but the equation

$$d(X_{nm},X_*)\leq d(X_{nm},Y_{nm})+d(Y_{nm},X_*)\leq\Delta+\epsilon$$

implies that it is r-convergent for $r=\Delta$. Similarly, a Cauchy sequence $Y=(Y_{nm})$ is approximated by $X=(X_{nm})$ with $\Delta>0$, then, for all $\epsilon>0$ there exists i_ϵ such that

$$d(X_{nm},X_{kl})\leq d(X_{nm},Y_{nm})+d(Y_{nm},Y_{kl})+d(Y_{kl},X_{kl})\leq 2\Delta+\epsilon,$$

i.e., (X_{nm}) is a ρ-Cauchy sequence for $\rho=2\Delta$.

Theorem 3.1. *If a sequence $X=(X_{nm})$ converges to X_*, then*

$$LIM^r X_{nm}:=\overline{B_r}(X_*).$$

Proof. Let $\epsilon>0$. Since X_{nm} converges to X_*, there is an integer i_ϵ provided that

$$d(X_{nm},X_*)<\epsilon,$$

whenever $n,m\geq i_\epsilon$. Assume that $Y\in\overline{B_r}(X_*)=\{Y\in L(\mathbb{R}):d(Y,X_*)\leq r\}$. Then, we have

$$d(X_{nm},Y)\leq d(X_{nm},X_*)+d(X_*,Y)<r+\epsilon,$$

for every $n,m\geq i_\epsilon^{'}$. It shows that $Y\in LIM^r X_{nm}$.

Let $Y\in LIM^r X_{nm}$. Then, there is an integer $i_\epsilon^{''}$ provided that

$$d(X_{nm},Y)<r+\epsilon,$$

for all $n,m\geq i_\epsilon^{''}$. Let $i_\epsilon=max\{i_\epsilon^{''},i_\epsilon^{'}\}$. For every $i>i_\epsilon$, we get

$$d(Y,X_*)\leq d(Y,X_{nm})+d(X_{nm},X_*)<r+2\epsilon.$$

Since ϵ is arbitrary, we have $d(Y,X_*)\leq r$ which shows that $Y\in\overline{B_r}(X_*)$. □

Theorem 3.2. *For a sequence $X=(X_{nm})$, we have $diam(LIM^r X_{nm})\leq 2r$.*

Proof. Assume that

$$diam(LIM^r X_{nm})=sup\{d(Y,Z):Y,Z\in LIM^r X_{nm}\}>2r.$$

Then, there exists $Y,Z\in LIM^r X_{nm}$ such that $d(Y,Z)>2r$. For any $\epsilon\in(0,\frac{d(Y,Z)}{2}-r)$, we have

$$\exists i_\epsilon^{'}\in N:\forall n,m\geq i_\epsilon^{'}\Rightarrow d(X_{nm},Y)<r+\epsilon$$

and

$$\exists i_\epsilon^{''} \in N : \forall n, m \geq i_\epsilon^{''} \Rightarrow d(X_{nm}, Z) < r + \epsilon.$$

Let $i_\epsilon := max\{i_\epsilon^{'}, i_\epsilon^{''}\}$. Then, we get

$$d(Y, Z) \leq d(X_{nm}, Y) + d(X_{nm}, Z) < 2(r + \epsilon) < 2r + 2\frac{d(Y, Z)}{2} < d(Y, Z).$$

This is a contradiction. Thus, we have $diam(LIM^r X_{nm}) \leq 2r$. □

For a convergent sequence (X_{nm}) with $limX_{nm} = X_*$, we have $LIM^r X_{nm} = \overline{B_r}(X_*)$. Since $diam(\overline{B_r}(X_*)) = 2r$, in general the upper bound $2r$ of the diameter of an $r - limitset$ cannot be decreased anymore.

Theorem 3.3. *A sequence $X = (X_{nm})$ is r-convergent to X_*, if there exists a double sequence $Y = (Y_{nm})$ of fuzzy number such that $Y_{nm} \to X_*$ as $n, m \to \infty$ and $d(X_{nm}, Y_{nm}) \leq r$ for every $n, m \in N$.*

Proof. Let $Y_{nm} \to X_*$ and $d(X_{nm}, Y_{nm}) \leq r$ for every $n, m \in N$. From assumption, for every $\epsilon > 0$, there exists an i_ϵ such that $d(Y_{nm}, X_*) < \epsilon$ for every $n, m \geq i_\epsilon$. Since $d(X_{nm}, Y_{nm}) \leq r$, we have

$$d(X_{nm}, X_*) \leq d(X_{nm}, Y_{nm}) + d(Y_{nm}, X_*) < r + \epsilon$$

for $n, m \geq i_\epsilon$. This implies that (X_{nm}) is r-convergent to X_*. □

Theorem 3.4. *If $(X_{n_i m_i})$ is a subsequence (X_{nm}), then $LIM^r X_{nm} \subset LIM^r X_{n_i m_i}$.*

Theorem 3.5. *The r-limit set of an arbitrary sequence $X = (X_{nm})$ is closed.*

Proof. Let $(Y_{nm}) \subset LIM^r X_{nm}$ such that $Y_{nm} \to Y_*$ as $n, m \to \infty$. We will show that $Y_* \in LIM^r X_{nm}$. We can write

$$d(Y_{n_0 m_0}, Y_*) < r + \frac{\epsilon}{2}$$

for chosen $n_0, m_0 \in N$ such that $n_0, m_0 \geq k$. Since $(Y_{nm}) \subset LIM^r X_{nm}$, we have $(Y_{n_0 m_0}) \in LIM^r X_{nm}$, i.e.

$$d(Y_{n_0 m_0}, Y_*) < r + \frac{\epsilon}{2}.$$

Therefore, we get

$$d(X_{nm}, Y_*) \leq d(X_{nm}, Y_{n_0 m_0}) + d(Y_{n_0 m_0}, Y_*) < r + \epsilon$$

for $n, m, n_0, m_0 \geq k_\epsilon$. Thus, we have $Y_* \in LIM^r X_{nm}$. □

Definition 3.3. The Pringsheim's limit inferior and the Pringsheim's limit superior of X are defined as follows:

$$\lim infX_{nm} := infM_X$$

and

$$\lim supX_{nm} := supN_X,$$

where

$$M_X := \{\mu \in L(R) : \{(n,m) \in N \times N : X_{nm} < \mu\} \text{ is infinite set}\}$$

and

$$N_X := \{\mu \in L(R) : \{(n,m) \in N \times N : X_{nm} > \mu\} \text{ is infinite set}\}.$$

Theorem 3.6. *If $X_* \in LIM^r X_{nm}$, then $d(limsupX_{nm}, X_*) \leq r$ and $d(liminfX_{nm}, X_*) \leq r$.*

Proof. Suppose that $d(liminfX_{nm}, X_*) > r$. Then, take $\epsilon := \frac{d(liminfX_{nm},X_*)-r}{2}$. By definition of limit inferior, for given $k_\epsilon^{'}$ there exists $(n,m) \in N \times N$ with $n, m \geq k_\epsilon^{'}$ provided that

$$d(liminfX_{nm}, X_{nm}) < \epsilon.$$

On the other hand, since $X_* \in LIM^r X_{nm}$, there is $k_\epsilon^{''}$ such that

$$d(X_{nm}, X_*) < r + \epsilon,$$

whenever $n, m \geq k_\epsilon^{''}$. Let $k_\epsilon := max\{k_\epsilon^{'}, k_\epsilon^{''}\}$. Hence, we get

$$\begin{aligned} d(liminfX_{nm}, X_*) \leq & \, d(liminfX_{nm}, X_{nm}) + \; d(X_{nm}, X_*) \\ < & r + 2\epsilon \\ = & r + \; d(liminfX_{nm}, X_*) - r \\ = & d(liminfX_{nm}, X_*) \end{aligned}$$

which is a contradiction. Similarly, the theorem's other part can be proved. □

Theorem 3.7. *If $LIM^r X_{nm} \neq \emptyset$, then $LIM^r X_{nm} \subseteq [(limsupX_{nm}) - r, (liminfX_{nm}) + r]$.*

Proof. Now, we will show that $(limsupX_{nm}) - r \preceq X_* \preceq (liminfX_{nm}) + r$ for any $X_* \in LIM^r X_{nm}$. Assume that $X_* \succeq (liminfX_{nm}) + r$. Then, there exists an $\alpha \in [0,1]$ provided that

$$\underline{X_*}^\alpha > (\underline{liminfX_{nm}}^\alpha) + r$$

or

$$\overline{X_*}^\alpha > (\overline{liminfX_{nm}}^\alpha) + r.$$

Thus, we can write the inequalities below:

$$\underline{X_*}^\alpha - (\underline{liminfX_{nm}}^\alpha) > r$$

or

$$\overline{X_*}^\alpha - (\overline{liminfX_{nm}}^\alpha) > r.$$

Also, from Theorem 3.6, we get

$$|(\underline{liminfX_{nm}}^\alpha) - \underline{X_*}^\alpha| \leq r$$

and

$$|(\overline{liminfX_{nm}}^\alpha) - \overline{X_*}^\alpha| \leq r.$$

This is a contradiction. □

Lemma 3.1. *Let Γ_X be the set of cluster point of a sequence $X = (X_{nm})$. If any $C \in \Gamma_X$, we have*

$$d(X_*, C) \leq r$$

for all $X_* \in LIM^r X_{nm}$.

Proof. Assume that $C \in \Gamma_X$ and $X_* \in LIM^r X_{nm}$ such that $d(X_*, C) > r$. Then, we have

$$r < d(X_*, C) \leq d(X_*, X_{nm}) + d(X_{nm}, C) < r + 2\epsilon$$

Hence, we get $d(X_*, C) < r$, where $\epsilon = \frac{d(X_*, C) - r}{3}$. So, the proof is completed. □

Theorem 3.8. *If C is a cluster point of a sequence $X = (X_{nm})$, then*

$$LIM^r X_{nm} \subseteq \overline{B_r}(C) \tag{3.1}$$

Also,

$$LIM^r X_{nm} = \bigcap_{C \in \Gamma_X} \overline{B_r}(C) = \{X_* \in L(\mathbb{R}^n) : \Gamma_X \subseteq \overline{B_r}(X_*)\}. \tag{3.2}$$

Proof. Let $C \in \Gamma_X$ and $X_* \in LIM^r X_{nm}$. Then, according to the Lemma 3.1, we write

$$d(X_*, C) \leq r$$

otherwise, there are infinite X_{nm} satisfying

$$d(X_{nm}, X_*) \geq r + \epsilon$$

for $\epsilon = \frac{d(X_*, C) - r}{2} > 0$. This contradicts with the fact that $X_* \in LIM^r X_{nm}$.

Now, we will show that another equality. From (3.1), we can write

$$LIM^r X_{nm} \subseteq \bigcap_{C \in \Gamma_X} \overline{B_r}(C). \tag{3.3}$$

Let

$$Y \in \bigcap_{C \in \Gamma_X} \overline{B_r}(C).$$

Then, we have

$$d(Y, C) \leq r$$

for all $C \in \Gamma_X$. Therefore,

$$\bigcap_{C \in \Gamma_X} \overline{B_r}(C) \subseteq \{X_* \in L(\mathbb{R}^n) : \Gamma_X \subseteq \overline{B_r}(X_*)\}. \tag{3.4}$$

Let $Y \notin LIM^r X_{nm}$. Then, there exists infinite X_{nm} such that $d(X_{nm}, Y) \geq r + \epsilon$, for an $\epsilon > 0$. This implies that there is a cluster point C of (X_{nm}) such as $d(Y, C) \leq r + \epsilon$, that is,

$$\Gamma_X \nsubseteq \overline{B_r}(Y) \text{ and } Y \notin \{X_* \in L(\mathbb{R}^n) : \Gamma_X \subseteq \overline{B_r}(X_*)\}.$$

Hence, $Y \in LIM^r X_{nm}$ follows from $Y \in \{X_* \in L(\mathbb{R}^n) : \Gamma_X \subseteq \overline{B_r}(X_*)\}$, that is,

$$\{X_* \in L(\mathbb{R}^n) : \Gamma_X \subseteq \overline{B_r}(X_*)\} \subseteq LIM^r X_{nm}. \tag{3.5}$$

Thus, the inclusions (3.3)-(3.5) show that the equality in (3.2) is true. □

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