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ANALYSIS OF THE TIME FRACTIONAL NONLINEAR DIFFUSION EQUATION FROM DIFFUSION PROCESS

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Abstract Under investigation in this paper is a time fractional nonlinear diffusion equation which can be utilized to express various diffusion processes. The symmetry of this considered equation has been obtained via fractional Lie group approach with the sense of Riemann-Liouville (R-L) fractional derivative. Based on the symmetry, this equation can be changed into an ordinary differential equation of fractional order. Moreover, some new invariant solutions of this considered equation are found. Lastly, utilising the Noether theorem and the general form of Noether type theorem, the conservation laws are yielded to the time fractional nonlinear diffusion equation, respectively. Our discovery that there are no conservation laws under the general form of Noether type theorem case. This result tells us the symmetry of this equation is not variational symmetry of the considered functional. These rich results can give us more information to interpret this equation.

Keywords Fractional Lie group approach, time fractional nonlinear diffusion equation, invariant solutions, conservation laws.

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1. Introduction

In this paper, we mainly focus on the time fractional nonlinear diffusion equation, which reads

$$D_t^{\alpha} u = au^p u_{xx} + bu^{p-1}u_x^2, 0 < \alpha < 1, \tag{1.1}$$

where a, b are non-zero constants and $p \in N$. This equation (1.1) usually is used to write rich nonlinear phenomena of diffusion process. Meanwhile, it can also regards as from the integer order nonlinear diffusion equation [19]

$$u_t = au^p u_{xx} + bu^{p-1}u_x^2,$$

by replacing its time derivative by fractional derivative. Fractional derivative can be liked as a generalization of integer order derivative. It may be better to emerge

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in different complex natural phenomena from mathematics, physics, biology and engineering [20, 28, 31, 40–42]. There exist many various definitions of fractional order [4, 16, 17, 27, 30, 34, 43], such as the R-L fractional derivative

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{\partial^m u(x,t)}{\partial t^m}, \alpha = m \in N; \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int\limits_0^t \frac{u(s,x)}{(t-s)^{\alpha+1-m}} ds, m-1 < \alpha < m, m \in N, \end{cases}$$

the Caputo fractional derivative

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{\partial^m u(x,t)}{\partial t^m}, \alpha = m \in N; \\ \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{u^{(m)}(s,x)}{(t-s)^{\alpha+1-m}} ds, m-1 < \alpha < m, m \in N, \end{cases}$$

the Caputo-Fabrizio fractional derivative

$$^{CF}D_{c^+}^{\alpha} f(x) := \frac{M(\alpha)}{1-\alpha} \frac{d}{dt} \int_{c^+}^{x} \exp[\frac{-\alpha}{1-\alpha}(x-y)] f(y) dy,$$

with $M(\alpha)$ being a normalisation function satisfying $M(0) = M(1) = 1$, and the Grünwald-Letnikov fractional derivative

$$D^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{m=0}^{x/h} (-1)^m \begin{pmatrix} \alpha \\ m \end{pmatrix} f(x - mh),$$

and so on.

Lie symmetry approach is a powerful mathematical tool to deal with integer order or fractional order partial differential equations and ordinary differential equations [1, 2, 7, 8, 15, 21, 22, 32, 33, 35–38]. For example, S.Y. Lou, et al used the non-Lie symmetry groups to study the (2+1)-dimensional nonlinear systems in 2005 [21], E. Buckwar, et al investigated the invariance of a partial differential equation of fractional order under the Lie group of scaling transformations [1], R. Sahadevan, et al analyzed the invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations [33], D. Baleanu, et al used this approach explored the time fractional third-order evolution equation [2]. Meanwhile, we had known this famous fact that the Lie symmetry method can obtain massive new exact solutions based on its reduced equation from the original equation. These new and accurate solutions can help us reveal the deeper mechanism of the evolution of things. Recently, many scholars applied different schemes to obtain various exact solutions [5, 9, 10, 12, 23, 24, 39, 44], such as multi-exponential wave solutions, soliton solutions, lump solutions, rogue wave type solutions, and so on.

Next we applied the Lie symmetry method to analyze Eq.(1.1). As a result, the symmetry, invariant solutions and invariant solutions can be found.

2. Lie group analysis of Eq.(1.1)

We used the Lie group method in this section to investigate the time fractional nonlinear diffusion equation. As a direct result of this behavior, the symmetry of this equation (1.1) can be obtained. Based on the symmetry, some invariant solutions are derived in the next section.

First of all, Eq.(1.1) keeps invariance under the following from one-parameter continuous transformations group [3, 15, 22, 29, 32, 36–38]

$$\begin{aligned}
&\frac{\partial^\alpha \tilde{u}}{\partial \tilde{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon U^{\alpha t}(x,t,u) + O(\varepsilon^2),\\
&\frac{\partial \tilde{u}}{\partial \tilde{x}} = \frac{\partial u}{\partial x} + \varepsilon U^{x}(x,t,u) + O(\varepsilon^2),\\
&\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} = \frac{\partial^2 u}{\partial x^2} + \varepsilon U^{2x}(x,t,u) + O(\varepsilon^2),\\
&...,\\
&\frac{\partial^k \tilde{u}}{\partial \tilde{t}^k} = \frac{\partial^k u}{\partial x^k} + \varepsilon U^{kx}(x,t,u) + O(\varepsilon^2),
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
&\tilde{t} = t + \varepsilon T(x,t,u) + O(\varepsilon^2),\\
&\tilde{x} = x + \varepsilon X(x,t,u) + O(\varepsilon^2),\\
&\tilde{u} = u + \varepsilon U(x,t,u) + O(\varepsilon^2),
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&U^x = D_x U - u_x D_x(X) - u_t D_t(T),\\
&U^{xx} = D_x U^x - u_{xx} D_x(X) - u_t D_{xt}(T),\\
&...,\\
&U^{kx} = D_x U^{(k-1)x} - u_{kx} D_x(X) - u_{(k-1)xt} D_t(T),
\end{aligned} \tag{2.3}$$

with D_x is the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + ...$$

Suppose that the infinitesimal operator V has the form

$$V = T(x,t,u)\frac{\partial}{\partial t} + X(x,t,u)\frac{\partial}{\partial x} + U(x,t,u)\frac{\partial}{\partial u}, \tag{2.4}$$

and moreover, has

$$T = \frac{d\tilde{t}}{d\varepsilon}|_{\varepsilon=0}, X = \frac{d\tilde{x}}{d\varepsilon}|_{\varepsilon=0}, U = \frac{d\tilde{u}}{d\varepsilon}|_{\varepsilon=0}.$$

Utilizing the infinitesimal invariance criterion [8], yields

$$pr^{(\alpha,t)}V(\triangle)|_{\triangle=0} = 0, \triangle = D_t^\alpha u - au^p u_{xx} - bu^{p-1}u_x^2, \tag{2.5}$$

where $pr^{(\alpha,t)}V$ is the prolongation of V. It can be given by

$$pr^{(\alpha,t)}V = V + \frac{\partial}{\partial D_t^\alpha u} + U^x \frac{\partial}{\partial u_x} + U^{xx}\frac{\partial}{\partial u_{xx}} + ... + U^{\alpha t}. \tag{2.6}$$

For the time fractional nonlinear diffusion equation (1.1), the above equation (2.6) becomes

$$U^{\alpha,t} - au^p U_{xx} - apu^{p-1}u_{xx}U - 2bu^{p-1}u_x U_x - bpu^{p-2}u_x^2 U + bu^{p-2}u_x^2 U = 0. \quad (2.7)$$

Taking the coefficients of linearly independent derivatives $u_x, u_{xx}...$ and $\partial_t^{\alpha-n}u_x$, $\partial_t^{\alpha-n}u$ to zero, solving these determining equations, we obtain the solutions of the form

$$T = 2c_1 t + c_4, X = c_3 + c_2 x, U = (c_1\alpha - c_2)u,$$

where c_1, c_2, c_3 and c_4 are free constants.

Meanwhile, as well to admit the structure of the R-L fractional derivative. The invariance condition needs to satisfy

$$T(x,t,u)|_{t=0} = 0. \quad (2.8)$$

So

$$c_4 = 0.$$

Therefore the vector fields are given by

$$V_1 = \frac{\partial}{\partial x}, V_2 = xp\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, V_3 = \frac{t}{\alpha}p\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \quad (2.9)$$

Meanwhile, we note that the vector fields (2.9) are closed by using Lie bracket, respectively.

$$[V_1, V_2] = pV_1, [V_1, V_3] = [V_2, V_3] = 0.$$

In the next section, we used these vector fields (2.9) to derive a few invariant solutions with fixed values α.

3. Invariant solutions and reduction equations of Eq.(1.1)

At the beginning of this section, the definition of invariant solution is concisely given in this part. Then the reduction equations and invariant solutions of Eq.(1.1) are found with some special values α.

Definition 3.1. Let $u = u(x,t)$ be an invariant solution of FPDE Eq.(1.1) if and only if

(i) $u = u(x,t)$ is an invariant surface, i.e. $Vu = 0 \Rightarrow T\frac{\partial u}{\partial t} + X\frac{\partial u}{\partial x} + U\frac{\partial u}{\partial u} = 0$.
(ii) $u = u(x,t)$ satisfies FPDE Eq.(1.1).

The invariant solutions of Eq.(1.1) can be expressed by two new invariants (p,q). Using them the original equation can be reduced into an ordinary differential equation of fractional order.

For the case of $V_1 = \frac{\partial}{\partial x}$, the characteristic equation can be shown as the following form

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \quad (3.1)$$

has two different invariants as follows:

$$P = t, Q = u. \quad (3.2)$$

The invariant solution of Eq.(1.1) is given by the form

$$Q = f(P) \Rightarrow u = f(t). \tag{3.3}$$

Substituting equation (3.3) into equation (1.1), we have

$$\frac{d^\alpha f(t)}{dt^\alpha} = 0. \tag{3.4}$$

Eq.(3.4) has an invariant solution by using Laplace transform

$$u(x,t) = f(t) = \frac{c_0}{\Gamma(\alpha)} t^{\alpha-1}, \tag{3.5}$$

where c_0 is an arbitrary parameter.

For the case of $V_2 = xp\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}$, we get the characteristic equation of the following expression

$$\frac{dx}{xp} = \frac{dt}{0} = \frac{du}{2u}, \tag{3.6}$$

yields two different invariants of the forms

$$P = t, Q = u^p x^2. \tag{3.7}$$

Hence, has

$$u = (x^{-2} f(t))^{\frac{1}{p}}. \tag{3.8}$$

Furthermore,

$$u_x = -\frac{2}{p} x^{\frac{2p-5}{p}} f(t)^{\frac{1}{p}}, u_{xx} = \frac{10-4p}{p^2} x^{1-\frac{5}{p}} f(t)^{\frac{1}{p}}. \tag{3.9}$$

Substituting Eqs.(3.8) and (3.9) into (1.1) has the following reduced equation of ordinary differential equation of fractional of the form

$$D_t^\alpha f(t) = \frac{2f(t)^{(1+\frac{1}{p})}(ap+2a+2b)}{x^4p^2}. \tag{3.10}$$

Hereafter, we give a few exact solutions with the special values $\alpha = 1/10, 1/3, 1/2, 2/3$, respectively.

$$u_1 = \frac{x^4p^2(10529610f^{'}(0)t^{\frac{9}{10}} + 5541900f^{''}(0)\,t^{\frac{19}{10}} + \Delta_1)^{\frac{p}{p+1}}}{18953298\,\Gamma\left(\frac{9}{10}\right)(ap+2\,a+2\,b)},$$

$$u_2 = \frac{x^4p^2(18480f^{'}(0)\,t^{\frac{2}{3}}+11088f^{''}(0)\,t^{\frac{5}{3}}+4158f^{'''}(0)\,t^{\frac{8}{3}}+1134f^{(4)}(0)\,t^{\frac{11}{3}}+243f^{(5)}(0)\,t^{\frac{14}{3}})^{\frac{p}{1+p}}}{24640\,\Gamma(2/3)\,(ap+2\,a+2\,b)},$$

$$u_3 = \frac{p^4x^8(1890\,f^{'}(0)\sqrt{t} + 1260f^{''}(0)\,t^{\frac{3}{2}} + 504f^{'''}(0)\,t^{\frac{5}{2}} + 144\,f^{(4)}(0)t^{\frac{7}{2}} + 32f^{(5)}(0)t^{\frac{9}{2}})^{\frac{p}{(1+p)}}}{3572100\,\pi\,(ap+2\,a+2\,b)^2},$$

$$u_4 = \left(\frac{x^4p^2(10920\,f^{'}(0)\sqrt[3]{t}\sqrt{3}\Gamma(2/3) + 8190f^{''}(0)\,t^{\frac{4}{3}}\sqrt{3}\Gamma(2/3) + 3510f^{'''}(0)\,t^{\frac{7}{3}}\sqrt{3}\Gamma(2/3) + \Delta_2)}{14560\,\pi\,(ap+2\,a+2\,b)}\right)^{\frac{p}{1+p}},$$

with

$$\Delta_1 = 1911000f^{'''}(0)\,t^{\frac{29}{10}} + 490000f^{(4)}(0)\,t^{\frac{39}{10}} + 100000f^{(5)}(0)\,t^{\frac{49}{10}}),$$
$$\Delta_2 = 1053f^{(4)}(0)\,t^{\frac{10}{3}}\sqrt{3}\Gamma(2/3) + 243\,f^{(5)}(0)t^{\frac{13}{3}}\sqrt{3}\Gamma(2/3),$$

where the $f(0)$ is an initial solution of Eq.(1.1).

In a similar matter, the vector field

$$V_3 = \frac{t}{\alpha} p \frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

corresponding to the form ordinary differential equation of fractional order as follows:

$$\frac{\Gamma(1-\frac{\alpha}{p})}{\Gamma(1-\frac{\alpha}{p}-\alpha)} f^{2-\alpha p}(x) + [a(1-\alpha) - b\alpha] f_x^2(x) + \frac{f^{''}(x)}{f^2(x)} a = 0. \tag{3.11}$$

4. Conservation laws of Eq.(1.1)

Basic information about conservation laws was firstly listed in this section [6,11,13, 14,18,25,26]. Then apply the Lie point symmetry, variational Lie point symmetry and nonlinear self-adjointness to construct conservation laws of Eq.(1.1).

4.1. Basic knowledge on the conservation laws

Taking the following n-th order time fractional nonlinear partial differential equations of the form

$$F(t, x, D_t^\alpha u, u^{'}, u^{''}, ..., u^{(n)}) = 0. \tag{4.1}$$

And its formal Lagrangian is

$$L = vF(t, x, D_t^\alpha u, u^{'}, u^{''}, ..., u^{(n)}), \tag{4.2}$$

where $v = v(t, x)$ is a new dependent variable.

Conservation laws need to admit the condition

$$D_t(C^t) + D_x(C^x) = 0, \tag{4.3}$$

where $C^t = C^t(x, t, u, ...), C^x = C^x(x, t, u, ...)$ are called conserved vectors of conservation laws.

Definition 4.1. The adjoint equation of equation (4.1) is given by

$$\frac{\delta L}{\delta u} := F^*(t, x, D_t^\alpha u, u^{'}, u^{''}, ..., u^{(n)}) = 0, \tag{4.4}$$

and the Euler-Lagrange operator [3,29] is shown by

$$\frac{\delta}{\delta u_j} = \frac{\partial}{\partial u_j} + (D_t^\alpha)^* \frac{\partial}{\partial(D_t^\alpha u_j)} + \sum_{k=1}^{\infty} (-1)^k D_{i_1 i_2 ... i_k} \frac{\partial}{\partial (u_j)_{i_1 i_2 ... i_k}}, \tag{4.5}$$

where $(D_t^\alpha)^*$ is adjoint operator of D_t^α and

$$({}_0D_t^\alpha)^* = (-1)^n {}_tI_T{}^{n-\alpha}(D_t^n) \equiv {}_t^CD_T^\alpha, ({}_0^CD_t^\alpha)^* = (-1)^n D_t^n({}_tI_T{}^{n-\alpha}) \equiv {}_tD_T^\alpha.$$

Definition 4.2. Eq.(4.1) is called to be nonlinearly self-adjoint if its adjoint equation (4.4) upon the substitution $v = \phi(t, x, u, u^{'}, ...)$, then it becomes equivalent to Eq.(4.1), for example,

$$F^*|_{v=\phi} = \lambda(t, x, u, u^{'}, ...)F. \tag{4.6}$$

If $\phi = \phi(u)$ and $\phi^{'}(u) \neq 0$, Eq.(4.1) is said to be quasi-adjoint.

Now we considering the case of $u = u(x,t)$, has

$$\bar{V} + D_t^{\alpha}(T)I + D_x^{\alpha}(X)I = W\frac{\delta}{\delta u} + D_t N^t + D_x N^x, \tag{4.7}$$

with

$$\begin{aligned} \bar{V} &= T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u} + U_{\alpha}^{0}\frac{\partial}{\partial D_t^{\alpha} u} + U^x\frac{\partial}{\partial u_x}, \\ W &= U - Tu_t - Xu_x, \end{aligned} \tag{4.8}$$

where I is the identity operator, $\frac{\delta}{\delta u}$ is the Euler-Lagrangian operator, N^t, N^x are the Noether operators.

The Noether operators N^t and N^x can be written as the forms with the R-L fractional derivative as follows:

$$\begin{aligned} N^x &= XI + W[\frac{\partial}{\partial u_x} - D_x(\frac{\partial}{\partial u_{xx}}) + D_x^2(\frac{\partial L}{\partial u_{xxx}}) - ...] + D_x(W)[\frac{\partial}{\partial u_{xx}} - D_x(\frac{\partial L}{\partial u_{xxx}}) + ...], \\ N^t &= TI + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial L}{\partial(D_t^{\alpha}u)} - (-1)^n J(W, D_t^n \frac{\partial L}{\partial(D_t^{\alpha}u)}), \end{aligned} \tag{4.9}$$

where J is the following integral operator

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(s,x)g(\upsilon,x)}{(\upsilon - s)^{\alpha+1-n}} d\upsilon ds. \tag{4.10}$$

The invariance condition for V of Eq.(4.1), becomes

$$(\bar{V}L + D_t(\tau)L + D_x(\xi)L)|_{Eq.(1)} = 0 \tag{4.11}$$

and Eq.(4.3) yields

$$D_t(N^t L) + D_x(N^x L) = 0. \tag{4.12}$$

Hence the x and t-components of conserved vectors with the different symmetries, respectively, are

$$C^x = XL + W[\frac{\partial L}{\partial u_x} - D_x(\frac{\partial L}{\partial u_{xx}}) + D_x^2(\frac{\partial L}{\partial u_{xxx}}) - ...] + D_x(W)[\frac{\partial L}{\partial u_{xx}} - D_x(\frac{\partial L}{\partial u_{xxx}}) + ...], \tag{4.13}$$

and

$$C^t = TL + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W) D_t^k \frac{\partial L}{\partial(D_t^{\alpha}u)} - (-1)^n J(W, D_t^n \frac{\partial L}{\partial(D_t^{\alpha}u)}). \tag{4.14}$$

In what follows we apply the processes to find the conservation laws to Eq.(1.1) in the next subsection.

4.2. Conservation laws via Noether's operator

Here we obtain the adjoint equation to Eq.(1.1) through the concepts and definitions express to conservation laws in the form

$$\frac{\delta L}{\delta u} = (D_t^{\alpha})^* v + 2(b-p)u^{p-1}u_{xx} + (p-1)(b-ap)u^{p-2}u_x^2. \tag{4.15}$$

Then, For V_1, V_2 and V_3, get

$$W_1 = -u_x, W_2 = 2u - xpu_x, W_3 = -u - \frac{t}{\alpha} pu_t. \tag{4.16}$$

Based on the subsection 4.1, the conservation laws can be obtained for V_1, V_2 and V_3, respectively.

For V_1, the conserved vector of Eq.(1.1) are

$$\begin{aligned} C_1^t &= -I_t^{1-\alpha}(u_x)v + J(-u_x, v_t), \\ C_1^x &= vD_t^\alpha u - bvu^{p-1}u_x^2 + (2vb - av_x - avp)u_x u^{p-1}. \end{aligned} \tag{4.17}$$

For V_2, the conserved vector of Eq.(1.1) are given by

$$\begin{aligned} C_2^t &= I_t^{1-\alpha}(2u - xpu_x)v + J(2u - xpu_x, v_t), \\ C_2^x &= xpvD_x^t u + xpu^{p-1}u_x^2(vb - p) + u^p u_x(3vpa - 4vb - xpav_x - 2va) + 2av_x u^{p+1}. \end{aligned} \tag{4.18}$$

For V_3, the conserved vector of Eq.(1.1) are the following forms

$$\begin{aligned} C_3^t &= \frac{t}{\alpha} vp(D_t^\alpha u - au^p u_{xx} - bu^{p-1}u_x^2) + I_t^{1-\alpha}(-u - \frac{t}{\alpha} pu_t)v + J(-u - \frac{t}{\alpha} pu_t, v_t), \\ C_3^x &= \frac{t}{\alpha} pu^{p-1}(2vbu_x u_t - au_t v_x - avpu_x u_t + auvu_{xt}) + vau_x u^p. \end{aligned} \tag{4.19}$$

Furthermore, if we taking $v = \mu(t, x) = \varpi(x)\omega(t) = c\omega(t)$, where c is an arbitrary constant, then we have

$$(D_t^\alpha)^*(\omega(t)) = {}^C D_T^\alpha(\omega(t)) = 0.$$

So

$$\mu(t, x) = A = constant.$$

We assume that $v = \mu(t, x) = 1$, then Eqs.(4.17),(4.18) and (4.19) can be rewritten as

$$\begin{aligned} C_1^t &= -I_t^{1-\alpha}(u_x), \\ C_1^x &= D_t^\alpha u - bu^{p-1}u_x^2 + (2b - ap)u_x u^{p-1}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} C_2^t &= I_t^{1-\alpha}(2u - xpu_x), \\ C_2^x &= xpD_x^t u + xpu^{p-1}u_x^2(b - p) + u^p u_x(3pa - 4b - 2a), \end{aligned} \tag{4.21}$$

$$\begin{aligned} C_3^t &= \frac{t}{\alpha} p(D_t^\alpha u - au^p u_{xx} - bu^{p-1}u_x^2) + I_t^{1-\alpha}(-u - \frac{t}{\alpha} pu_t), \\ C_3^x &= \frac{t}{\alpha} pu^{p-1}(2bu_x u_t - apu_x u_t + auu_{xt}) + au_x u^p, \end{aligned} \tag{4.22}$$

respectively.

4.3. Conservation laws via fractional generalizations of Noether's theorem

In the present subsection we will apply the generalization of Nother's theorem for fractional Lagrangian densities to find conservation laws with in the sense of R-L fractional derivative [13, 18]. The a one-parameter group of transformation has a variational symmetry group of fractional variational problem as follows:

$$L(u)=\int_a^b \Im(t,x,u^{(n)},{}_aD_t^\alpha u)dt. \tag{4.23}$$

Some useful Definitions and Theorems are listed in this text as follows [18]:

Definition 4.3. Assume that functions f and g are of class C^1 in the interval $[a,b]$, we have

$$D_t^\alpha(f,g)=-g_tD_b^\alpha f+f_aD_t^\alpha g,t\in[a,b]. \tag{4.24}$$

If $\alpha=1$, then $D_t^1(f,g)=\frac{d}{dt}(fg)$.

Definition 4.4. The fractional conserved quantity $\Theta(t,x)$ needs to admit the relation of the from

$$\Theta(t,x,{}_aD_t^\alpha u,{}_tD_b^\beta u)=\sum_{i=1}^{n}\Theta_i^1(t,x,{}_aD_t^\alpha u,{}_tD_b^\beta u)\Theta_i^2(t,x,{}_aD_t^\alpha u,{}_tD_b^\beta u),n\in N, \tag{4.25}$$

where each pairs Θ_i^1 and Θ_i^2 satisfy

$$D_t^{\gamma_i}(\Theta_i^{j_i^1}(t,x,{}_aD_t^\alpha u,{}_tD_b^\beta u)\Theta_i^{j_i^2}(t,x,{}_aD_t^\alpha u,{}_tD_b^\beta u))=0,\gamma_i\in\{\alpha,\beta,1\}, \tag{4.26}$$

where $j_i^1=1$ and $j_i^2=2$ or $j_i^1=2$ and $j_i^1=2$, along all the solutions of the fractional Euler-Lagrange equations [18].

Theorem 4.5. *Make $L(u)$ be the set of functions $u(t,x)$ which have continuous left and right R-L fractional derivative of order α and β in interval $[a,b]$, respectively. Meanwhile, it with the boundary conditions*

$$u(a)=C_a \quad and \quad u(b)=C_b. \tag{4.27}$$

Then a necessary condition for $L(u)$ to have an extreme for a given function $u(t)$ is that $u(t)$ admits the Euler-Lagrange equation in the following expression

$$E_k(\Im)=0,1\le k\le q, \tag{4.28}$$

such that the k-th Euler operator is shown by

$$E_k=\sum_J(-D)_J\frac{\partial}{\partial u_J^\alpha}+{}_tD_b^\alpha\frac{\partial}{\partial_aD_t^\alpha u}+{}_aD_t^\beta u\frac{\partial}{\partial_tD_b^\beta}, \tag{4.29}$$

where $J=(j_1,j_2,...,j_k),(1\le j_k\le p,1\le k\le q)$.

Theorem 4.6. *A connect group of transformations G acting on M is a variational symmetry group of the fractional Lagrangian (4.23) if and only if*

$$pr^{(\alpha)}V(\Im)+\Im Div(\zeta)=0, \tag{4.30}$$

for every infinitesimal general

$$V=\sum_{i=1}^{p}\zeta^i\frac{\partial}{\partial x^i}+\sum_{j=1}^{q}\eta^j\frac{\partial}{\partial u^j}$$

of G, Div denotes the total divergence of the p-tuple $\zeta = (\zeta^1, \zeta^2, ..., \zeta^n)$.

Theorem 4.7. *Let V and G be the infinitesimal generator of G and a variational symmetry group of equation (4.23), respectively. There exists a p-tuple $P(t, x, \vartheta^\alpha) = (P_1, P_2, ..., P_p)$ such that*

$$Div(P) = W \cdot E(\Im) = \sum_{\upsilon=1}^{q} W_\upsilon \cdot E_\upsilon(\Im), \tag{4.31}$$

where

$$W_\upsilon(x, u^{(1)}) = \eta_\upsilon - \sum_{i=1}^{p} \zeta^i \frac{\partial u^\upsilon}{\partial x_i}, \tag{4.32}$$

and $W = (W_1, W_2, ..., W_q)$ is the characteristic of a conservation law for Euler-Lagrange equation

$$E(\Im) = 0.$$

Corollary 4.8. *By means of equations (4.30) and (4.31), Eq.(4.33) is given by*

$$pr^{(\alpha)}V(\Im) + \Im Div(\zeta) = \sum_{\upsilon=1}^{q} W_\upsilon E_\upsilon(\Im) + Div(P), \tag{4.33}$$

so, first integral of Euler-Lagrange equation or conservation laws of Eq.(4.1) is given by

$$Div(P) = \mathrm{P}^{(\alpha)}rV(\Im) + \Im Div(\zeta) - W \cdot E(\Im). \tag{4.34}$$

In what follows we apply these definitions and theorems of the above state to construct conservation laws.

Considering the Lagrangian $L = c(D_t^\alpha u - au^p u_{xx} - bu^{p-1}u_x^2)$. For V_1, V_2 and V_3, we first check the vector fields $V_i, i = 1, 2, 3$ whether are variational symmetry. We find that

(i)

$$pr^{(\alpha)}V_1(\Im) + \Im(D_t(T) + D_x(X)) \neq 0, \tag{4.35}$$

(ii)

$$pr^{(\alpha)}V_2(\Im) + \Im(D_t(T) + D_x(X)) \neq 0, \tag{4.36}$$

(iii)

$$pr^{(\alpha)}V_3(\Im) + \Im(D_t(T) + D_x(X)) \neq 0, \tag{4.37}$$

for V_1, V_2 and V_3 with W_1, W_2 and W_3, respectively. Thus, there no exist conservation laws by using this scheme.

5. Conclusion and Discussions

With the sense of Riemann-Liouville fractional derivative, the Lie point symmetry was obtained through group analysis scheme for the time fractional nonlinear diffusion equation, which usually be utilized to show various diffusion processes from various fields of natural science. Then some new invariant solutions can be found according to the ordinary differential equation of fractional order. Moreover, we applied the Noether theorem and the general form of Noether type theorem to deal with the equation (1.1). As a result, the conservation laws yield this time fractional nonlinear diffusion equation, respectively. However, our discovery that there are no conservation laws under the general form of Noether type theorem case. This result implies that there no exist the variational symmetry of this considered functional. There are two different issues here that deserve further consideration. For example,

(I) Finding an effective way to get the solutions of fractional ordinary differential equation;

(II) How to apply the general form of Noether type theorem to find more conservation laws for others nonlinear fractional evolution differential equations.

These problems will be considered in the subsequent papers.

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