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# EXISTENCE OF SOLUTIONS FOR A FRACTIONAL ADVECTION-DISPERSION EQUATION WITH IMPULSIVE EFFECTS VIA VARIATIONAL APPROACH*

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**Abstract** In this paper, based on the variational approach and iterative technique, the existence of nontrivial weak solutions is derived for a fractional advection-dispersion equation with impulsive effects, and the nonlinear term of fractional advection-dispersion equation contain the fractional order derivative. In addition, an example is presented as an application of the main result.

**Keywords** Fractional advection-dispersion equation, variational approach, impulsive, iterative technique.

**MSC(2010)** 26A33, 35A15, 34B15.

## 1. Introduction

In this paper, we investigate the existence of nontrivial weak solutions for following a fractional advection-dispersion equation (FADE for short) with impulsive effects

$$\begin{cases} -\dfrac{d}{dt}\Big[\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big] = \varrho u(t) + f(t, u(t),\ {}_0^cD_t^{\alpha}u(t)), \\ a.e.\ t \in [0,T],\ \ t \neq t_j, \\ \Delta\Big(\dfrac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j)) - \dfrac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j))\Big) = I_j(u(t_j)),\ \ j = 1, 2, \dots, n, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Loiuville fractional integrals of order $1-\alpha$ respectively, ${}_0^cD_t^{\alpha}$ and ${}_t^cD_T^{\alpha}$ are the left and right Caputo fractional derivatives of order $0 < \alpha \leq 1$ respectively, $\varrho$ is a parameter, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, \dots, n)$ are continuous functions,

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and

$$
\begin{aligned}
&\Delta\Big(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j))\Big)\\
=&\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j^+))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j^+))\\
&-\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j^-))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j^-))\right),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j^+))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j^+))\\
=&\lim_{t\to t_j^+}\Big(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j))\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j^-))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j^-))\\
=&\lim_{t\to t_j^-}\Big(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j))-\frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j))\Big).
\end{aligned}
$$

A large number of scholars have been attracted to study the fractional advection-dispersion equations since it can simulate anomalous diffusion on certain conditions, and describes nonsymmetric or symmetric transition and solute transportation and so on. For instance, Ervin and Roop in [5] considered the following form FADE

$$
-\frac{d}{dt}(p_0D_t^{-\beta}+(1-p)_tD_T^{-\beta})u'(t)+b(t)u'(t)+c(t)u(t)=\nabla F(t,u(t)), a.e. t\in[0,T], \tag{1.2}
$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Loiuville fractional integral operators respectively, $0\le\beta\le1$, $p\in[0,1]$ is a constant describing the skewness of the transport process, $b,c,F$ satisfies some suitable conditions. If taking $p=\frac{1}{2}$ in (1.2), then the FADE (1.2) describes symmetric transitions. Sun and Zhang in [20] investigate the FADE (1.2) with $b(t)=c(t)=0, T=1$, and the boundary conditions $u(0)=u(1)=0$. For more background information on FADE, see [1, 2, 4, 13, 14, 18, 21, 25–27] and so on. Recently, by the critical point theory, Jiao and Zhou in [9] consider the symmetric FADE of the following form

$$
\begin{cases}
\dfrac{d}{dt}\Big(\dfrac{1}{2}\ {}_0D_t^{-\beta}+\dfrac{1}{2}\ {}_tD_T^{-\beta}\Big)u'(t)+\nabla F(t,u(t))=0, \quad a.e.\ t\in[0,T],\\
u(0)=u(T)=0,
\end{cases} \tag{1.3}
$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Loiuville fractional integral operators respectively, $0\le\beta\le1$, and $\nabla F(t,x)$ is the gradient of $F$ at $x$. The existence of solution and nontrivial solution for FADE are obtained.

Li et al. in [12] study the existence of solutions to fractional boundary-value problems with a parameter by using critical point theory and variational methods

$$
\begin{cases}
-\dfrac{d}{dt}\Big(\dfrac{1}{2}\ {}_0D_t^{-\beta}+\dfrac{1}{2}\ {}_tD_T^{-\beta}\Big)u'(t)=\lambda u(t)+\nabla F(t,u(t)), \quad a.e.\ t\in[0,T],\\
u(0)=u(T)=0,
\end{cases} \tag{1.4}
$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Loiuville fractional integral operators respectively, $0 \le \beta \le 1$, $\lambda \in \mathbb{R}$ is a parameter, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla F(t, x)$ is the gradient of $F$ with respect to $x$.

Especially, differential equations with impulsive effects are intensively investigated recently. It can be used to describe discontinuous jumps and sudden changes of their states in optimal control and so on. Therefore, it is worth to study. There are few works that the existence of solutions for fractional advection-dispersion equations with impulsive effects and impulsive fractional differential equations. Chai and Chen in [3] investigated the following impulsive fractional boundary problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = f(t, u(t),\ {}_0^cD_t^{\alpha}u(t)),\ t \neq t_j \quad a.e.\ t \in [0, T], \\ \triangle({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = I_j(u(t_j)),\ j = 1, 2, \ldots, n, \\ u(0) = u(T) = 0, \end{cases} \tag{1.5}$$

where $\alpha \in (\frac{1}{2}, 1]$, $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = T$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, \ldots, n$, are continuous functions, $a \in C[0, T]$. Under the condition $0 < a_1 \le a(t) \le a_2$, the authors proved the existence of at least one nontrivial solution by using the variational method and iterative technique.

Nyamoradi and Tayyebi in [17] study the existence of weak solutions for following impulsive fractional differential equations by using critical point theory and variational methods

$$\begin{cases} \dfrac{d}{dt}\Big[\dfrac{1}{2}\ {}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\ {}_tD_T^{-\beta}(u'(t))\Big] = \nabla F(t, u(t)), \quad a.e.\ t \in [0, T],\ t \neq t_j, \\ \Delta({}_0D_t^{-\beta}((u^i)'(t_j)) +\ {}_tD_T^{-\beta}((u^i)'(t_j))) = I_{ij}(u^i(t_j)),\ i \in A,\ j \in B, \\ u(0) = u(T) = 0, \end{cases} \tag{1.6}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Loiuville fractional integrals of order $0 \le \beta < 1$ respectively, $A = \{1, 2, \ldots, N\}$, $B = \{1, 2, \ldots, L\}$, $0 = t_0 < t_1 < t_2 < \ldots < t_L < t_{L+1} = T$, $\nabla F(t, x)$ denotes the gradient of $F(t, x)$ in $x$, and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$ $(i \in A,\ j \in B)$ are continuous functions. In early time, Wang et al. in [22] apply Minimax principle and saddle point theorem to study the existence of weak solutions of problem (1.6).

Obviously, if we choose $\alpha = 1$, and $I_j = 0$ $(j = 1, 2, \ldots, n)$, then the FADE (1.1) reduces to the second-order FADE of the following form

$$\begin{cases} -u''(t) = \varrho u(t) + f(t, u(t), u'(t)), \quad a.e.\ t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.7}$$

There have been many methods to investigate the existence of solutions of problem (1.7) such as fixed point theory and monotone iterative method and so on. (see [7, 8, 24] and references therein).

Inspired by the works described above, we aim to investigate the existence of nontrivial weak solutions for a fractional advection-dispersion equation with impulsive effects. Different from the previous paper, the main characteristics of the present paper are as follows. Firstly, the nonlinear term of fractional advection-dispersion equation contain the fractional order derivative. As far as we know, there are no works for the impulsive fractional advection-dispersion equation with nonlinearity involving fractional derivatives of unknown function, although many excellent results about impulsive fractional differential equation are obtained. Secondly, the approach is different from the [6, 9, 12, 16, 17, 22, 23]. The tool of this article is variational method and iterative technique, which has been adopted in [3, 20]. Comparison with [20], the assumed conditions in this paper are different from the conditions in [20], and the result depends on the parameter. Finally, comparisons with [3] and [11], the hypothetical conditions are weaker than those in [3, 11]. For example, functions $\varphi, \psi \in L^2([0,T])$ contain constants $M_1, M_2$, functions $b(t), c(t), d(t), l(t), m(t) \in L^1([0,T])$ contain constants $s_1, s_2, l, m, d$. The parameter $\varrho$ in [11] is a non-negative real, but, the parameter $\varrho$ can be either positive or negative in this paper.

The paper consists of four sections. In sect. 2, we present some preliminaries and lemmas to be used later. In sect. 3, we discuss the existence of nontrivial weak solutions for FADE (1.1). In sect. 4, we take an example to illustrate our main results.

## 2. Preliminaries and lemmas

In this section, some definitions and lemmas are presented, which are to be used to prove our main results.

**Definition 2.1** ( [10]). Let $f$ be a function defined on $[a,b]$. Then the left and right Riemann-Liouville fractional derivatives of order $\gamma > 0$ for function $f$ denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, are represented by

$${}_aD_t^{\gamma} f(t) = \frac{d^n}{dt^n}\, {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\gamma-1} f(s)ds,$$

and

$${}_tD_b^{\gamma} f(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tD_b^{\gamma-n} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_t^b (t-s)^{n-\gamma-1} f(s)ds,$$

for every $t \in [a,b]$, where $n-1 \le \gamma < n$ and $n \in \mathbf{N}$. In particular, if $0 \le \gamma < 1$, then

$${}_aD_t^{\gamma} f(t) = \frac{d}{dt}\, {}_aD_t^{\gamma-1} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_a^t (t-s)^{-\gamma} f(s)ds,\ t \in [a,b],$$

and

$${}_tD_b^{\gamma} f(t) = (-1) \frac{d}{dt}\, {}_tD_b^{\gamma-1} f(t) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_t^b (s-t)^{-\gamma} f(s)ds,\ t \in [a,b].$$

**Definition 2.2** ( [10]). Let $\gamma \ge 0$ and $n \in \mathbb{N}$. If $\gamma \in (n-1, n)$ and $f \in AC^n([a,b], \mathbb{R})$, Then the left and right Caputo fractional derivatives of order $\gamma$ for function $f$ denoted by ${}_a^cD_t^{\gamma} f(t)$ and ${}_t^cD_b^{\gamma} f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^{\gamma} f(t)$ and ${}_t^cD_b^{\gamma} f(t)$ are represented by

$${}_a^cD_t^{\gamma} f(t) = {}_aD_t^{\gamma-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)} \int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)ds,$$

and

$$ {}_t^cD_b^\gamma f(t) = (-1)^n {}_tD_b^{\gamma-n} f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)ds, $$

respectively, where $t \in [a,b]$. In particular, if $0 < \gamma < 1$, then

$$ {}_a^cD_t^\gamma f(t) = {}_aD_t^{\gamma-1} f'(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t (t-s)^{-\gamma} f'(s)ds, \ t \in [a,b], $$

and

$$ {}_t^cD_b^\gamma f(t) = -{}_tD_b^{\gamma-1} f'(t) = -\frac{1}{\Gamma(1-\gamma)} \int_t^b (s-t)^{-\gamma} f'(s)ds, \ t \in [a,b]. $$

If $\gamma = n-1$ and $f \in AC^{n-1}([a,b],\mathbb{R}^N)$, then ${}_a^cD_t^{n-1}f(t)$ and ${}_t^cD_b^{n-1}f(t)$ are represented by ${}_a^cD_t^{n-1}f(t) = f^{(n-1)}(t)$ and ${}_t^cD_b^{n-1}f(t) = (-1)^{n-1}f^{(n-1)}(t), \ t \in [a,b]$. In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t), \ t \in [a,b]$.

**Proposition 2.1** ( [10]). *If $f \in L^p([a,b],\mathbb{R}^N)$, $g \in L^q([a,b],\mathbb{R}^N)$ and $p \geq 1, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1 + \gamma$ or $p \neq 1, q \neq 1, \frac{1}{p} + \frac{1}{q} = 1 + \gamma$, then*

$$ \int_a^b [{}_aD_t^{-\gamma} f(t)] g(t)dt = \int_a^b [{}_tD_b^{-\gamma} g(t)] f(t)dt, \ \gamma > 0. $$

**Definition 2.3.** Let $0 < \alpha \leq 1$. We define the fractional derivative space $J_0^\alpha$ as the completion of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$ \|u\|_\alpha = \Big( \int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \Big)^{\frac{1}{2}}, \ \forall u \in J_0^\alpha. \tag{2.1} $$

**Remark 2.1.** From [9], we know that the fractional derivative space $J_0^\alpha$ is the space of functions $u \in L^2([0,T],\mathbb{R}^N)$ having an $\alpha$-order fractional derivative ${}_0^cD_t^\alpha u(t) \in L^2([0,T],\mathbb{R}^N)$ and $u(0) = u(T) = 0$.

**Proposition 2.2** ( [9]). *Let $0 < \alpha \leq 1$, the fractional derivative space $J_0^\alpha$ is reflexive and separable Banach space.*

**Proposition 2.3** ( [9]). ${}_0D_t^\alpha u(t) = {}_0^cD_t^\alpha u(t), \ {}_tD_T^\alpha u(t) = {}_t^cD_T^\alpha u(t), \ \forall u \in J_0^\alpha, \ t \in [0,T]$.

**Lemma 2.1** (Proposition 3.2, [9]). *Let $\frac{1}{2} < \alpha \leq 1$. For any $x \in J_0^\alpha$, one has*

*(i)*

$$ \|x\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^cD_t^\alpha x\|_{L^2}; \tag{2.2} $$

*(ii)*

$$ \|x\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \|{}_0^cD_t^\alpha x\|_{L^2}. \tag{2.3} $$

By (2.2), we can consider $J_0^\alpha$ under the norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^cD_t^\alpha u(t)|^2 dt\Big)^{\frac{1}{2}},\ \forall u \in J_0^\alpha, \tag{2.4}$$

which is equivalent to (2.1).

**Lemma 2.2** (Proposition 4.1, [9]). *If $\frac{1}{2} < \alpha \leq 1$, then for any $u \in J_0^\alpha$, we have*

*(i)*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leq -\int_0^T \Big(\ {}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t)\Big)dt \leq \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2; \tag{2.5}$$

*(ii)*

$$\int_0^T |{}_t^cD_T^\alpha u(t)|^2 dt \leq \frac{1}{|\cos(\pi\alpha)|^2}\|u\|_\alpha^2. \tag{2.6}$$

**Definition 2.4.** A function $u \in J_0^\alpha$ is known as a weak solution of FADE (1.1) if

$$\begin{aligned} &-\frac{1}{2}\int_0^T \Big(\ {}_0^cD_t^\alpha u(t)\cdot\ {}_t^cD_T^\alpha v(t) +\ {}_t^cD_T^\alpha u(t)\cdot\ {}_0^cD_t^\alpha v(t)\Big)dt + \Sigma_{j=1}^n I_j(u(t_j))v(t_j) \\ =&\int_0^T f(t,u(t),\ {}_0^cD_t^\alpha u(t))v(t)dt + \int_0^T \varrho u(t)v(t)dt \end{aligned}$$

holds for every $v \in J_0^\alpha$.

**Proposition 2.4** ( [9]). *Let $0 < \alpha \leq 1$, $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to $u$ in $J_0^\alpha$, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.*

Define functional $I_w : J_0^\alpha \to \mathbb{R}$ for given $w \in J_0^\alpha$ as

$$\begin{aligned} I_w(u) =& -\frac{1}{2}\int_0^T \Big(\ {}_0^cD_t^\alpha u(t)\cdot\ {}_t^cD_T^\alpha u(t)\Big)dt - \int_0^T F(t,u(t),\ {}_0^cD_t^\alpha w(t))dt \\ &-\frac{1}{2}\varrho\int_0^T u^2(t)dt + \Sigma_{j=1}^n \int_0^{u(t_j)} I_j(s)ds,\ \forall u \in J_0^\alpha, \end{aligned} \tag{2.7}$$

where $F(t,u,z) = \int_0^u f(t,s,z)ds$. Owing to the continuity of $f$ and $I_j$, the functional $I_w \in C^1(J_0^\alpha,\mathbb{R})$ and

$$\begin{aligned} I_w'(u)v =& -\frac{1}{2}\int_0^T \Big(\ {}_0^cD_t^\alpha u(t)\cdot\ {}_t^cD_T^\alpha v(t) +\ {}_0^cD_t^\alpha v(t)\cdot\ {}_t^cD_T^\alpha u(t)\Big)dt \\ &-\int_0^T f(t,u(t),\ {}_0^cD_t^\alpha w(t))v(t)dt - \varrho\int_0^T u(t)v(t)dt \\ &+\Sigma_{j=1}^n I_j(u(t_j))v(t_j),\ \forall u,v \in J_0^\alpha. \end{aligned} \tag{2.8}$$

**Lemma 2.3** (Theorem 3.2, [22]). *If $u \in J_0^\alpha$ is a critical point of $I_u$ in $J_0^\alpha$, i.e. $I_u'(u) = 0$, then, $u$ is a weak solution of FADE (1.1).*

**Definition 2.5** ( [15]). Suppose that $X$ is a Banach space and $\phi \in C^1(X,\mathbb{R})$. We say that $\phi$ satisfies the Palais-Smale (P.S.) condition if any sequence $\{u_n\} \subset X$ such that $\phi(u_n)$ is bounded and $\phi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence in $X$.

**Lemma 2.4** (Theorem 2.2, [19]). *Let $X$ be a real Banach space and $\phi \in C^1(X,\mathbb{R})$ satisfying P.S. condition. Suppose $\phi(0) = 0$, and*

*(i) there are constants $\rho, \beta > 0$ such that $\phi|_{\partial B_\rho} \geq \beta$, where $B_\rho = \{x \in X : \|x\| < \rho\}$;*

*(ii) there is an $e \in X\backslash\overline{B}_\rho$ such that $\phi(e) \leq 0$.*

*Then $\phi$ possesses a critical value $c \geq \beta$. Moreover $c$ can be characterized as*

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} \phi(u),$$

*where $\Gamma = \{g \in C([0,1], X)|g(0) = 0, g(1) = e\}$.*

For convenience, put

$$\begin{aligned}
&A = \frac{T^\alpha}{\Gamma(\alpha+1)}, \quad B = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)\sqrt{2\alpha-1}}, \quad \overline{Q}_j = \frac{Q_j}{\kappa_j+1}B^{\kappa_j+1}, \quad \overline{M}_j = \nu(t_j)M_j,\\
&\overline{N}_1 = N_1\|\nu\|^\iota_{L^\iota}, \quad \overline{B} = B\|N_2(t)\|_{L^{\frac{2}{2-\kappa}}}, \quad \overline{s} = \frac{2-\kappa}{2}(\overline{B})^{\frac{2}{2-\kappa}}\left[\frac{8\kappa\zeta}{(\zeta-2)|\cos(\pi\alpha)|}\right]^{\frac{\kappa}{2-\kappa}},\\
&\overline{d} = \|d(t)\|_{L^{\frac{2-\tau}{2}}} A^\tau, \qquad \overline{l} = \|l(t)\|_{L^{\frac{2-\xi}{2}}}, \qquad \overline{m} = \|m\|_{L^1} + \Sigma^n_{j=1} l_j,\\
&d^* = \frac{2-\tau}{2}\overline{d}^{\frac{2}{2-\tau}}\left[\frac{4\tau}{(\zeta-2)|\cos(\pi\alpha)|}\right]^{\frac{\tau}{2-\tau}}, \qquad l^* = \frac{2-\xi}{2}\overline{l}^{\frac{2}{2-\xi}}\left[\frac{8\xi}{(\zeta-2)|\cos(\pi\alpha)|}\right]^{\frac{\xi}{2-\xi}},\\
&d^*_j = \frac{2-\tau_j}{2}\overline{d}_j^{\frac{2}{2-\tau_j}}\left[\frac{4n\tau_j}{(\zeta-2)|\cos(\pi\alpha)|}\right]^{\frac{\tau_j}{2-\tau_j}}, \quad \widetilde{C} = \max\left\{\widetilde{A}, \widetilde{B}\right\},\\
&P_1 = \left(\frac{\Sigma^n_{j=1} d^*_j + l^* + \zeta\widetilde{C} + \zeta\|h\|_{L^1} + \overline{m} + d^*}{\frac{(\zeta-2)|\cos(\pi\alpha)|}{8} - \varrho\frac{\zeta-2}{2}A^2}\right)^{\frac{1}{2}}, \quad \overline{d}_j = d_j B^{\tau_j},\\
&\widetilde{B} := \widetilde{A}\left(\frac{2\widetilde{A}}{\overline{N}_1\iota}\right)^{\frac{2}{\iota-2}} - \overline{N}_1\left(\frac{2\widetilde{A}}{\overline{N}_1\iota}\right)^{\frac{\iota}{\iota-2}}, \quad \overline{\varrho} = \frac{|\varrho|}{2}A^2,\\
&\widetilde{A} := \left(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho}\right) + \overline{s} + \Sigma^n_{j=1}\overline{Q}_j + \Sigma^n_{j=1}\overline{M}_j \ (j = 1, 2, \ldots, n).
\end{aligned}$$

# 3. Main result

We are now in a position to give some conditions that will be used in the proof of our main result.
$(R_1)$ There exist constants $\delta > 0, \iota > 2,\ \eta > 2,\ 0 < \mu < 2,\ \sigma_j > 1, a_j \geq 0,\ j = 1, 2, \ldots, n$, and functions $b(t), c(t) \in L^1([0,T],\mathbb{R})$ with $b(t) \geq 0$, $c(t) \geq 0$, such that

$$\begin{aligned}
&F(t,x,y) \leq b(t)|x|^\iota + c(t)|x|^\eta|y|^\mu, \text{ for } |x| \leq \delta,\ y \in \mathbb{R},\ a.e.\ t \in [0,T],\\
&I_j(u) \geq -a_j|u|^{\sigma_j},\ j = 1, 2, \ldots, n, \text{ for } |u| \leq \delta;
\end{aligned}$$

$(R_2)$ There exist nonnegative functions $\varphi, \psi \in L^2([0,T],\mathbb{R})$, and constants $\gamma_j > 0$, $j = 1,2,\ldots,n$, such that

$$|f(t,\widetilde{x},\widetilde{y}) - f(t,x,y)| \le \varphi(t)|\widetilde{x}-x| + \psi(t)|\widetilde{y}-y|,\ a.e.\ t\in[0,T],$$

for $x,\widetilde{x} \in [-\widetilde{P}_1, \widetilde{P}_1], \widetilde{y}, y \in \mathbb{R}$, and

$$|I_j(u_1) - I_j(u_2)| \le \gamma_j |u_1 - u_2|,\ j = 1,2,\ldots,n,\ for\ u_1,u_2 \in [-\widetilde{P}_1, \widetilde{P}_1];$$

$(R_3)$ There exist constants $\zeta > 2$, $0 < \tau_j, \tau, \xi < 2$, $d_j \ge 0, l_j \ge 0$, $j = 1,2,\ldots,n$, and functions $d(t), l(t), m(t) \in L^1([0,T],\mathbb{R})$ with $d(t) \ge 0$, $l(t) \ge 0$, $m(t) \ge 0$, such that

$$xf(t,x,y) - \zeta F(t,x,y) \ge -d(t)|x|^{\tau} - l(t)|y|^{\xi} - m(t),\ \text{for } x,y\in\mathbb{R},\ a.e.\ t\in[0,T],$$
$$\zeta\int_0^u I_j(s)ds - uI_j(u) \ge -d_j|u|^{\tau_j} - l_j,\ j=1,2,\ldots,n,\ \text{for } u\in R;$$

$(R_4)$ There exist nonnegative constant $N_1$ and functions $h(t), N_2(t) \in L^1([0,T],\mathbb{R})$ with $h(t) \ge 0$, $N_2(t) \ge 0$, and $\iota > 2$, $0 < \kappa < 1$, $0 < \kappa_j < 1, Q_j \ge 0, M_j \ge 0$, $j = 1,2,\ldots,n$, such that

$$F(t,x,y) \ge N_1|x|^{\iota} - N_2(t)|x||y|^{\kappa} - h(t),\ for\ \ x,y\in\mathbb{R},\ a.e.\ t\in[0,T],$$
$$I_j(u) \le Q_j|u|^{\kappa_j} + M_j,\ j=1,2,\ldots,n,\ for\ 0\le u<\infty.$$

**Theorem 3.1.** *Suppose that* $(R_1)-(R_4)$ *hold, and*

$$-\frac{\|\varphi\|_{L^1} + \|\psi\|_{L^2}B^{-1} + \Sigma_{j=1}^n\gamma_j}{T} < \varrho < \frac{|\cos(\pi\alpha)|}{4A^2}. \tag{3.1}$$

*Then, the FADE* (1.1) *has a nontrivial weak solution.*

**Proof.** We give the proof of this theorem by five steps.

*Step* 1. We certificate that there exist $\omega_1, \rho > 0$ such that $I_w(u) \ge \omega_1$ for $u \in \{u \in J_0^\alpha : \|u\|_\alpha = \rho\}$.

For given $w \in J_0^\alpha$ with $\|w\|_\alpha \le P_1$. Choose $\widetilde{\delta} = \frac{\Gamma(\alpha)\sqrt{2\alpha-1}\delta}{T^{\alpha-\frac{1}{2}}}$, for any $u \in J_0^\alpha$ with $\|u\|_\alpha \le \widetilde{\delta}$, by (2.3), we have $|u(t)| \le \delta$. In view of $(R_1)$, (2.2), (2.3), (2.5)–(2.7), and Holder's inequality, we have

$$\begin{aligned} I_w(u) =& -\frac{1}{2}\int_0^T \Big({}_0^cD_t^\alpha u(t)\cdot {}_t^cD_T^\alpha u(t)\Big)dt - \int_0^T F(t,u(t),{}_0^cD_t^\alpha w(t))dt \\ &-\frac{1}{2}\varrho\int_0^T u^2(t)dt + \Sigma_{j=1}^n\int_0^{u(t_j)} I_j(s)ds \\ \ge& \frac{1}{2}|\cos(\pi\alpha)||{}_0^cD_t^\alpha u(t)|_{L^2}^2 - \int_0^T b(t)|u|^\iota dt - \int_0^T c(t)|u|^\eta|{}_0^cD_t^\alpha w(t)|^\mu dt \\ &-\frac{1}{2}\varrho\|u(t)\|_{L^2}^2 - \Sigma_{j=1}^n\frac{a_j}{\sigma_j+1}\|u\|_\infty^{\sigma_j+1} \\ \ge& \frac{1}{2}|\cos(\pi\alpha)|\|u\|_\alpha^2 - B^\iota\|u\|_\alpha^\iota\|b\|_{L^1} - B^\eta\|u\|_\alpha^\eta\|c\|_{L^{\frac{2}{2-\mu}}}P_1^\mu \\ &-\frac{\varrho}{2}A^2\|u\|_\alpha^2 - \Sigma_{j=1}^n\frac{a_j}{\sigma_j+1}B^{\sigma_j+1}\|u\|_\alpha^{\sigma_j+1}, \forall u\in J_0^\alpha. \end{aligned} \tag{3.2}$$

Since $\iota, \eta > 2$, $0 < \mu < 2$, $\sigma_j > 1$, $j = 1, 2, \ldots, n$, and (3.1), we can take $\rho$ small enough, then we can gain a constant $\omega_1, \rho > 0$ such that $I_w(u) \geq \omega_1$ with $\|u\|_\alpha = \rho$.

*Step* 2. We testify that there exists $e \in J_0^\alpha$ with $\|e\|_\alpha > \rho$ such that $I_w(e) < 0$.

Choosing $\nu = \frac{\widehat{u}}{\|\widehat{u}\|}$, where

$$\widehat{u}(t) = \begin{cases} \dfrac{4t}{T}, & t \in \Big[0, \dfrac{T}{4}\Big), \\ 1, & t \in \Big[\dfrac{T}{4}, \dfrac{3T}{4}\Big), \\ \dfrac{4(T-t)}{T}, & t \in \Big[\dfrac{3T}{4}, T\Big]. \end{cases}$$

For fixed $\nu \in J_0^\alpha$, we have $\nu(t) > 0, \forall t \in (0,T)$, and $\|\nu\|_\alpha = 1$. For $\Lambda > 0$, owing to $(R_4)$, (2.2), (2.3), (2.5)–(2.7), and Holder's inequality, we obtain that

$$\begin{aligned} I_w(\Lambda\nu(t)) =& -\frac{1}{2}\Lambda^2 \int_0^T \Big( {}_0^cD_t^\alpha \nu(t) \cdot {}_t^cD_T^\alpha \nu(t)\Big) dt - \int_0^T F(t, \Lambda\nu(t), {}_0^cD_t^\alpha w(t)) dt \\ & - \frac{1}{2}\varrho\Lambda^2 \int_0^T \nu^2(t) dt + \Sigma_{j=1}^n \int_0^{\Lambda\nu(t_j)} I_j(s) ds \\ \leq & \frac{1}{2}\Lambda^2 \frac{1}{|\cos(\pi\alpha)|} |{}_0^cD_t^\alpha \nu(t)|_{L^2}^2 - \int_0^T N_1 |\Lambda\nu|^\iota dt \\ & + \int_0^T N_2(t) |\Lambda\nu| |{}_0^cD_t^\alpha w(t)|^\kappa dt + \int_0^T h(t) dt - \frac{\varrho}{2}\Lambda^2 \|\nu\|_{L^2}^2 \\ & + \Sigma_{j=1}^n \Big( \frac{Q_j}{\kappa_j + 1} \Lambda^{\kappa_j+1} \|\nu\|_\infty^{\kappa_j+1} + \Lambda\nu(t_j) M_j \Big) \\ \leq & \frac{1}{2}\Lambda^2 \frac{1}{|\cos(\pi\alpha)|} \|\nu\|_\alpha^2 - N_1\Lambda^\iota \|\nu\|_{L^\iota}^\iota + \Lambda\|\nu\|_\infty \|N_2(t)\|_{L^{\frac{2}{2-\kappa}}} P_1^\kappa + \|h\|_{L^1} \\ & - \frac{\varrho}{2}\Lambda^2 \|\nu\|_{L^2}^2 + \Sigma_{j=1}^n \Big( \frac{Q_j}{\kappa_j+1} \Lambda^{\kappa_j+1} B^{\kappa_j+1} \|\nu\|_\alpha^{\kappa_j+1} + \Lambda\nu(t_j) M_j \Big) \\ \leq & \frac{1}{2}\Lambda^2 \frac{1}{|\cos(\pi\alpha)|} \|\nu\|_\alpha^2 - N_1\Lambda^\iota \|\nu\|_{L^\iota}^\iota + \Lambda B \|\nu\|_\alpha \|N_2(t)\|_{L^{\frac{2}{2-\kappa}}} P_1^\kappa + \|h\|_{L^1} \\ & - \frac{\varrho}{2}\Lambda^2 \|\nu\|_{L^2}^2 + \Sigma_{j=1}^n \Big( \frac{Q_j}{\kappa_j+1} \Lambda^{\kappa_j+1} B^{\kappa_j+1} \|\nu\|_\alpha^{\kappa_j+1} + \Lambda\nu(t_j) M_j \Big). \end{aligned} \tag{3.3}$$

Due to $\iota > 2$, $0 < \kappa < 1$, $0 < \kappa_j < 1$, we know that there exists a $\Lambda_0 > 0$ large enough such that $I_w(\Lambda_0\nu(t)) < 0$ with $\|\Lambda_0\nu\|_\alpha > \rho$. Choice $e(t) = \Lambda_0\nu(t)$, then $I_w(e(t)) < 0$ with $\|e\|_\alpha > \rho$. Clealy, $I_w(0) = 0$.

*Step* 3. We show $I_w$ satisfies P.S. condition. Let $\{u_n\} \subset J_0^\alpha$ be a P.S. sequence, that is $\{I_w(u_n)\}$ is bounded and $I_w'(u_n) \to 0$ as $n \to \infty$. From $(R_3)$, (2.2), (2.3), (2.5)–(2.7), and Holder's inequality, we assert

$$\begin{aligned} & \zeta I_w(u_n) - I_w'(u_n) u_n \qquad (3.4) \\ =& -\frac{1}{2}\zeta \int_0^T \Big( {}_0^cD_t^\alpha u_n(t) \cdot {}_t^cD_T^\alpha u_n(t) \Big) dt - \zeta \int_0^T F(t, u_n(t), {}_0^cD_t^\alpha w(t)) dt \\ & - \frac{1}{2}\zeta\varrho \int_0^T u_n^2(t) dt + \zeta\Sigma_{j=1}^n \int_0^{u_n(t_j)} I_j(s) ds + \int_0^T \Big( {}_0^cD_t^\alpha u_n(t) \cdot {}_t^cD_T^\alpha u_n(t) \Big) dt \\ & + \int_0^T f(t, u_n(t), {}_0^cD_t^\alpha w(t)) u_n(t) dt + \varrho \int_0^T u_n^2(t) dt - \Sigma_{j=1}^n I_j(u_n(t_j)) u_n(t_j) \end{aligned}$$

$$
\begin{aligned}
=&\Big(1-\frac{1}{2}\zeta\Big)\int_0^T\Big({}_0^cD_t^\alpha u_n(t)\cdot{}_t^cD_T^\alpha u_n(t)\Big)dt+\varrho\Big(1-\frac{\zeta}{2}\Big)\int_0^T u_n^2(t)dt\\
&+\int_0^T f(t,u_n(t),\,{}_0^cD_t^\alpha w(t))u_n(t)-\zeta F(t,u_n(t),\,{}_0^cD_t^\alpha w(t))dt\\
&+\Sigma_{j=1}^n\Big(\zeta\int_0^{u_n(t_j)} I_j(s)ds-I_j(u_n(t_j))u_n(t_j)\Big)\\
\geq&\Big(\frac{1}{2}\zeta-1\Big)|\cos(\pi\alpha)|\|{}_0^cD_t^\alpha u_n(t)|_{L^2}^2-\Big(\int_0^T d(t)|u_n(t)|^\tau dt+\int_0^T l(t)|{}_0^cD_t^\alpha w(t)|^\xi dt\\
&+\int_0^T m(t)dt\Big)-\varrho\frac{\zeta-2}{2}\|u_n(t)\|_{L^2}^2-\Sigma_{j=1}^n(d_j|u_n(t_j)|^{\tau_j}+l_j)\\
\geq&\Big(\frac{1}{2}\zeta-1\Big)|\cos(\pi\alpha)|\|u_n\|_\alpha^2-(\|d(t)\|_{L^{\frac{2-\tau}{2}}}A^\tau\|u_n\|_\alpha^\tau+\|l(t)\|_{L^{\frac{2-\xi}{2}}}\|w\|_\alpha^\xi+\|m\|_{L^1})\\
&-\varrho\frac{\zeta-2}{2}A^2\|u_n\|_\alpha^2-\Big(\Sigma_{j=1}^n d_jB^{\tau_j}\|u_n\|_\alpha^{\tau_j}+\Sigma_{j=1}^n l_j\Big),\ \forall u_n\in J_0^\alpha.
\end{aligned}
$$

In view of $0<\tau_j,\tau,\xi<2$, $\zeta>2$, and $\{I_w(u_n)\}$ is bounded and $I_w'(u_n)\to 0$ as $n\to\infty$, we know that $\{u_n\}\subset J_0^\alpha$ is bounded. Moreover, it has a weakly convergent subsequence $u_{n_{k_i}}\rightharpoonup u\in J_0^\alpha$ in view of the reflexivity of $J_0^\alpha$. It follows from Proposition 2.4 we know that $u_n\to u$ in $C[0,T]$. We still denote $\{u_{n_{k_i}}\}$ by $\{u_n\}$. Since $f$ and $I_j$ $(j=1,2,\ldots,n)$ are continuous, and $u_n\to u$ in $C[0,T]$, we have

$$
\begin{cases}
\displaystyle\int_0^T(u_n(t)-u(t))^2dt\to 0,\\
\Sigma_{j=1}^n[I_j(u_n(t_j))-I_j(u(t_j))](u_n(t_j)-u(t_j))\to 0,\\
\displaystyle\int_0^T[f(t,u_n(t),\,{}_0^cD_t^\alpha w(t))-f(t,u(t),\,{}_0^cD_t^\alpha w(t))](u_n(t)-u(t))\to 0.
\end{cases}
\tag{3.5}
$$

In view of the fact that $I_w'(u_n)\to 0, u_n\rightharpoonup u$ as $n\to\infty$, the boundedness of the sequence $\{u_n-u\}$, we obtain

$$|(I_w'(u_n)-I_w'(u))(u_n-u)|\leq|I_w'(u_n)|\|u_n-u\|+|I_w'(u)(u_n-u)|\to 0$$

as $n\to\infty$. Thus, we observe that

$$
\begin{aligned}
|\cos(\pi\alpha)|\|u_n-u\|_\alpha^2\leq&-\int_0^T\Big({}_0^cD_t^\alpha(u_n(t)-u(t))\cdot{}_t^cD_T^\alpha(u_n(t)-u(t))\Big)dt\\
=&(I_w'(u_n)-I_w'(u))(u_n-u)+\varrho\int_0^T(u_n(t)-u(t))^2dt\\
&-\Sigma_{j=1}^n[I_j(u_n(t_j))-I_j(u(t_j))](u_n(t_j)-u(t_j))\\
&+\int_0^T[f(t,u_n(t),\,{}_0^cD_t^\alpha w(t))-f(t,u(t),\,{}_0^cD_t^\alpha w(t))]\\
&(u_n(t)-u(t))dt\\
\to&0,
\end{aligned}
\tag{3.6}
$$

as $n\to\infty$. So, $u_n\to u$ in $J_0^\alpha$. Hence, functional $I_w$ satisfies the P.S. condition. From Lemma 2.4, we know that there exists a point $\widehat{u}\in J_0^\alpha$ satisfying $I_w'(\widehat{u})=0$ and $I_w(\widehat{u})\geq\omega_1>0$.

*Step* 4. we prove that we can construct a sequence $\{u_n\}$ in $J_0^\alpha$ satisfying $I'_{u_{n-1}}(u_n) = 0$ and $I_{u_{n-1}}(u_n) \geq \omega_1 > 0$ with $\|u_n\|_\alpha \leq P_1$, $n \in N$. For a given $u_1 \in J_0^\alpha$ with $\|u_1\|_\alpha \leq P_1$, from the previous conclusions, we know that $I_{u_1}$ has a nontrivial critical point $u_2$. If we can show $\|u_2\|_\alpha \leq P_1$, then, from the previous conclusions, we also obtain $I_{u_2}$ has a nontrivial critical point $u_3$. So, by the same process as above, we suppose $\|u_{n-1}\|_\alpha \leq P_1$, we can get the nontrivial critical point $u_n$ of $I_{u_{n-1}}$, and $\|u_n\|_\alpha \leq P_1$. Now, we show $\|u_2\|_\alpha \leq P_1$. On the basis of (3.3), we have

$$\begin{aligned}
I_{u_1}(u_2) \leq & \max_{\Lambda\in[0,\infty)} I_{u_1}(\Lambda\nu(t)) \\
\leq & \max_{\Lambda\in[0,\infty)} \frac{1}{2}\Lambda^2 \frac{1}{|\cos(\pi\alpha)|} - N_1\Lambda^\iota \|\nu\|_{L^\iota}^\iota + \Lambda B \|N_2(t)\|_{L^{\frac{2}{2-\kappa}}} P_1^\kappa + \|h\|_{L^1} \\
& - \frac{\varrho}{2}\Lambda^2\|\nu\|_{L^2}^2 + \Sigma_{j=1}^n \Big(\frac{Q_j}{\kappa_j+1}\Lambda^{\kappa_j+1}B^{\kappa_j+1} + \Lambda\nu(t_j)M_j\Big) \\
\leq & \max_{\Lambda\in[0,\infty)} \Lambda^2\Big(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho}\Big) - \overline{N}_1\Lambda^\iota + \Lambda\overline{B}P_1^\kappa + \|h\|_{L^1} \\
& + \Sigma_{j=1}^n \overline{Q}_j \Lambda^{\kappa_j+1} + \Sigma_{j=1}^n \Lambda\overline{M}_j,
\end{aligned} \tag{3.7}$$

where $\overline{Q}_j = \frac{Q_j}{\kappa_j+1}B^{\kappa_j+1}$, $\overline{M}_j = \nu(t_j)M_j$, $\overline{\varrho} = \frac{|\varrho|}{2}A^2$, $\overline{N}_1 = N_1\|\nu\|_{L^\iota}^\iota$, $\overline{B} = B \times \|N_2(t)\|_{L^{\frac{2}{2-\kappa}}}$. By Young inequality, we have

$$\begin{aligned}
\Lambda\overline{B}P_1^\kappa \leq & \frac{1}{q'}\Big(\frac{1}{\varepsilon_0}\Lambda\overline{B}\Big)^{q'} + \frac{1}{p'}\Big(\varepsilon_0 P_1^\kappa\Big)^{p'} \\
= & \frac{2-\kappa}{2}(\Lambda\overline{B})^{\frac{2}{2-\kappa}}\Big[\frac{8\kappa\zeta}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\kappa}{2-\kappa}} + \frac{\kappa}{2}\frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{4\kappa\zeta}P_1^2,
\end{aligned}$$

where $p' = \frac{2}{\kappa}, q' = \frac{2}{2-\kappa}$, and $\varepsilon_0 = \Big(\frac{(\zeta-2)|\cos(\pi\alpha)|}{8\kappa\zeta}\Big)^{\frac{\kappa}{2}}$.

Denote $\overline{s} = \frac{2-\kappa}{2}(\overline{B})^{\frac{2}{2-\kappa}}\Big[\frac{8\kappa\zeta}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\kappa}{2-\kappa}}$, from (3.7), we have

$$\begin{aligned}
I_{u_1}(u_2) \leq & \max_{\Lambda\in[0,\infty)} \Lambda^2\Big(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho}\Big) - \overline{N}_1\Lambda^\iota + \Lambda^{\frac{2}{2-\kappa}}\overline{s} + \frac{\Big(\frac{\zeta-2}{2}\Big)|\cos(\pi\alpha)|}{8\zeta}P_1^2 \\
& + \|h\|_{L^1} + \Sigma_{j=1}^n \overline{Q}_j\Lambda^{\kappa_j+1} + \Sigma_{j=1}^n \Lambda\overline{M}_j.
\end{aligned} \tag{3.8}$$

Put

$$Z(\Lambda) = \Lambda^2\Big(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho}\Big) - \overline{N}_1\Lambda^\iota + \Lambda^{\frac{2}{2-\kappa}}\overline{s} + \Sigma_{j=1}^n \overline{Q}_j\Lambda^{\kappa_j+1} + \Sigma_{j=1}^n \Lambda\overline{M}_j. \tag{3.9}$$

Then,

$$I_{u_1}(u_2) \leq \max_{\Lambda\in[0,\infty)} Z(\Lambda) + \|h\|_{L^1} + \frac{\Big(\frac{\zeta-2}{2}\Big)|\cos(\pi\alpha)|}{8\zeta}P_1^2. \tag{3.10}$$

If $0 \leq \Lambda < 1$, and $0 < \kappa, \kappa_j < 1$, then

$$Z(\Lambda) \leq \Big(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho}\Big) + \overline{s} + \Sigma_{j=1}^n \overline{Q}_j + \Sigma_{j=1}^n \overline{M}_j := \widetilde{A}. \tag{3.11}$$

If $1 \le \Lambda < \infty$, $0 < \kappa, \kappa_j < 1$, $\iota > 1$, then

$$\begin{aligned} Z(\Lambda) \le& \Lambda^2\Big(\frac{1}{2}\frac{1}{|\cos(\pi\alpha)|} + \overline{\varrho} + \overline{s} + \Sigma_{j=1}^n \overline{Q}_j + \Sigma_{j=1}^n \overline{M}_j\Big) - \overline{N}_1\Lambda^\iota \\ =& \widetilde{A}\Lambda^2 - \overline{N}_1\Lambda^\iota := \widetilde{Z}(\Lambda), \end{aligned} \tag{3.12}$$

and $\widetilde{Z}'(\Lambda) = 2\Lambda\widetilde{A} - \iota\overline{N}_1\Lambda^{\iota-1}$. By a simple calculation, we know that $\widetilde{Z}(\Lambda)$ has a maximum at $\overline{\Lambda} = \left(\frac{2\widetilde{A}}{\overline{N}_1\iota}\right)^{\frac{1}{\iota-2}}$, and $\widetilde{Z}(\overline{\Lambda}) = \max_{\Lambda\in[0,\infty)} Z(\Lambda) := \widetilde{B}$.

Choose $\widetilde{C} = \max\left\{\widetilde{A}, \widetilde{B}\right\}$, we derive

$$I_{u_1}(u_2) \le \widetilde{C} + \|h\|_{L^1} + \frac{\left(\frac{\zeta-2}{2}\right)|\cos(\pi\alpha)|}{8\zeta} P_1^2. \tag{3.13}$$

For another, by (3.4),

$$\begin{aligned} &\zeta I_{u_1}(u_2) - I'_{u_1}(u_2)u_2 \\ \ge& \Big(\frac{1}{2}\zeta - 1\Big)|\cos(\pi\alpha)|\|u_2\|_\alpha^2 - \Big(\|d(t)\|_{L^{\frac{2-\tau}{2}}} A^\tau \|u_2\|_\alpha^\tau + \|l(t)\|_{L^{\frac{2-\xi}{2}}} P_1^\xi + \|m\|_{L^1}\Big) \\ &- \varrho\frac{\zeta-2}{2}A^2\|u_2\|_\alpha^2 - \Big(\Sigma_{j=1}^n d_j B^{\tau_j}\|u_2\|_\alpha^{\tau_j} + \Sigma_{j=1}^n l_j\Big). \end{aligned} \tag{3.14}$$

In view of $I'_{u_1}(u_2) = 0$, and (3.13), we have

$$\begin{aligned} &\Big[\Big(\frac{1}{2}\zeta - 1\Big)|\cos(\pi\alpha)| - \varrho\frac{\zeta-2}{2}A^2\Big]\|u_2\|_\alpha^2 \\ \le& \zeta\Big(\widetilde{C} + \|h\|_{L^1} + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{8\zeta}P_1^2\Big) + \Big(\Sigma_{j=1}^n d_j B^{\tau_j}\|u_2\|_\alpha^{\tau_j} + \Sigma_{j=1}^n l_j\Big) \\ &+ \Big(\|d(t)\|_{L^{\frac{2-\tau}{2}}} A^\tau\|u_2\|_\alpha^\tau + \|l(t)\|_{L^{\frac{2-\xi}{2}}} P_1^\xi + \|m\|_{L^1}\Big) \\ \le& \frac{\left(\frac{\zeta-2}{2}\right)|\cos(\pi\alpha)|}{8}P_1^2 + \overline{d}\|u_2\|_\alpha^\tau + \overline{l}P_1^\xi + \zeta\widetilde{C} + \overline{m} + \zeta\|h\|_{L^1} + \Sigma_{j=1}^n \overline{d}_j\|u_2\|_\alpha^{\tau_j}, \end{aligned} \tag{3.15}$$

where $\overline{d} = \|d(t)\|_{L^{\frac{2-\tau}{2}}} A^\tau$, $\overline{l} = \|l(t)\|_{L^{\frac{2-\xi}{2}}}$, $\overline{d}_j = d_j B^{\tau_j}$, $\overline{m} = \|m\|_{L^1} + \Sigma_{j=1}^n l_j$. By Young inequality, we have

$$\begin{aligned} \overline{d}\|u_2\|_\alpha^\tau \le& \frac{2-\tau}{2}\overline{d}^{\frac{2}{2-\tau}}\Big[\frac{4\tau}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\tau}{2-\tau}} + \frac{\left(\frac{\zeta-2}{2}\right)|\cos(\pi\alpha)|}{4}\|u_2\|_\alpha^2 \\ =& d^* + \frac{\left(\frac{\zeta-2}{2}\right)|\cos(\pi\alpha)|}{4}\|u_2\|_\alpha^2, \end{aligned}$$

$$\begin{aligned} \overline{l}P_1^\xi \le& \frac{2-\xi}{2}\overline{l}^{\frac{2}{2-\xi}}\Big[\frac{8\xi}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\xi}{2-\xi}} + \frac{\left(\frac{\zeta-2}{2}\right)|\cos(\pi\alpha)|}{8}P_1^2 \\ =& l^* + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{8}P_1^2, \end{aligned}$$

$$\begin{aligned}\overline{d}_j\|u_2\|_\alpha^{\tau_j} &\leq \frac{2-\tau_j}{2}\overline{d}_j^{\frac{2}{2-\tau_j}}\Big[\frac{4n\tau_j}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\tau_j}{2-\tau_j}} + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{4n}\|u_2\|_\alpha^2\\ &= d_j^* + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{4n}\|u_2\|_\alpha^2,\end{aligned}$$

where $d^* = \frac{2-\tau}{2}\overline{d}^{\frac{2}{2-\tau}}\Big[\frac{4\tau}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\tau}{2-\tau}}$, $l^* = \frac{2-\xi}{2}\overline{l}^{\frac{2}{2-\xi}}\Big[\frac{8\xi}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\xi}{2-\xi}}$, $d_j^* = \frac{2-\tau_j}{2}\overline{d}_j^{\frac{2}{2-\tau_j}} \cdot \Big[\frac{4n\tau_j}{(\zeta-2)|\cos(\pi\alpha)|}\Big]^{\frac{\tau_j}{2-\tau_j}}$. So, by (3.15), we have

$$\begin{aligned}&\Big[\Big(\frac{1}{2}\zeta-1\Big)|\cos(\pi\alpha)| - \varrho\frac{\zeta-2}{2}A^2\Big]\|u_2\|_\alpha^2\\ \leq&\frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{8}P_1^2 + \Sigma_{j=1}^n d_j^* + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{4}\|u_2\|_\alpha^2 + l^*\\ &+\frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{8}P_1^2 + \zeta\widetilde{C} + \zeta\|h\|_{L^1} + \overline{m} + d^* + \frac{(\frac{\zeta-2}{2})|\cos(\pi\alpha)|}{4}\|u_2\|_\alpha^2, \quad (3.16)\end{aligned}$$

that is,

$$\begin{aligned}&\Big[\Big(\frac{\zeta-2}{4}\Big)|\cos(\pi\alpha)| - \varrho\frac{\zeta-2}{2}A^2\Big]\|u_2\|_\alpha^2\\ \leq&\frac{(\zeta-2)|\cos(\pi\alpha)|}{8}P_1^2 + \Sigma_{j=1}^n d_j^* + l^* + \zeta\widetilde{C} + \zeta\|h\|_{L^1} + \overline{m} + d^*. \quad (3.17)\end{aligned}$$

So, it follows from (3.17), we have

$$\begin{aligned}\|u_2\|_\alpha^2 \leq&\Big(\frac{(\zeta-2)|\cos(\pi\alpha)|}{8}P_1^2 + \Sigma_{j=1}^n d_j^* + l^* + \zeta\widetilde{C} + \zeta\|h\|_{L^1} + \overline{m} + d^*\Big)\\ &\cdot\frac{1}{\Big[(\frac{\zeta-2}{4})|\cos(\pi\alpha)| - \varrho\frac{\zeta-2}{2}A^2\Big]}. \quad (3.18)\end{aligned}$$

Since

$$P_1 = \Big(\frac{\Sigma_{j=1}^n d_j^* + l^* + \zeta\widetilde{C} + \zeta\|h\|_{L^1} + \overline{m} + d^*}{\frac{(\zeta-2)|\cos(\pi\alpha)|}{8} - \varrho\frac{\zeta-2}{2}A^2}\Big)^{\frac{1}{2}},$$

so we have $\|u_2\|_\alpha \leq P_1$. So, by the same process as above, we suppose $\|u_{n-1}\|_\alpha \leq P_1$, we can get the nontrivial critical point $u_n$ of $I_{u_{n-1}}$, and $\|u_n\|_\alpha \leq P_1$.

*Step* 5. We certificate that the iterative sequence $\{u_n\}$ constructed in the previous step is convergent to a nontrivial weak solution $\widetilde{u}$ of FADE (1.1). Assume that the sequence $\{u_n\}$ is divergent on $J_0^\alpha$, that is, there exists a number $\widetilde{\varepsilon} > 0$, for any positive number $N$ such that for each $n, n+1 > N$, we have $\|u_{n+1} - u_n\|_\alpha \geq \widetilde{\varepsilon}$.

In view of $\|u_n\|_\alpha \leq P_1$, and Lemma 2.1, we have $\|u_n\|_\infty \leq BP_1 = \widetilde{P}_1$. Thus, from $(R_2)$, we get

$$\begin{aligned}&\Big|\int_0^T [f(t, u_{n+1}(t),\ {}_0^cD_t^\alpha u_n(t)) - f(t, u_n(t),\ {}_0^cD_t^\alpha u_{n-1}(t))](u_{n+1}(t) - u_n(t))dt\Big|\\ \leq&\int_0^T [\varphi(t)|u_{n+1}(t) - u_n(t)| + \psi(t)|{}_0^cD_t^\alpha (u_n - u_{n-1})(t)|]|u_{n+1}(t) - u_n(t)|dt\end{aligned}$$

$$
\begin{aligned}
&\le \|u_{n+1}(t)-u_n(t)\|_\infty\Big(|u_{n+1}(t)-u_n(t)|_\infty\int_0^T\varphi(t)\\
&\quad+\int_0^T\psi(t)|{}_0^cD_t^\alpha(u_n-u_{n-1})(t)|dt\Big)\\
&\le \|u_{n+1}(t)-u_n(t)\|_\infty(\|\varphi\|_{L^1}|u_{n+1}(t)-u_n(t)|_\infty+\|\psi\|_{L^2}\|{}_0^cD_t^\alpha(u_n-u_{n-1})\|_{L^2})\\
&\le \|u_{n+1}(t)-u_n(t)\|_\infty\Big(2\|\varphi\|_{L^1}\widetilde{P}_1+2\|\psi\|_{L^2}\frac{\widetilde{P}_1}{B}\Big),
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&|I_j(u_{n+1}(t_j))-I_j(u_n(t_j))||u_{n+1}(t_j)-u_n(t_j)|\\
\le&\gamma_j|u_{n+1}(t_j)-u_n(t_j)|^2\\
\le&\gamma_j\|u_{n+1}-u_n\|_\infty^2\\
\le&2\gamma_j\|u_{n+1}-u_n\|_\infty\widetilde{P}_1,\ \ j=1,2,\dots,n.
\end{aligned}
\tag{3.20}
$$

Since $I'_{u_n}(u_{n+1})(u_{n+1}-u_n)=0, I'_{u_{n-1}}(u_n)(u_{n+1}-u_n)=0$, and combing with (3.19), (3.20), we observe that

$$
\begin{aligned}
&|\cos(\pi\alpha)|\|u_{n+1}-u_n\|_\alpha^2\\
\le&-\int_0^T\Big(\ {}_0^cD_t^\alpha(u_{n+1}(t)-u_n(t))\cdot\ {}_t^cD_T^\alpha(u_{n+1}(t)-u_n(t)\Big)dt\\
=&(I'_{u_n}(u_{n+1})-I'_{u_{n-1}}(u_n))(u_{n+1}-u_n)+\varrho\int_0^T(u_{n+1}(t)-u_n(t))^2dt\\
&-\Sigma_{j=1}^n[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))](u_{n+1}(t_j)-u_n(t_j))\\
&+\int_0^T f(t,u_{n+1}(t),\ {}_0^cD_t^\alpha u_n(t))-f(t,u_n(t),\ {}_0^cD_t^\alpha u_{n-1}(t))(u_{n+1}(t)-u_n(t))\\
\le&2\|u_{n+1}-u_n\|_\infty\Big(\|\varphi\|_{L^1}\widetilde{P}_1+\|\psi\|_{L^2}\frac{\widetilde{P}_1}{B}\Big)+2\Sigma_{j=1}^n\gamma_j\|u_{n+1}-u_n\|_\infty\widetilde{P}_1\\
&+2\varrho T\|u_{n+1}-u_n\|_\infty\widetilde{P}_1\\
=&2\|u_{n+1}-u_n\|_\infty\Big(\widetilde{P}_1\varrho T+\|\varphi\|_{L^1}\widetilde{P}_1+\|\psi\|_{L^2}\frac{\widetilde{P}_1}{B}+\widetilde{P}_1\Sigma_{j=1}^n\gamma_j\Big),
\end{aligned}
\tag{3.21}
$$

as $n\to\infty$. Put $\varsigma=2\Big(\widetilde{P}_1\varrho T+\|\varphi\|_{L^1}\widetilde{P}_1+\|\psi\|_{L^2}\frac{\widetilde{P}_1}{B}+\widetilde{P}_1\Sigma_{j=1}^n\gamma_j\Big)$, which implies that $\|u_{n+1}-u_n\|_\infty\ge\frac{|\cos(\pi\alpha)|}{\varsigma}\|u_{n+1}-u_n\|_\alpha^2\ge\frac{|\cos(\pi\alpha)|}{\varsigma}\widetilde{\varepsilon}^2=\varepsilon$, that is, there exists a number $\varepsilon>0$, for any positive number $N$ such that for each $n,n+1>N$, we have $\|u_{n+1}-u_n\|_\infty\ge\varepsilon$. It is contradict with the fact that $\{u_n(t)\}$ strongly converges to $\widetilde{u}$ in $C([0,T],\mathbb{R})$ as $k\to\infty$.

Next, we claim that $I'_{\widetilde{u}}(\widetilde{u})=0$. In fact, by Lemma 2.2, we derive

$$
\begin{aligned}
&\int_0^T\Big(\ {}_0^cD_t^\alpha(u_n-\widetilde{u})(t)\cdot\ {}_t^cD_T^\alpha v(t)+\ {}_0^cD_t^\alpha v(t)\cdot\ {}_t^cD_T^\alpha(u_n-\widetilde{u})(t)\Big)dt\\
\le&\|{}_0^cD_t^\alpha(u_n-\widetilde{u})\|_{L^2}\|{}_t^cD_T^\alpha v\|_{L^2}+\|{}_0^cD_t^\alpha v\|_{L^2}\|{}_t^cD_T^\alpha(u_n-\widetilde{u})\|_{L^2}\\
\le&\|u_n-\widetilde{u}\|_\alpha\|\frac{1}{|\cos(\pi\alpha)|}\|v\|_\alpha+\|v\|_\alpha\|\|u_n-\widetilde{u}\|_\alpha\|\frac{1}{|\cos(\pi\alpha)|}\\
=&2\|u_n-\widetilde{u}\|_\alpha\|\frac{1}{|\cos(\pi\alpha)|}\|v\|_\alpha.
\end{aligned}
\tag{3.22}
$$

Moreover, $u_n \to \widetilde{u}$ in $J_0^\alpha$ means that

$$\begin{aligned}&\int_0^T \Big( {}_0^cD_t^\alpha u_n(t)\cdot {}_t^cD_T^\alpha v(t) + {}_0^cD_t^\alpha v(t)\cdot {}_t^cD_T^\alpha u_n(t)\Big)dt \\ \to &\int_0^T \Big( {}_0^cD_t^\alpha \widetilde{u}(t)\cdot {}_t^cD_T^\alpha v(t) + {}_0^cD_t^\alpha v(t)\cdot {}_t^cD_T^\alpha \widetilde{u}(t)\Big)dt. \end{aligned}\tag{3.23}$$

Cleary,

$$\Sigma_{j=1}^n I_j(u_n(t_j))v(t_j) \to \Sigma_{j=1}^n I_j(\widetilde{u}(t_j))v(t_j). \tag{3.24}$$

From $(R_2)$, we have

$$\begin{aligned}&\Big|\int_0^T [f(t,u_n(t),\ {}_0^cD_t^\alpha u_{n-1}(t)) - f(t,\widetilde{u}(t),\ {}_0^cD_t^\alpha \widetilde{u}(t))]v(t))dt\Big| \\ \le &\int_0^T [\varphi(t)|u_n(t)-\widetilde{u}(t)| + \psi(t)|{}_0^cD_t^\alpha(u_{n-1}-\widetilde{u})(t)|]|v(t)|dt \\ \le &\|v\|_\infty(\|u_n-\widetilde{u}\|_{L^2}\|\varphi\|_{L^2} + \|\psi\|_{L^2}\|{}_0^cD_t^\alpha(u_{n-1}-\widetilde{u})\|_{L^2} \\ \le &\|v\|_\infty(\|\varphi\|_{L^2}A\|u_n-\widetilde{u}\|_\alpha + \|\psi\|_{L^2}\|u_{n-1}-\widetilde{u}\|_\alpha).\end{aligned}$$

Therefore, by $u_n \to \widetilde{u}$ in $J_0^\alpha$, we have

$$f(t,u_n(t),\ {}_0^cD_t^\alpha u_{n-1}(t)) \to f(t,\widetilde{u}(t),\ {}_0^cD_t^\alpha \widetilde{u}(t))), \tag{3.25}$$

as $n\to\infty$. In addition,

$$\begin{aligned}\varrho\Big|\int_0^T u_n(t)v(t)dt - \int_0^T \widetilde{u}(t)v(t)dt\Big| &\le \varrho\int_0^T |u_n(t)-\widetilde{u}(t)||v(t)|dt \\ &\le \varrho A^2\|u_n-\widetilde{u}\|_\alpha\|v\|_\alpha.\end{aligned}$$

So, we also have

$$\int_0^T u_n(t)v(t)dt \to \int_0^T \widetilde{u}(t)v(t)dt, \tag{3.26}$$

since $u_n\to\widetilde{u}$ in $J_0^\alpha$. Moreover, by (2.8) and $I'_{u_{n-1}}(u_n)v=0$, we obtain

$$\begin{aligned}0 = I'_{u_{n-1}}(u_n)v = &-\frac{1}{2}\int_0^T \Big( {}_0^cD_t^\alpha u_n(t)\cdot {}_t^cD_T^\alpha v(t) + {}_0^cD_t^\alpha v(t)\cdot {}_t^cD_T^\alpha u_n(t)\Big)dt \\ &-\int_0^T f(t,u_n(t),\ {}_0^cD_t^\alpha u_{n-1}(t))v(t)dt \\ &-\varrho\int_0^T u_n(t)v(t)dt + \Sigma_{j=1}^n I_j(u_n(t_j))v(t_j),\ \forall v\in J_0^\alpha.\end{aligned}\tag{3.27}$$

The above equality combined with (3.23)–(3.26) indicates that

$$\begin{aligned}0 = I'_{\widetilde{u}}(\widetilde{u})v = &-\frac{1}{2}\int_0^T \Big( {}_0^cD_t^\alpha \widetilde{u}(t)\cdot {}_t^cD_T^\alpha v(t) + {}_0^cD_t^\alpha v(t)\cdot {}_t^cD_T^\alpha \widetilde{u}(t)\Big)dt \\ &-\int_0^T f(t,\widetilde{u}(t),\ {}_0^cD_t^\alpha \widetilde{u}(t))v(t)dt \\ &-\varrho\int_0^T \widetilde{u}(t)v(t)dt + \Sigma_{j=1}^n I_j(\widetilde{u}(t_j))v(t_j),\ \forall v\in J_0^\alpha,\end{aligned}\tag{3.28}$$

and

$$\lim_{n\to\infty} I'_{u_{n-1}}(u_n) = I'_{\widetilde{u}}(\widetilde{u}).$$

This implies that $\widetilde{u}$ is a weak solution of FADE (1.1). Similarly, we can certificate that $\lim_{n\to\infty} I_{u_{n-1}}(u_n) = I_{\widetilde{u}}(\widetilde{u})$. Because $I_{u_{n-1}}(u_n) \geq \omega_1 > 0$, we conclude that $I_{\widetilde{u}}(\widetilde{u}) \geq \omega_1 > 0$, it indicates that $\widetilde{u}$ is a nontrivial weak solution of FADE (1.1). Hence, our claim is proved. $\square$

# 4. An Example

**Example 4.1.** Investigate the existence of nontrivial weak solutions for following a fractional advection-dispersion equation (FADE for short) with impulsive effects

$$\begin{cases} -\dfrac{d}{dt}\Big[\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t))\Big] = \varrho u(t) + f(t, u(t),\ {}_0^cD_t^{\alpha}u(t)), \\ t \neq t_j,\ a.e.\ t \in [0, T], \\ \Delta\Big(\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t_j)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t_j))\Big) = I_j(u(t_j)),\ j = 1, 2, \ldots, n, \\ u(0) = u(T) = 0, \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{4}, T = 1,\ I_1(s) = -b_1 s|s|, s \in \mathbb{R}, b_1 > 0$.

Define

$$g(x) = \begin{cases} 0,\ x \leq 0, \\ x,\ 0 \leq x \leq 1, \\ 1,\ 1 \leq x < \infty, \end{cases} \qquad h(y) = \begin{cases} 0,\quad y \leq 0, \\ y^3,\ 0 \leq y \leq 1, \\ y^{\frac{1}{3}},\ 1 \leq y \leq \widetilde{L}_0, \\ \widetilde{L}_0^{\frac{1}{3}},\ \widetilde{L}_0 \leq y < \infty, \end{cases}$$

$$f(t, x, y) = 4be^{-t}x^3 + 4c_1t^2x^3(\sin y)^{\frac{4}{3}} - c_2(\cos t)g(x)h(y),$$

with $b > 0, c_1, c_2 > 0, \widetilde{L}_0 > 0$. We verify that all conditions of Theorem 3.1 are valid.

$(R_1)$

$$F(t, x, y) \leq bx^4 + c_1x^4|y|^{\frac{4}{3}},\ a.e.\ t \in [0, 1], x, y \in \mathbb{R},$$

with $b(t) = b, c(t) = c_1, \iota = \eta = 4, \mu = \frac{4}{3}$;

$$I_1(s) \geq -b_1|s|^2, s \in \mathbb{R}, b_1 > 0,$$

with $a_j = b_1, \sigma_j = 2$.

$(R_2)$

$$|f(t, \widetilde{x}, \widetilde{y}) - f(t, x, y)| \leq \varphi(t)|\widetilde{x} - x| + \psi(t)|\widetilde{y} - y|,\ a.e.\ t \in [0, 1],$$

for $x, \widetilde{x} \in [-\widetilde{P}_1, \widetilde{P}_1], \widetilde{y}, y \in \mathbb{R}$, where

$$\varphi(t) = 12be^{-t}\widetilde{P}_1^2 + 12c_1\widetilde{P}_1^2t^2 + c_2\widetilde{L}_0^{\frac{1}{3}}\cos t,$$

$$\psi(t) = \frac{16}{3} c_1 \widetilde{P}_1^3 t^2 + 3c_2 |\cos t|, \ t \in [0,1];$$

moreover,

$$|I_1(u_1) - I_1(u_2)| \leq 2b_1 \widetilde{P}_1 |u_1 - u_2|, \ for \ u_1, u_2 \in [-\widetilde{P}_1, \widetilde{P}_1],$$

where $\gamma_1 = 2b_1\widetilde{P}_1$, $d^* = l^* = d_j^* = \overline{m} = \overline{Q}_1 = \overline{M}_1 = \overline{B} = 0$.
$(R_3)$

$$xf(t,x,y) - 3F(t,x,y) \geq 0, \ for \ x, y \in \mathbb{R}, \ a.e. \ t \in [0,1],$$

where $\zeta = 3, d(t) = l(t) = m(t) = 0$;

$$3\int_0^u I_1(s)ds - I_1(u)u \geq 0, \ for \ u \in R,$$

where $d_j = l_j = 0$.
$(R_4)$

$$F(t,x,y) \geq \frac{bx^4}{e} - c_2|x||y|^{\frac{1}{3}}, \ a.e. \ t \in [0,1], \ x, y \in \mathbb{R},$$

where $N_1 = \frac{b}{e}, h(t) = 0, N_2(t) = c_2, \iota = 4, \kappa = \frac{1}{3}$;

$$I_1(s) \leq 0, s \geq 0, b_1 > 0,$$

where $Q_1 = M_1 = 0$.

Choose $\varrho = \frac{|\cos(\frac{3\pi}{4})|}{16\left(\frac{1}{\Gamma(\frac{7}{4})}\right)^2}$, it is found that all conditions are satisfied, thus, FADE (4.1) has a nontrivial weak solution.

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