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MULTIWAVE SOLUTIONS TO THE NEGATIVE-ORDER KDV EQUATION IN (3+1)-DIMENSIONS

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Abstract This work aims to study the negative-order KdV equation in (3+1)-dimensions which is developed via using the recursion operator of the KdV equation by employing the three-wave methods. As a consequence, a variety of novel multiwave solutions with several arbitrary parameters to the considered equation are presented. Moreover, selecting particular values for the parameters, some graphs are plotted to show the spatial structures and dynamics of the resulting solutions. These results enrich the variety of the dynamics in the field of nonlinear waves.

Keywords Negative-order KdV equation, three-wave methods, multiwave solutions.

MSC(2010) 35Q53, 35A25, 35C99.

1. Introduction

Many complex nonlinear phenomena arising in diverse areas of science, such as plasma physics, nonlinear optics, optical communication, solid state physics, lattice dynamics, etc, can be mainly described by nonlinear evolution equations (NLEEs). Investigating on exact solutions to NLEEs has important help for the understanding of nonlinear interaction [25]. Therefore, it is a very meaningful and interesting work to construct novel solutions to NLEEs. Plenty of approaches currently exist for the derivation of exact solutions, some of which include the inverse scattering method [1, 26], the Hirota's bilinear method [2, 7, 14, 15], the Darboux transformation method [16, 22, 27–29], the Lie symmetry method [18, 23], the Riemann-Hilbert method [10, 17, 24], the variable separation method [5, 12, 30] and the multiple exp-function method [6,9]. Among these methods, the Hirota's bilinear method provides a direct and powerful way to seek multi-soliton solutions of NLEEs. Inspired by this approach, extensive research has been carried out on probing various kinds of interaction solutions among solitary wave solutions (kink or bell), periodic wave solutions, etc. For example, Tang et al. investigated interaction phenomena including a lump and two kink solitons [13], a lump and other multi-solitons [3] for the (2+1)-dimensional BLMP equation. Liu et al. [8] obtained a lump and a kink soliton for a new (3+1)-dimensional generalized Kadomtsev-Petviashvili equation, and analyzed the interaction process in detail. In Ref. [19], Wang et al. proposed

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an efficient approach by using Hirota's bilinear form and a generalized three-wave type of ansatz method, and studied the (2+1)-dimensional and (3+1)-dimensional KdV-type equations. As a result, the periodic type of three-wave solutions including the periodic two-solitary wave solution, the doubly periodic solitary wave solution as well as the breather type of solitary wave solution for both equations were presented. Subsequently, many NLEEs were investigated via adopting this approach, such as the (2+1)-dimensional Ito equation [31], the (2+1)-dimensional breaking soliton equation [32], the (2+1)-dimensional Kadomtsev-Petviashvili equation [20], the (3+1)-dimensional Sharma-Tasso-Olver-like equation [4] and others.

In 2017, depending on using the recursion operator of the KdV equation, Wazwaz [21] developed some negative-order KdV equations in (3+1)-dimensions, and showed distinct solitons structures for these equations. In the current work, we focus on the following equation

$$-u_{xxt}+4uu_t+2u_x\partial_x^{-1}(u_t)=u_x+u_y-u_{xxz}+4uu_z+2u_x\partial_x^{-1}(u_z), \tag{1.1}$$

where u is a function of the spatial variables x, y, z and temporal variable t, ∂_x means the total derivative with respect to x, and ∂_x^{-1} is its integration operator. Eq. (1.1) is referred to as the (3+1)-dimensional negative-order KdV equation for model II. The principal aim of this study is to work out a series of exact solutions to the above newly introduced equation by implementing the three-wave methods.

2. Application of three-wave methods

In this section, we start with a variable transformation

$$u(x,y,z,t)=\phi_x(x,y,z,t), \tag{2.1}$$

under which Eq. (1.1) is cast into the following equation in $\phi(x,y,z,t)$:

$$-\phi_{xxxt}+4\phi_x\phi_{xt}+2\phi_{xx}\phi_t=\phi_{xx}+\phi_{xy}-\phi_{xxxz}+4\phi_x\phi_{xz}+2\phi_{xx}\phi_z. \tag{2.2}$$

We introduce an auxiliary function $f(x,y,z,t)$ having the link with $\phi(x,y,z,t)$ as

$$\phi(x,y,z,t)=-2[\ln f(x,y,z,t)]_x. \tag{2.3}$$

By substituting (2.3) into Eq. (2.2), we obtain

$$\begin{aligned}
&4f_{xxx}f_tf_x-4f_{xt}f_xf_{xx}-4f_{xxx}f_xf_z+4f_{xx}f_xf_{xz}-f_{xx}f_yf-2f_xf_{xy}f\\
&+2f_x^2f_y+f_{xxxx}f_zf+4f_{xxxz}f_xf-4f_{xxz}f_x^2-2f_{xx}f_{xxz}f+2f_{xx}^2f_z\\
&-3f_{xx}f_xf+2f_{xxt}f_{xx}f-4f_{xxxt}f_xf-f_{xxxx}f_tf+4f_{xxt}f_x^2-2f_{xx}^2f_t\\
&+2f_x^3+f_{xxxxt}f^2+f_{xxx}f^2+f_{xxy}f^2-f_{xxxxz}f^2=0.
\end{aligned} \tag{2.4}$$

Apparently, the potential function $u(x,y,z,t)$ is determined by (2.1) provided the auxiliary function $f(x,y,z,t)$ is a solution to Eq. (2.4). In the subsections that follow, we shall treat Eq. (2.4) by considering the auxiliary function $f(x,y,z,t)$ in two forms.

2.1. Ansatz method I

Following the description of method in Ref. [19], the auxiliary function f is supposed to be of the form

$$f = a_1\cos\eta_1 + a_2\cosh\eta_2 + a_3\mathrm{e}^{\eta_3} + a_4\mathrm{e}^{-\eta_3}, \tag{2.5}$$

where

$$\eta_1 = k_1x + k_2y + k_3z + k_4t, \eta_2 = l_1x + l_2y + l_3z + l_4t, \eta_3 = c_1x + c_2y + c_3z + c_4t,$$

and the parameters $a_j, k_j, l_j, c_j, j = 1, 2, 3, 4$, need to be known later. Insertion of the assumption (2.5) into Eq. (2.4) directly results in a system of overdetermined algebraic equations in the unknowns. By straightforward calculation of the resulting system, seven classes of the parameters' relations are listed below:

Case 1.

$$a_1 = a_2 = 0, c_2 = 4c_1^2c_3 - 4c_1^2c_4 - c_1.$$

Through substitution of these results into (2.5), we derive

$$f = a_3\mathrm{e}^{\eta_3} + a_4\mathrm{e}^{-\eta_3},$$

where

$$\eta_3 = c_1x + \left(4c_1^2c_3 - 4c_1^2c_4 - c_1\right)y + c_3z + c_4t.$$

Hence, setting $a_3a_4 > 0$, we acquire the solitary wave solution of Eq. (1.1) as follows

$$u = -2c_1^2\mathrm{sech}^2\left(\eta_3 + \frac{1}{2}\ln\frac{a_3}{a_4}\right). \tag{2.6}$$

Case 2.

$$a_1 = 0, a_3 = \frac{a_2^2}{4a_4}, c_1 = \epsilon l_1, c_2 = 4\epsilon l_1^2l_3 + 4l_1^2c_3 - 4l_1^2c_4 - 4\epsilon l_1^2l_4 - 2\epsilon l_1 - \epsilon l_2, \epsilon = \pm 1.$$

In this case, carrying these results into (2.5) under the restriction of $a_2a_4 > 0$ leads to

$$f = a_2\cosh\eta_2 + \frac{a_2^2}{4a_4}\mathrm{e}^{\eta_3} + a_4\mathrm{e}^{-\eta_3},$$

with η_2, η_3 being given by

$$\begin{aligned}\eta_2 &= l_1x + l_2y + l_3z + l_4t,\\ \eta_3 &= \epsilon l_1x + \left(4\epsilon l_1^2l_3 + 4l_1^2c_3 - 4l_1^2c_4 - 4\epsilon l_1^2l_4 - 2\epsilon l_1 - \epsilon l_2\right)y + c_3z + c_4t.\end{aligned}$$

Therefore, the expression of solitary wave solution for Eq. (1.1) reads

$$u = -2l_1^2 + \frac{2\left(a_2l_1\sinh\eta_2 + \epsilon\sqrt{a_2^2}l_1\sinh(\eta_3 + \xi_1)\right)^2}{\left(a_2\cosh\eta_2 + \sqrt{a_2^2}\cosh(\eta_3 + \xi_1)\right)^2},$$

where $\xi_1 = \ln\frac{a_2}{2a_4}$.

Case 3.

$$
\begin{aligned}
&a_2=0, a_3=-\frac{a_1^2k_1^2}{4c_1^2a_4},\\
&c_2=\frac{2k_1^2k_4c_1^2-2k_1^2k_3c_1^2-k_1^2k_2-k_1^4k_3+k_1^4k_4-k_1c_1^2+k_2c_1^2-k_3c_1^4+k_4c_1^4-k_1^3}{2k_1c_1},\\
&c_3=\frac{k_1^2k_3-k_1^2k_4+k_1+2k_1c_1c_4-k_3c_1^2+k_4c_1^2+k_2}{2k_1c_1}.
\end{aligned}
$$

In this case, under the assumption of $a_3=-a_5, a_5>0$, the auxiliary function takes the form

$$
f=a_1\cos\eta_1-a_5\mathrm{e}^{\eta_3}+a_4\mathrm{e}^{-\eta_3},
$$

with η_1, η_3 being given by

$$
\begin{aligned}
\eta_1&=k_1x+k_2y+k_3z+k_4t,\\
\eta_3&=c_1x\\
&\quad+\frac{(2k_1^2k_4c_1^2-2k_1^2k_3c_1^2-k_1^2k_2-k_1^4k_3+k_1^4k_4-k_1c_1^2+k_2c_1^2-k_3c_1^4+k_4c_1^4-k_1^3)y}{2k_1c_1},\\
&\quad+\frac{\left(k_1^2k_3-k_1^2k_4+k_1+2k_1c_1c_4-k_3c_1^2+k_4c_1^2+k_2\right)z}{2k_1c_1}+c_4t.
\end{aligned}
$$

Hence, we write down the periodic solitary wave solution to Eq. (1.1)

$$
\begin{aligned}
u=&\frac{2\left(a_1k_1^2\cos\eta_1+2c_1^2\sqrt{a_4a_5}\sinh(\eta_3+\xi_2)\right)}{a_1\cos\eta_1-2\sqrt{a_4a_5}\sinh(\eta_3+\xi_2)}\\
&+\frac{2\left(a_1k_1\sin\eta_1+2c_1\sqrt{a_4a_5}\cosh(\eta_3+\xi_2)\right)^2}{\left(a_1\cos\eta_1-2\sqrt{a_4a_5}\sinh(\eta_3+\xi_2)\right)^2},
\end{aligned}
$$

where $\xi_2=\frac{1}{2}\ln\frac{a_5}{a_4}$.

Case 4.

$$
\begin{aligned}
&a_1=\frac{\epsilon i a_2 l_1}{k_1}, a_3=a_4=0,\\
&k_2=\frac{2k_1^2l_1^2l_3-2k_1^2l_1^2l_4+k_1^2l_2+l_1^4l_3-l_1^4l_4-k_1^2l_1-l_1^2l_2-k_1^4l_4+k_1^4l_3-l_1^3}{2k_1l_1},\\
&k_3=-\frac{k_1^2l_3-k_1^2l_4-2k_1k_4l_1+l_1-l_1^2l_3+l_1^2l_4+l_2}{2k_1l_1}, \epsilon=\pm1.
\end{aligned}
$$

Regarding this case, we set $k_1=i\tilde{k}_1$ and $k_4=i\tilde{k}_4$, where $\tilde{k}_1,\tilde{k}_4\in\mathbb{R}$, $k_1l_1\neq0$. Then plugging these results into (2.5) gives rise to

$$
f=\frac{\epsilon a_2 l_1}{\tilde{k}_1}\cosh\tilde{\eta}_1+a_2\cosh\eta_2,
$$

with $\tilde{\eta}_1, \eta_2$ being given by

$$
\begin{aligned}
\tilde{\eta}_1=&\tilde{k}_1x+\frac{\left(2\tilde{k}_1^2l_1^2l_3-2\tilde{k}_1^2l_1^2l_4+\tilde{k}_1^2l_2-l_1^4l_3+l_1^4l_4-\tilde{k}_1^2l_1+l_1^2l_2+\tilde{k}_1^4l_4-\tilde{k}_1^4l_3+l_1^3\right)y}{2\tilde{k}_1l_1}\\
&-\frac{\left(\tilde{k}_1^2l_3-\tilde{k}_1^2l_4-2\tilde{k}_1\tilde{k}_4l_1-l_1+l_1^2l_3-l_1^2l_4-l_2\right)z}{2\tilde{k}_1l_1}+\tilde{k}_4t,\\
\eta_2=&l_1x+l_2y+l_3z+l_4t.
\end{aligned}
$$

Thus, we obtain the solitary wave solution to Eq. (1.1)

$$u=-\frac{2\big(\epsilon\tilde{k}_1 l_1\cosh\tilde{\eta}_1+l_1^2\cosh\eta_2\big)}{\frac{\epsilon l_1}{\tilde{k}_1}\cosh\tilde{\eta}_1+\cosh\eta_2}+\frac{2(\epsilon l_1\sinh\tilde{\eta}_1+l_1\sinh\eta_2)^2}{\big(\frac{\epsilon l_1}{\tilde{k}_1}\cosh\tilde{\eta}_1+\cosh\eta_2\big)^2}.$$

Case 5.

$$k_2=-k_1,k_3=k_4,l_2=-l_1,l_3=l_4,c_2=-c_1,c_3=c_4.$$

Insertion of these results into (2.5) generates

$$f=a_1\cos\eta_1+a_2\cosh\eta_2+a_3\mathrm{e}^{\eta_3}+a_4\mathrm{e}^{-\eta_3},$$

where η_1,η_2,η_3 are determined by

$$\eta_1=k_1x-k_1y+k_4z+k_4t,\eta_2=l_1x-l_1y+l_4z+l_4t,\eta_3=c_1x-c_1y+c_4z+c_4t.$$

If $a_3a_4>0$, then the expression of periodic solitary wave solution for Eq. (1.1) reads

$$\begin{aligned}u=&-\frac{2\big(-a_1k_1^2\cos\eta_1+a_2l_1^2\cosh\eta_2+2c_1^2\sqrt{a_3a_4}\cosh(\eta_3+\xi_3)\big)}{a_1\cos\eta_1+a_2\cosh\eta_2+2\sqrt{a_3a_4}\cosh(\eta_3+\xi_3)}\\&+\frac{2\big(-a_1k_1\sin\eta_1+a_2l_1\sinh\eta_2+2c_1\sqrt{a_3a_4}\sinh(\eta_3+\xi_3)\big)^2}{\big(a_1\cos\eta_1+a_2\cosh\eta_2+2\sqrt{a_3a_4}\cosh(\eta_3+\xi_3)\big)^2},\end{aligned}\tag{2.7}$$

where $\xi_3=\frac{1}{2}\ln\frac{a_3}{a_4}$.

Further, under the conditions of $a_2=0$ and $k_1=c_1$, the periodic solitary wave solution (2.7) can be reduced to

$$u=-\frac{2c_1^2\big(-a_1^2+4a_3a_4+4a_1\sqrt{a_3a_4}\sin\eta_1\sinh(\eta_3+\xi_3)\big)}{\big(a_1\cos\eta_1+2\sqrt{a_3a_4}\cosh(\eta_3+\xi_3)\big)^2},\tag{2.8}$$

where

$$\eta_1=c_1x-c_1y+k_4z+k_4t,\eta_3=c_1x-c_1y+c_4z+c_4t.$$

Case 6.

$$\begin{aligned}&k_1=\epsilon ic_1,k_2=\epsilon ic_1(4l_3c_1-4l_4c_1-1),k_3=\epsilon il_3-\epsilon il_4+k_4,l_1=c_1,\\&l_2=c_1(4l_3c_1-4l_4c_1-1),c_2=c_1(4c_1c_3-4c_1c_4-1),\epsilon=\pm1.\end{aligned}$$

In this case, if we posit that $a_3a_4>0$ and $k_4=i\tilde{k}_4$, where $\tilde{k}_4$ is a real constant, then we substitute these results into (2.5) and find

$$f=a_1\cosh\tilde{\eta}_1+a_2\cosh\eta_2+a_3\mathrm{e}^{\eta_3}+a_4\mathrm{e}^{-\eta_3},$$

in which $\tilde{\eta}_1,\eta_2,\eta_3$ are determined by

$$\begin{aligned}&\tilde{\eta}_1=\epsilon c_1x+\epsilon c_1(4l_3c_1-4l_4c_1-1)y+\big(\epsilon l_3-\epsilon l_4+\tilde{k}_4\big)z+\tilde{k}_4t,\\&\eta_2=c_1x+c_1(4l_3c_1-4l_4c_1-1)y+l_3z+l_4t,\\&\eta_3=c_1x+c_1(4c_1c_3-4c_1c_4-1)y+c_3z+c_4t.\end{aligned}$$

Therefore, the solitary wave solution of Eq. (1.1) is represented as

$$u=-2c_1^2+\frac{2\left(\epsilon a_1c_1\sinh\tilde{\eta}_1+a_2c_1\sinh\eta_2+2c_1\sqrt{a_3a_4}\sinh(\eta_3+\xi_4)\right)^2}{\left(a_1\cosh\tilde{\eta}_1+a_2\cosh\eta_2+2\sqrt{a_3a_4}\cosh(\eta_3+\xi_4)\right)^2},$$

where $\xi_4=\frac{1}{2}\ln\frac{a_3}{a_4}$.

Case 7.

$$k_1=\epsilon ic_1,k_2=4k_3c_1^2-4k_4c_1^2-\epsilon ic_1,l_1=-c_1,l_2=4\epsilon ik_3c_1^2-4\epsilon ik_4c_1^2+c_1,$$
$$l_3=\epsilon ik_3-\epsilon ik_4+l_4,c_2=8\epsilon ik_3c_1^2-8\epsilon ik_4c_1^2-c_1,c_3=2\epsilon ik_3-2\epsilon ik_4+c_4,\epsilon=\pm1.$$

Particularly, in this case, if we take $a_3=a_4=1$ and $l_4=i\tilde{l}_4,c_1=i\tilde{c}_1,c_4=i\tilde{c}_4,\tilde{l}_4,\tilde{c}_1,\tilde{c}_4\in\mathbb{R}$, then the auxiliary function (2.5) is of the form

$$f=a_1\cos\eta_1+a_2\cos\tilde{\eta}_2+\mathrm{e}^{\eta_3}+\mathrm{e}^{-\eta_3},$$

where $\eta_3=i\tilde{\eta}_3$, and $\eta_1,\tilde{\eta}_2,\tilde{\eta}_3$ are given by

$$\eta_1=-\epsilon\tilde{c}_1x+\left(\epsilon\tilde{c}_1-4k_3\tilde{c}_1^2+4k_4\tilde{c}_1^2\right)y+k_3z+k_4t,$$
$$\tilde{\eta}_2=-\tilde{c}_1x+\left(4\epsilon k_4\tilde{c}_1^2-4\epsilon k_3\tilde{c}_1^2+\tilde{c}_1\right)y+\left(\epsilon k_3-\epsilon k_4+\tilde{l}_4\right)z+\tilde{l}_4t,$$
$$\tilde{\eta}_3=\tilde{c}_1x+(8\epsilon k_4\tilde{c}_1^2-8\epsilon k_3\tilde{c}_1^2-\tilde{c}_1)y+(2\epsilon k_3-2\epsilon k_4+\tilde{c}_4)z+\tilde{c}_4t.$$

Thus, the periodic wave solution of Eq. (1.1) is obtained as

$$u=2\tilde{c}_1^2+\frac{2(\epsilon a_1\tilde{c}_1\sin\eta_1+a_2\tilde{c}_1\sin\tilde{\eta}_2-2\tilde{c}_1\sin\tilde{\eta}_3)^2}{(a_1\cos\eta_1+a_2\cos\tilde{\eta}_2+2\cos\tilde{\eta}_3)^2}.$$

2.2. Ansatz method II

In this subsection, we take an interest in generating double resonance-type solutions to Eq. (1.1) based on a different ansatz method [11]. Assuming that the expression of the auxiliary function f is

$$f=a_1\sin^2\eta_1+a_2\sinh^2\eta_1+a_3\mathrm{e}^{\eta_2}+a_4\mathrm{e}^{-\eta_2}, \tag{2.9}$$

in which

$$\eta_1=k_1x+k_2y+k_3z+k_4t,\eta_2=c_1x+c_2y+c_3z+c_4t,$$

and $a_\iota,k_\iota,c_\iota,\iota=1,2,3,4$, are several undetermined parameters. Performing the similar operations as before, we get two sets of solutions:

Case 1.

$$a_4=k_1=0,k_2=k_3c_1^2-k_4c_1^2,c_2=c_1^2c_3-c_1^2c_4-c_1.$$

Substituting these results into (2.9) yields

$$f=a_1\sin^2\eta_1+a_2\sinh^2\eta_1+a_3\mathrm{e}^{\eta_2},$$

where

$$\eta_1=\left(k_3c_1^2-k_4c_1^2\right)y+k_3z+k_4t,\eta_2=c_1x+\left(c_1^2c_3-c_1^2c_4-c_1\right)y+c_3z+c_4t.$$

Therefore, the solution to Eq. (1.1) is expressed as

$$u=-\frac{2a_3c_1^2\mathrm{e}^{\eta_2}}{a_1\sin^2\eta_1+a_2\sinh^2\eta_1+a_3\mathrm{e}^{\eta_2}}+\frac{2a_3^2c_1^2\mathrm{e}^{2\eta_2}}{\left(a_1\sin^2\eta_1+a_2\sinh^2\eta_1+a_3\mathrm{e}^{\eta_2}\right)^2}. \tag{2.10}$$

Case 2.

$$k_2=-k_1, k_3=k_4, c_2=-c_1, c_3=c_4.$$

Carrying these results into (2.9), it can be seen that the auxiliary function

$$f=a_1\sin^2\eta_1+a_2\sinh^2\eta_1+a_3\mathrm{e}^{\eta_2}+a_4\mathrm{e}^{-\eta_2},$$

where

$$\eta_1=k_1x-k_1y+k_4z+k_4t, \eta_2=c_1x-c_1y+c_4z+c_4t.$$

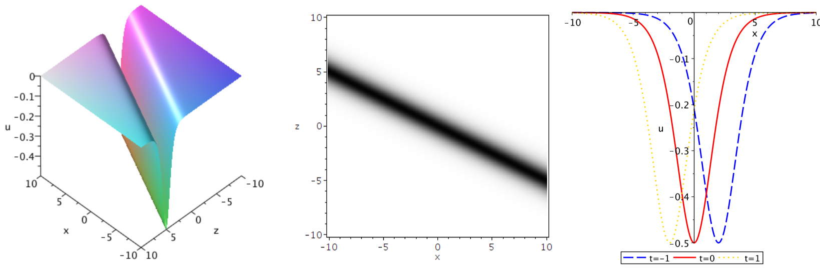
Hence, Eq. (1.1) possesses the double resonance-type solution

$$u=-\frac{2\left(2a_1k_1^2\cos(2\eta_1)+2a_2k_1^2\cosh^2\eta_1+2a_2k_1^2\sinh^2\eta_1+2c_1^2\sqrt{a_3a_4}\cosh(\eta_2+\xi_5)\right)}{a_1\sin^2\eta_1+a_2\sinh^2\eta_1+2\sqrt{a_3a_4}\cosh(\eta_2+\xi_5)}$$
$$+\frac{2\left(a_1k_1\sin(2\eta_1)+2a_2k_1\sinh\eta_1\cosh\eta_1+2c_1\sqrt{a_3a_4}\sinh(\eta_2+\xi_5)\right)^2}{\left(a_1\sin^2\eta_1+a_2\sinh^2\eta_1+2\sqrt{a_3a_4}\cosh(\eta_2+\xi_5)\right)^2},$$

where $\xi_5=\frac{1}{2}\ln\frac{a_3}{a_4}$ and $a_3a_4>0$.

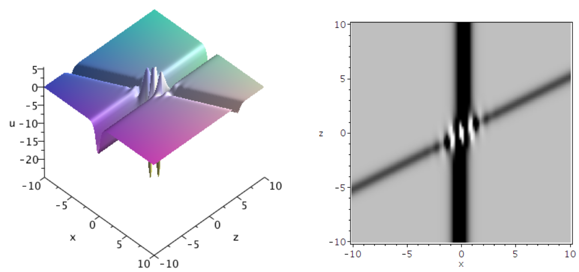
3. Summary and discussion

In this work, we concentrate on the study of the negative-order KdV equation in (3+1)-dimensions upon applying the three-wave methods. Finally, a series of multiwave solutions to the equation under examination are gained. Additionally, via selecting appropriate values for the parameters, we depict some obtained solutions to show their localized coherent structures. Figure 1 indicates the localization of solution (2.6) in (x, z)-plane which is a typical anti-bell-shaped solitary wave solution. Figure 2 presents the periodic solitary wave solution (2.7). Manifestly, from the expression (2.7), it is revealed that this solution is a complexiton solution or an interaction solution combining trigonometric and hyperbolic waves. Due to the amplitude of trigonometric waves involved oscillating periodically as time evolves, it is actually a type of breather solitary wave solution. Under a proper choice of parameters, the localized features are clearly displayed in Figure 2(a). It can be known that a pair of anti-bell-shaped solitons interact mutually. As time evolves, two solitons gradually fuse to generate a novel soliton with oscillations. And later they seperate and recover their original shapes. Figure 2(b) is the density plot of this wave. In particular, a reduced solution (2.8) arises from the solution (2.7), which is also a type of breather wave solution with periodicity (see Figure 3). In Figure 4, one can evidently observe the dynamics of the Y-type resonance solitary wave solution (2.10). As time goes by, this wave propagates towards the positive direction of the y-axis. In the future, more research should be undertaken to explore abundant multiwave solutions of nonlinear evolution equations in nonlinear science.



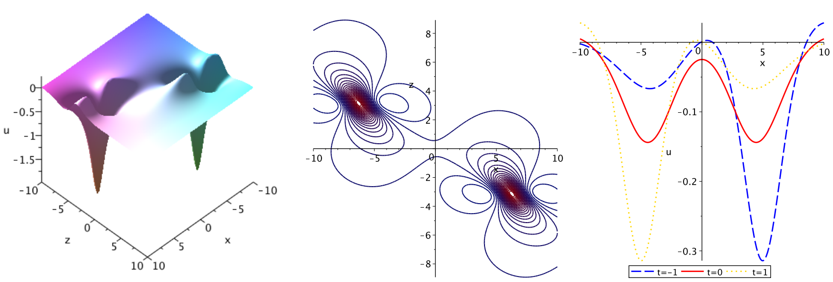
(a) (b) (c)

Figure 1. Plots of solution (2.6) with the parameters being fixed at $a_3 = 2, a_4 = 2, c_1 = \frac{1}{2}, c_3 = 1, c_4 = 1$. (a) Perspective view of the wave with $y = t = 0$; (b) Density plot corresponding to Figure 1(a); (c) The wave along the x-axis with $y = z = 0$ at different times.



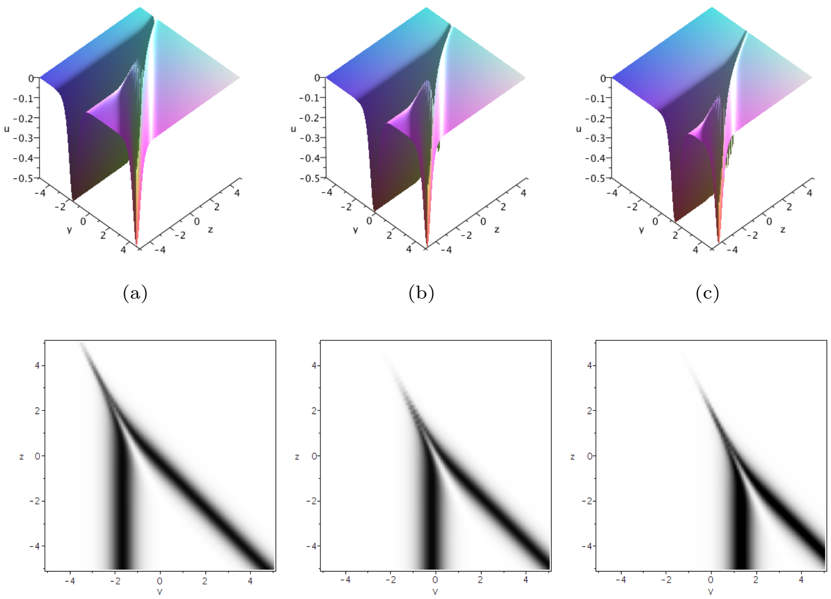
(a) (b)

Figure 2. Plots of solution (2.7) with the parameters being fixed at $a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, k_1 = 6, k_4 = 1, l_1 = 1, l_4 = 2, c_1 = -3, c_4 = 2$. (a) Perspective view of the wave with $y = t = 0$; (b) Density plot corresponding to Figure 2(a).



(a) (b) (c)

Figure 3. Plots of solution (2.8) with the parameters being fixed at $a_1 = \frac{4}{5}, a_3 = \frac{1}{2}, a_4 = \frac{1}{2}, k_4 = -\frac{1}{3}, c_1 = \frac{1}{3}, c_4 = \frac{2}{3}$. (a) Perspective view of the wave with $y = t = 0$; (b) Contour plot corresponding to Figure 3(a); (c) The wave along the x-axis with $y = z = 0$ at different times.



(a) (b) (c) (d) (e) (f)

Figure 4. Plots of solution (2.10) with the parameters being fixed at $a_1 = 1, a_2 = 2, a_3 = 1, c_1 = -1, c_3 = -2, c_4 = -1, k_3 = 1, k_4 = -1, x = 0$. (a) Perspective view of the wave at $t = -2$; (b) Perspective view of the wave at $t = 0$; (c) Perspective view of the wave at $t = 2$; (d) Density plot corresponding to Figure 4(a); (e) Density plot corresponding to Figure 4(b); (f) Density plot corresponding to Figure 4(c).

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