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GLOBAL STABILITY OF PERIODIC SOLUTIONS OF PREDATOR-PREY SYSTEM WITH HOLLING TYPE III FUNCTIONAL RESPONSE

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Abstract This paper studied some properties of a predator-prey system with Holling type III functional response. Based on Mawhin's Continuation Theorem, some sufficient conditions for the existence of periodic solutions are obtained. Moreover, the global stability of the periodic solution is built with the help of a suitable Lyapunov function.

Keywords Periodic solution, Mawhin's Continuous Theorem, global stability, Lyapunov function.

MSC(2010) 34C25, 34D23.

1. Introduction

Predator-prey systems are the basic and important models in mathematic biology, and massive works have been focus on them. In consideration of predator-prey capacity, Holling proposes three different predations with functional response based on experiments. A general Holling type functional response f, which depends on the prey population x, takes the form

$$f(x) = \frac{\lambda x^n}{1 + \lambda m x^n},$$

where λ is the attack efficiency of predator to prey population, m denotes the handling time of predators, and the exponent n illustrates the shape of the functional response.

The research on the existence and stability of the periodic solution is one of the most important directions. And many authors investigated the existence and stability of the periodic solutions by different methods. For examples, in Ref. [1], Cheng, Zhang and Wang used differential equation geometry theory and the method of successor functions to verify the existence of the periodic solution of the system with holling type I

$$\begin{cases} x'(t) = rx(t) - cx(t)y(t), \quad x \le x_0; \\ y'(t) = -dy(t) + ecx(t)y(t), \quad x \le x_0; \\ x'(t) = rx(t) - cx_0 y(t), \quad x > x_0; \\ y'(t) = -dy(t) + ecx_0 y(t), \quad x > x_0. \end{cases}$$

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And they prove the attractivity of the periodic solution by sequence convergence rules and qualitative analysis.

In Ref. [5], Lisennna verified the existence of the period solutions of the system through the comparison theorem and Brouwer fixed-point theorem

$$\begin{cases} u'(t) = u(a(t) - u - \dfrac{u}{u + m(t)}), \\ v'(t) = v(b(t) - \dfrac{v}{r(t)u}). \end{cases}$$

And they construct a Liapunov function to prove the global asymptotic stability of the periodic solution.

The results in Ref. [2], O. Diop, A. Moussaoui and A. Sene established sufficient conditions for existence of a periodic solution of the system with Holling type II

$$\begin{cases} x'(t) = r(t)x(1 - \dfrac{x}{k(t)}) - \dfrac{\alpha xy}{x + D}, \\ y'(t) = -d(t)y + \dfrac{\beta xy}{x + D} + \Lambda(t). \end{cases}$$

Readers can refer to Refs. [3, 4, 6–11] to get more methods which can solve out the existence of periodic solution or stability of the solution.

Motivated by the above discussion, in this paper, we mainly study the predator-prey model with Holling type III described by

$$\begin{cases} \dfrac{dx(t)}{dt} = r(t)x(t)(1 - \dfrac{x(t)}{k}) - \dfrac{m(t)x^2(t)y(t)}{1 + cx(t) + bx^2(t)}, \\ \dfrac{dy(t)}{dt} = sy(t)(\dfrac{m(t)x^2(t)}{1 + cx(t) + bx^2(t)} - d(t)). \end{cases} \tag{1.1}$$

Where $x(t), y(t)$ are the densities of the prey population and predator population at time t, and they are all positive numbers. The other parameters have the following biological meanings: c, b are positive, $r(t)$ is the intrinsic per capita growth rate of prey population at time t; k is the prey environmental carrying capacity; s is the efficiency with which predators convert consumed prey into new predators; $d(t)$ is the per capita death rate of predators at time t, $m(t)$ is the attack efficiency of predator to prey population at time t, and r, m, d are continuous functions of period T. We first denote:

$$\bar{r} = \frac{1}{T}\int_0^T r(t)dt, \bar{m} = \frac{1}{T}\int_0^T m(t)dt, \bar{d} = \frac{1}{T}\int_0^T d(t)dt,$$
$$\gamma = \frac{1}{T}\int_0^T |r(t)|dt, D = \frac{1}{T}\int_0^T |d(t)|dt.$$

Then let

$$a = (\gamma + \bar{r})T - ln\frac{\bar{d}b}{\bar{m}}.$$

This paper is organized as follows. In section 2, we first summarize a few concepts which are needed, then we construct some conditions to prove the existence of the periodic solution by continuation theorem. A suitable Lyapunov function is estabished to investigate the global stability in the last section.

2. Existence of periodic solution

To achieve the existence of periodic solutions, we shall summarize a few concepts in the following.

Let X and Z be Banach spaces. $L : DomL \subset X \to Z$ be a linear mapping and $N : X \to Z$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = codimImL < \infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = KerL$ and $KerQ = ImL = Im(I - Q)$. It follows that $L|DomL \bigcap KerP : (I - P)X \to ImL$ is invertible and its inverse is denoted by k_p. If Ω is a bounded open subset of X,the mapping N is called $L-$compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact, where I is the identity.Because ImQ is isomorphic to $KerL$, there exits an isomorphism $J : ImQ \to KerL$.

Lemma 2.1 (Continuation theorem). *Let L be a Fredholm mapping of index zero and let N be $L-$compact on $\bar{\Omega}$. Assume*

(i) for each $\lambda \in (0, 1)$,every solution x of $Lx = \lambda Nx$ is such that $x \in \partial\Omega \bigcap Dom(L)$,

(ii) $QNx \neq 0$ for each $x \in \partial\Omega \bigcap Ker(L)$,

(iii) $deg\{JQN, \Omega \bigcap Ker(L), 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \bigcap Dom(L)$.

Theorem 2.1. *If the coefficients in system (1.1) satisfies $c \gg b$ and $k > e^a$. And the algebraic equations*

$$\begin{cases} r(t) - \dfrac{r(t)}{k} e^{u_1(t)} - \dfrac{m(t)e^{u_1(t)+u_2(t)}}{1 + ce^{u_1(t)} + be^{u_1(t)^2}} = 0, \\ \dfrac{sm(t)e^{2u_1(t)}}{1 + ce^{u_1(t)} + be^{u_1(t)^2}} - sd(t) = 0, \end{cases}$$

has a unique solution $(u_1^*, u_2^*) \in intR_+^2 = \{(u_1^*, u_2^*)^T | u_i^* > 0\}$. Then the system (1.1) has at least one positive T-periodic solution.

Proof. Make the change of variables

$$x(t) = e^{u_1(t)}, y(t) = e^{u_2(t)}$$

in (1.1), we get a simplified system,

$$\begin{cases} \dfrac{du_1(t)}{dt} = r(t) - \dfrac{r(t)}{k} e^{u_1(t)} - \dfrac{m(t)e^{u_1(t)+u_2(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}}, \\ \dfrac{du_2(t)}{dt} = \dfrac{sm(t)e^{2u_1(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}} - sd(t). \end{cases} \tag{2.1}$$

Define

$$X = Z = \{u(t) = (u_1(t), u_2(t))^T \in C(R, R^2) : u_i(T + t) = u_i(t), i = 1, 2\}$$

with the norm $\|u\| = \|(u_1(t), u_2(t))^T\| = \max |u_1(t)| + \max |u_2(t)|,\ u \in X$, then it is easy to prove that X and Z are Banach space.

Let

$$Nu = \begin{bmatrix} r(t) - \frac{r(t)}{k}e^{u_1(t)} - \frac{m(t)e^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}} \\ \frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(s)}+be^{2u_1(t)}} - sd(t) \end{bmatrix}.$$

$$Lu = \frac{du(t)}{dt},\ Pu = \frac{1}{T}\int_0^T u(t)dt, u \in X,\ Qz = \frac{1}{T}\int_0^T z(t)dt, z \in Z.$$

Then it follows that $KerL = R^2, ImL = \{z \in Z : \int_0^T z(t)dt = 0\}$ is closed in Z, $dimkerL = codimImL = 2$ and P, Q are continuous projects such that

$$ImP = KerL \quad \text{and} \quad KerQ = ImL = Im(I-Q).$$

So, L can be proved a Fredholm mapping of index zero.

We find that the inverse $K_p : ImL \to KerP \bigcap DomL$ exists and is given by

$$K_p(z) = \int_0^T z(s)ds - \frac{1}{T}\int_0^T \int_0^t z(s)dsdt.$$

Thus

$$QNu = \begin{bmatrix} \frac{1}{T}\int_0^T [r(t) - \frac{r(t)}{k}e^{u_1(s)} - \frac{m(t)e^{u_1(s)+u_2(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}}]ds \\ \frac{1}{T}\int_0^T [\frac{sm(t)e^{2u_1(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}} - sd(t)]ds \end{bmatrix}.$$

and

$$K_p(I-Q)Nu = \begin{bmatrix} \int_0^T [r(t) - \frac{r}{k}e^{u_1(s)} - \frac{m(t)e^{u_1(s)+u_2(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}}]ds \\ \int_0^T [\frac{sm(t)e^{2u_1(s)}}{1+ce^{u_1(s)})+be^{2u_1(s)}} - sd(t)]ds \end{bmatrix}$$
$$- \begin{bmatrix} \frac{1}{T}\int_0^T \int_0^t [r(t) - \frac{r(t)}{k}e^{u_1(s)} - \frac{m(t)e^{u_1(s)+u_2(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}}]dsdt \\ \frac{1}{T}\int_0^T \int_0^t [\frac{sm(t)e^{2u_1(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}} - sd(t)]dsdt \end{bmatrix}.$$

Clearly, QN and $K_p(I-Q)N$ are continuous. By the Arzela-Ascoli theorem, it is not difficult to show that $K_p(I-Q)N(\bar{\Omega})$ is compact for any open bounded $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is clearly bounded.Thus, N is $L-$compact on $\bar{\Omega}$.

Now we are in the position of searching for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lu = \lambda Nu, \lambda \in (0,1)$, we obtain

$$\begin{cases} \frac{du_1(t)}{dt} = \lambda[r(t) - \frac{r(t)}{k}e^{u_1(t)} - \frac{m(t)e^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}], \\ \frac{du_2(t)}{dt} = \lambda[\frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}} - sd(t)]. \end{cases} \tag{2.2}$$

Assume that $(u_1(t), u_2(t))$ is a solution of (2.2) for a certain λ. Integrating (2.2) over the interval $[0, T]$, we have

$$\int_0^T [r(t) - \frac{r}{k}e^{u_1(t)} - \frac{m(t)e^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}]dt = 0,$$

and

$$\int_0^T [\frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}} - sd(t)]dt = 0,$$

Thus

$$\int_0^T \frac{r(t)}{k}e^{u_1(t)} + \frac{m(t)e^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}dt = \bar{r}T, \tag{2.3}$$

and

$$\int_0^T \frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}dt = s\bar{d}T. \tag{2.4}$$

From (2.2) and (2.3) we can get

$$\begin{aligned}\int_0^T |u_1'(t)|dt &= \lambda\int_0^T |r(t) - \frac{r}{k}e^{u_1(t)} - \frac{m(t)e^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}|dt \\ &< \int_0^T |r|dt + \int_0^T \frac{r}{k}e^{u_1(t)} + \frac{me^{u_1(t)+u_2(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}dt \\ &= (\gamma+\bar{r})T.\end{aligned}$$

From (2.2) and (2.4) we similar can have

$$\begin{aligned}\int_0^T |u_2'(t)|dt &= \lambda\int_0^T |\frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}} - sd(t)|dt \\ &< s(D+\bar{d})T.\end{aligned}$$

In view of (2.4), we will find

$$s\bar{d}T = \int_0^T \frac{sm(t)e^{2u_1(t)}}{1+ce^{u_1(t)}+be^{2u_1(t)}}dt \le \int_0^T \frac{sm(t)e^{2u_1(t)}}{ce^{u_1(t)}}dt = \int_0^T \frac{sm(t)e^{u_1(t)}}{c}dt.$$

By mean value theorem, we know there exists $\xi_1 \in [0,T]$ such that

$$e^{u_1(\xi_1)} \ge s\bar{d}T \times \frac{c}{s\bar{m}T} = \frac{\bar{d}c}{\bar{m}},$$

then we have

$$u_1(\xi_1) \ge \ln\frac{\bar{d}c}{\bar{m}},$$

thus

$$u_1(t) \ge u_1(\xi_1) - \int_0^T |u_1'(t)|dt = \ln\frac{\bar{d}c}{\bar{m}} - (\gamma+\bar{r})T \triangleq H_{11}. \tag{2.5}$$

Through the equation (2.4), we can easily get

$$\int_0^T \frac{m(t)e^{-u_1(t)}}{b} \ge \int_0^T \frac{m(t)e^{2u_1(t)}}{c+be^{3u_1(t)}}dt \ge \int_0^T \frac{m(t)e^{2u_1(t)}}{ce^{u_1(t)}+be^{2u_1(t)}}dt \ge \bar{d}T,$$

Then the mean value theorem implies that there exists $\xi_2 \in [0,T]$ such that

$$u_1(\xi_2) \le -\ln\frac{\bar{d}b}{\bar{m}}.$$

Hence

$$u_1(t) \le u_1(\xi_2) + \int_0^T |u_1'(t)|dt = (\gamma + \bar{r})T - \ln \frac{\bar{d}b}{\bar{m}} \triangleq H_{12} = a, \tag{2.6}$$

Thus, in view of (2.5) and (2.6) we obtain

$$\max |u_1(t)| \le \max\{|H_{11}|, |H_{12}|\} := H_1. \tag{2.7}$$

Furthermore, from (2.3) we also have

$$\int_0^T \frac{m(t)e^{u_1(t)+u_2(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}} dt = \int_0^T r(t) - \frac{r(t)}{k} e^{u_1(t)} dt.$$

By the basic inequality, we will get

$$\int_0^T \frac{m(t)e^{u_2(t)}}{2\sqrt{b} + c} \ge \int_0^T r(t) - \frac{r(t)}{k} e^{u_1(t)} dt.$$

Then using the mean value theorem, there exists $\eta_1 \in [0, T]$ Such that

$$u_2(\eta_1) \ge \ln[(\bar{r} - \frac{\bar{r}}{k} e^{H_{12}}) \frac{2\sqrt{b} + c}{\bar{m}}].$$

Thus

$$u_2(t) \ge u_2(\eta_1) - \int_0^T |u_2'(t)|dt = \ln[(\bar{r} - \frac{\bar{r}}{k} e^{H_{12}}) \frac{2\sqrt{b} + c}{\bar{m}}] - s(D + \bar{d})T \triangleq H_{21}. \tag{2.8}$$

From (2.3), we have

$$\int_0^T \frac{m(t)e^{u_1(t)+u_2(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}} dt \le \int_0^T r(t)dt.$$

Then we make use of the equation (2.4), and the mean value theorem, we can get that there exists $\eta_2 \in [0, 1]$ such that

$$\begin{aligned} e^{u_2(\eta_2)} \int_0^T \frac{m(t)e^{2u_1(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}} dt &\le e^{u_2(\eta_2)} \int_0^T e^{u_1(t)} dt \int_0^T \frac{m(t)e^{u_1(t)}}{1 + ce^{u_1(t)} + be^{2u_1(t)}} dt \\ &\le \bar{r}T \int_0^T e^{u_1(t)} dt. \end{aligned}$$

So we obtain

$$u_2(\eta_2) \le \ln \frac{\bar{r}Te^{H_{12}}}{\bar{d}}.$$

Thus

$$u_2(t) \le u_2(\eta_2) + \int_0^T |u_2'(t)|dt = \ln \frac{\bar{r}Te^{H_{12}}}{\bar{d}} + s(D + \bar{d})T \triangleq H_{22}. \tag{2.9}$$

In view of (2.8) and (2.9) we have

$$\max |u_2(t)| \le \max\{|H_{21}|, |H_{22}|\} := H_2. \tag{2.10}$$

Under the assumptions in Theorem 2.1, it is easy to show that the algebraic equation has a unique solution $(u_1^*, u_2^*) \in intR_+^2 = \{(u_1^*, u_2^*)^T | u_i^* > 0\}$. Let $H = H_1 + H_2 + H_3$, where $H_3 > 0$ is large enough that

$$\|(lg\{u_1^*(t)\}, \lg\{u_2^*(t)\})^T\| = |\lg\{u_1^*(t)\}| + |\lg\{u_2^*(t)\}| < H_3.$$

Define $\Omega = \{u(t) = (u_1(t), u_2(t))^T \in X : \|u\| < H\}$. Thus Ω satisfies the demand in Lemma 2.1. When $u \in \partial\Omega \bigcap R^2$, that is $\|u\| = H$. Then

$$QNu = \begin{bmatrix} \frac{1}{T}\int_0^T [r(t) - \frac{r(t)}{k}e^{u_1(s)} - \frac{m(t)e^{u_1(s)+u_2(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}}]ds \\ \frac{1}{T}\int_0^T [\frac{sm(t)e^{2u_1(s)}}{1+ce^{u_1(s)}+be^{2u_1(s)}} - sd(t)]ds \end{bmatrix} \neq 0.$$

Because of $ImP = KerL$, J can be the identity mapping,so

$$\begin{aligned} &\deg\{JQN, \Omega \bigcap KerL, 0\} \\ =&sgn \begin{vmatrix} A & -\frac{\bar{m}e^{u_1^*+u_2^*}}{1+ce^{u_1^*}+be^{2u_1^*}} \\ \frac{2s\bar{m}e^{2u_1^*}(1+ce^{u_1^*}+be^{2u_1^*}) - s\bar{m}e^{2u_1^*}(ce^{u_1^*}+2be^{2u_1^*})}{(1+ce^{u_1^*}+be^{2u_1^*})^2} & 0 \end{vmatrix} \\ =&sgn(\frac{s\bar{m}^2e^{3u_1^*+u_2^*}(2+ce^{u_1^*})}{(1+ce^{u_1^*}+be^{2u_1^*})^3}) \neq 0. \end{aligned}$$

And there is no need to calculate the exact form of A. By now the conditions in Lemma 2.1 are all satisfied. Hence the system (2.1) has at least one solution $(u_1^*(t), u_2^*(t))^T$ in $DomL \bigcap \bar{\Omega}$. Then $(x^*(t), y^*(t))^T$ is a positive T-periodic solution of (1.1). □

3. The stability of the periodic solution

Definition 3.1. Let $(x^*(t), y^*(t))$ be a positive T-periodic solution of system (1.1). We say that it is globally asymptotically stable if any other positive solution $(x(t), y(t))$ has the property

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = 0 = \lim_{t\to+\infty} |y(t) - y^*(t)|.$$

Let $(x^*(t), y^*(t))$ be a positive T-periodic solution of system (1.1). under the substitution

$$u(t) = \frac{x(t)}{x^*(t)} - 1, v(t) = \frac{y(t)}{y^*(t)} - 1.$$

system (1.1) turns into

$$\begin{cases} \frac{du(t)}{dt} = (1+u)\{-\frac{r(t)}{k}ux^*(t) - [\frac{y^*(t)}{x^*(t)}(\frac{m(t)}{\eta(t)} - \frac{m(t)(1+u+v+uv)}{\theta(t)})]\}, \\ \frac{dv(t)}{dt} = (1+v)[\frac{sm(t)(1+2u+u^2)}{\theta(t)} - \frac{sm(t)}{\eta(t)}]. \end{cases} \tag{3.1}$$

Where

$$\theta(t) = \frac{1}{x^*(t)^2} + c(1+u)\frac{1}{x^*(t)} + b(1+u)^2, \eta(t) = \frac{1}{x^*(t)^2} + c\frac{1}{x^*(t)} + b.$$

Denote by $\tilde{x}(t)$ the positive T-periodic solution of the equation

$$x' = x[r(t) - \frac{r(t)}{k}x], \tag{3.2}$$

and by $\tilde{y}(t)$ the positive solution of the equation

$$y' = y[\frac{sm(t)\tilde{x}^2(t)}{1 + c\tilde{x}(t) + b\tilde{x}^2(t)} - sd(t)], \tag{3.3}$$

moreover, we can define the $x\bar{(t)}$ is the positive periodic solution of the equation

$$x' = x[r(t) - \frac{r(t)}{k}x - \frac{m(t)x\tilde{y}(t)}{1 + cx + bx^2}], \tag{3.4}$$

and $\bar{y}(t)$ is the periodic solution to

$$y' = y[\frac{sm(t)\bar{x}^2(t)}{1 + c\bar{x}(t) + b\bar{x}^2(t)} - sd(t)]. \tag{3.5}$$

Then the system (1.1) has at least a positive, T-periodic solution $(x^*(t), y^*(t))$ such that

$$\bar{x}(t) \leq x^*(t) \leq \tilde{x}(t),\ \ y(t) \leq y^*(t) \leq \tilde{y}(t), t \in [0, T]$$

by using the comparison theorem and the Brouwer fixed-point theorem. Thus for any positive solution $(x(t), y(t))$, for $t > \bar{t}$, we get

$$\begin{aligned}
\bar{u}(t) &= \frac{\bar{x}(t)}{x^*(t)} - 1 \leq u(t) \leq \frac{\tilde{x}(t)}{x^*(t)} - 1 = \tilde{u}(t), \\
\bar{v}(t) &= \frac{\bar{y}(t)}{y^*(t)} - 1 \leq v(t) \leq \frac{\tilde{y}(t)}{y^*(t)} - 1 = \tilde{v}(t),
\end{aligned}$$

and

$$(u(t), v(t)) \in Q(t), t > \bar{t},$$

where $Q(t)$ is the rectangle that

$$Q(t) = [\bar{u}(t), \tilde{u}(t)] \times [\bar{v}(t), \tilde{v}(t)].$$

Next we define $\mu = [\theta(t)], \nu = [\eta(t)]$, and $\alpha = \max\{\mu, \nu\}$.

Let $G(t; u, v)$ be the function defined by

$$\begin{aligned}
G(t; u, v) =& u^2(u + 1 + \alpha)[-\frac{r(t)}{k}x^*u - \frac{y^*}{x^*}(\frac{m(t)}{\eta(t)} - \frac{m(t)(1+u)(1+v)}{\theta(t)})] \\
&+ v^2[\frac{sm(t)(1+u)}{\theta(t)} - \frac{sm(t)}{(1+u)\eta(t)}],
\end{aligned}$$

and a suitable Lyapunov function will be introduced through the following function

$$H(x, y) = \int_1^{1+x} (1 - \frac{1}{s})(1 + \frac{\alpha}{s})(s - 1)ds + \int_1^{1+y} (1 - \frac{1}{t})(t - 1)dt. \tag{3.6}$$

It is easy to find $H(0, 0) = 0$ and $H(u, v)$ is positive for all other value of (u, v).

Theorem 3.1. *Assume the conditions in Theorem 2.1 are hold and let $(x^*(t), y^*(t))$ be a positive periodic solution of (1.1) such the solution is globally asymptotically stable if*

$$\Psi(t) < 0,$$

where

$$\Psi(t) = \max\{\frac{G(t;u,v)}{(1+u)H(u,v)}\}, (u,v) \in Q(t), t \in [0,1].$$

Proof. Let $(x(t), y(t))$ be a solution of (1.1), and $(u(t), v(t))$ be the corresponding solution of system (3.1). Introduce the following Lyapunov function

$$V(t) = H(u(t), v(t)).$$

By the equation (3.6), we can get

$$\begin{aligned}
V'(t) =&(1-\frac{1}{u+1})(1+\frac{\alpha}{1+u})uu'(t) + (1-\frac{1}{1+v})vv'(t)\\
=&(\frac{u}{1+u})(u+1+\alpha)u\{-\frac{r(t)}{k}ux^*(t) - [\frac{y^*(t)}{x^*(t)}(\frac{m(t)}{\eta(t)} - \frac{m(t)(1+u+v+uv)}{\theta(t)})]\}\\
&+ v^2[\frac{sm(t)(1+2u+u^2)}{\theta(t)} - \frac{sm(t)}{\eta(t)}]\\
=&\frac{1}{1+u}\{u^2(u+1+\alpha)[-\frac{r(t)}{k}x^*u - \frac{y^*}{x^*}(\frac{m(t)}{\eta(t)} - \frac{m(t)(1+u)(1+v)}{\theta(t)})]\\
&+ v^2[\frac{sm(t)(1+u)}{\theta(t)} - \frac{sm(t)}{(1+u)\eta(t)}\}.
\end{aligned}$$

Thus we can easily find

$$V'(t) = \frac{G(t;u(t),v(t))}{u(t)+1}.$$

Let us investigate the behaviour of the ratio

$$\frac{G(t;u,v)}{(1+u)(H(u,v))}$$

in a neighborhood of $(0,0)$.

Through Taylor expansion, we have $H(u,v) = o(u^2+v^2)$. Furthermore, we can verify that

$$G(t;0,v) < 0 \quad \text{for} \quad v \neq 0, \quad G(t;0,0) = 0,$$

and for each $n \in R$,

$$\lim_{t\to 0} \frac{G(t;u,nu)}{(u+1)H(u,nu)}$$

is finite, this outcomes shows that $\frac{G(t;u,v)}{(1+u)H(u,v)}$ is bounded near the origin. The following is to consider the function $\Phi(t)$ defined above.

$$\frac{V'(t)}{V(t)} = \frac{G(t;u(t),v(t))}{(u(t)+1)H(u(t),v(t))} \leq \max\frac{G(t;u,v)}{(1+u)H(u,v)}, (u,v) \in Q(t),$$

that is

$$V'(t) \leq \Phi(t)V(t).$$

Obviously, if $\Phi(t) < 0$, we have $\lim_{t\to+\infty} V(t) = 0$, which suggests that

$$\lim_{t\to+\infty} |u(t)| = 0 = \lim_{t\to+\infty} |v(t)|.$$

Going back to the solution $(x(t), y(t))$, from the above we can deduce

$$\lim_{t\to+\infty} |x(t) - x^*(t)| = 0 = \lim_{t\to+\infty} |y(t) - y^*(t)|.$$

So the positive T-periodic solution $(x^*(t), y^*(t))$ of the system (1.1) is global asymptotically stable. □

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