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ON INVERSES AND EIGENPAIRS OF PERIODIC TRIDIAGONAL TOEPLITZ MATRICES WITH PERTURBED CORNERS*

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Abstract In this paper, we derive explicit determinants, inverses and eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners of Type I. The Mersenne numbers play an important role in these explicit formulas derived. Our main approaches include clever uses of the Schur complement and matrix decomposition with the Sherman-Morrison-Woodbury formula. Besides, the properties of Type II matrix can be also obtained, which benefits from the relation between Type I and II matrices. Lastly, we give three algorithms for these basic quantities and analyze them to illustrate our theoretical results.

Keywords Determinant, inverse, eigenpair, periodic tridiagonal Toeplitz matrix.

MSC(2010) 15A09, 15A15, 15A69, 65F05.

1. Introduction

The main research object of this paper is an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^{n}$, which is called a periodic tridiagonal Toeplitz matrix with perturbed corners of type I and defined as follows

$$A = \begin{pmatrix} \alpha_1 & 2\beta & 0 & \cdots & 0 & \gamma_1 \\ 0 & -3\beta & \ddots & \ddots & & 0 \\ 0 & \beta & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2\beta & 0 \\ 0 & & \ddots & \ddots & -3\beta & 0 \\ \alpha_n & 0 & \cdots & 0 & \beta & \gamma_n \end{pmatrix}_{n\times n}, \tag{1.1}$$

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where $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \beta$ are complex numbers with $\beta \neq 0$. Let $\hat{I}_n$ be the $n \times n$ "reverse unit matrix", which has ones along the secondary diagonal and zeros elsewhere. Let A be defined as in (1.1). A matrix of the form $B := \hat{I}_n A \hat{I}_n$ is called a periodic tridiagonal Toeplitz matrix with perturbed corners of type II. In this case, we say B is induced by A. It is readily seen that if A is a periodic tridiagonal Toeplitz matrix with perturbed corners of type I if and only if its transpose A^T is a periodic tridiagonal Toeplitz matrix with perturbed corners of type II.

General tridiagonal matrices appear not only in pure linear algebra, but also in many practical applications, such as, computer graphics [1], image denoising [26] and partial differential equations [9, 34, 35, 40], etc. One takes the one-dimensional linear hyperbolic equation

$$\frac{\partial u(x,t)}{\partial t} + v\frac{\partial u(x,t)}{\partial x} = g$$

considered by Holmgren and Otto [13] as an example to study certain matrices occur in discretized partial differential equations, where $0 < x \leq 1,\ t > 0,\ u(0,t) = f(-at),\ u(x,0) = f(x), g = (v-a)f'$. Here v and a are positive constants and f is a scalar function with derivative f'. Let k and h denote the time step and spatial step respectively. The linear hyperbolic equations discretized based on trapezoidal rule in time and center difference in space, respectively, whose coefficient matrix is a tridiagonal matrix with perturbed last row [2]:

$$C = \begin{pmatrix} 4 & \alpha & 0 & \cdots & \cdots & 0 \\ -\alpha & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -\alpha & 4 & \alpha \\ 0 & \cdots & \cdots & 0 & -2\alpha & 4+2\alpha \end{pmatrix}_{n\times n},$$

where $\alpha = vk/h$. On the other hand, some parallel computing algorithms are also designed for solving tridiagonal systems on graphics processing unit (GPU), which are parallel cyclic reduction [14] and partition methods [39]. Recently, Yang et al. [41] presented a parallel solving method which mixes direct and iterative methods for block-tridiagonal equations on CPU-GPU heterogeneous computing systems, while Myllykoski et al. [27] proposed a generalized graphics processing unit implementation of partial solution variant of the cyclic reduction (PSCR) method to solve certain types of separable block tridiagonal linear systems. Compared to an equivalent CPU implementation that utilizes a single CPU core, PSCR method indicated up to 24-fold speedups.

Many studies have been conducted for tridiagonal matrices [10, 16–19, 43]. Typical results for their inverses include Usmani's algorithm [38] based on rudimentary matrix analysis, El-Mikkawy and Atlan's two symbolic algorithms [5, 6] based on the Doolittle LU factorization of the k-tridiagonal matrix, Jia et al.'s algorithms [20, 21] based on block diagonalization technique, and so on. There are also some studies on the inverse of nonsingular periodic tridiagonal matrices [7, 22]. Tim and Emrah [37] used backward continued fractions to derive the LU factorization of periodic tridiagonal matrix and then derived the explicit formula for its inverse. Dow [4] discussed some special Toeplitz matrices including periodic tridiagonal Toeplitz matrices, while Shehawey [8] generalized Huang and McColl's [15] work and put forward the inverse formula for periodic tridiagonal Toeplitz matrices. Furthermore, some authors have done some research on the eigenpairs of tridiagonal matrices or periodic tridiagonal matrices based on the method of symbolic calculus for difference equations [3, 11, 42].

The rest of the paper is organized as follows: Section 2 illustrates the importance of the Mesenne numbers in the main results. Section 3 describes the detailed derivations of the determinants, inverses and eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners of Type I. Specifically, the formulas on representation of the determinants and inverses of these typies matrices in the form of products of Mersenne numbers and some initial values. Our main approaches include clever uses of the Schur complement [44, p.10] and matrix decomposition with Sherman-Morrison-Woodbury formula [12]. Besides, we calculate the eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners based on the eigenpairs of the symmetric tridiagonal Toeplitz matrix [23]. Furthermore, the properties of the periodic tridiagonal Toeplitz matrices with perturbed corners of Type II can be also obtained. Section 4 presents three algorithms for these basic quantities and analyze them to illustrate our theoretical results. The final conclusions are given in Section 5.

2. Mersenne Numbers and Applications

In this section, we introduce the Mersenne number M_n, which satisfies the following recurrence [32]:

$$M_{n+1} = 3M_n - 2M_{n-1} \quad \text{where} \quad M_0 = 0,\ M_1 = 1,\ n \geq 1; \tag{2.1}$$

$$M_{-(n+1)} = \frac{3}{2}M_{-n} - \frac{1}{2}M_{-(n-1)} \quad \text{where} \quad M_0 = 0,\ M_{-1} = -\frac{1}{2},\ n \geq 1. \tag{2.2}$$

It is known that the nth Mersenne number has the Binet formula $M_n = 2^n - 1$. Mersenne numbers are ubiquitous in combinatorics, number theory, group theory, chaos, geometry, physics, etc [25]. More specifically, Mersenne numbers play an important role in digital signal processing, which stems from arithmetic operations modulo Mersenne numbers can be implemented relatively simply in digital hardware [28]. Especially, Mersenne Number Transforms are often used to deal with problems of digital filtering and convolution of discrete signals [31, 36]. For example, Nussbaumer proposed some digital filtering using pseudo-Mersenne transforms in [29] and pseudo-Fermat number transforms in [30].

It is clearly seen that the elements in the diagonal and superdiagonal of matrix A satisfy the recursive relationship of the Mersenne sequence. Those elements chosen in the diagonal and superdiagonal reflect the neat applications of Mersenne numbers. Besides, the structure of the matrix itself determines some basic quantities of the matrix such as determinant and inverse. Therefore, the Mersenne numbers play a huge role in our main results.

3. The Determinants, Inverses and Eigenpairs Formulas of Periodic Tridiagonal Toeplitz Matrices with Perturbed Corners

In this section, we derive explicit formulas for the determinants, eigenpairs and inverses of periodic tridiagonal Toeplitz matrix with perturbed corners. Main effort is made for working out those for periodic tridiagonal Toeplitz matrix with perturbed corners of type I, since the results for type II matrices would follow immediately.

Theorem 3.1. *Let $A=(a_{i,j})_{i,j=1}^{n}$ $(n\geq 3)$ be given as in (1.1). Then*

$$\det A=(-\beta)^{n-2}\big(\alpha_1\gamma_n-\alpha_n\gamma_1\big)M_{n-1},\tag{3.1}$$

where M_{n-1} is the $(n-1)$th Mersenne number.

Proof. Define a circulant matrix

$$\rho=(\rho_{i,j})_{i,j=1}^{n},\tag{3.2}$$

where

$$\rho_{i,j}=\begin{cases}1, & i=n,j=1,\\ 1, & j=i+1,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, ρ is invertible and

$$\det\rho=(-1)^{n-3}.\tag{3.3}$$

Multiply A by ρ from right and then partition $A\rho$ into four blocks:

$$A\rho=\left(\begin{array}{cc|cccccc}
\gamma_1 & \alpha_1 & 2\beta & 0 & \cdots & \cdots & \cdots & 0\\
0 & 0 & -3\beta & 2\beta & 0 & \cdots & \cdots & 0\\ \hline
0 & 0 & \beta & -3\beta & 2\beta & 0 & \cdots & 0\\
\vdots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & 2\beta\\
0 & 0 & \vdots & & & \ddots & \ddots & -3\beta\\
\gamma_n & \alpha_n & 0 & \cdots & \cdots & \cdots & 0 & \beta
\end{array}\right)=\left(\begin{array}{c|c} A_{11} & A_{12}\\ \hline A_{21} & A_{22}\end{array}\right).\tag{3.4}$$

Since A_{22} is upper triangular, its determinant is clear which is

$$\det A_{22}=\beta^{n-2}.\tag{3.5}$$

As we assume $\beta\neq 0$, so A_{22} is invertible. It is known [45, Lemma 2.5] that $A_{22}^{-1}=(\hat{a}_{i,j})_{i,j=1}^{n}$, where

$$\hat{a}_{i,j}=\begin{cases}\frac{M_{j-i+1}}{\beta}, & i\leq j,\\ 0, & i>j,\end{cases}$$

and M_i is the ith Mersenne number.

Next, taking the determinants for both sides of (3.4) and by [44, p.10], we get

$$\det(A\rho) = \det A_{22}\det(A_{11} - A_{12}A_{22}^{-1}A_{21}). \tag{3.6}$$

Therefore,

$$\det A = \frac{\det A_{22}\det(A_{11} - A_{12}A_{22}^{-1}A_{21})}{\det\rho}. \tag{3.7}$$

To find $\det A$, we need to evaluate the determinant of $(A_{11} - A_{12}A_{22}^{-1}A_{21})$. From (3.4), we have

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{pmatrix} \gamma_1 - 2M_{n-2}\gamma_n & \alpha_1 - 2M_{n-2}\alpha_n \\ M_{n-1}\gamma_n & M_{n-1}\alpha_n \end{pmatrix},$$

and so

$$\det\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right) = M_{n-1}\left(\alpha_n\gamma_1 - \alpha_1\gamma_n\right). \tag{3.8}$$

Finally, applying (3.3), (3.5) and (3.8) to (3.7), we get the determinant of A, which completes the proof. □

Theorem 3.2. *Let $A = (a_{i,j})_{i,j=1}^n (n \geq 3)$ be given as in (1.1) and assume A to be nonsingular. Then $A^{-1} = (\breve{a}_{i,j})_{i,j=1}^n$, where*

$$\breve{a}_{i,j} = \begin{cases} \frac{\gamma_n}{\alpha_1\gamma_n - \alpha_n\gamma_1}, & i = 1, j = 1, \\ \frac{2M_{n-2}\gamma_n - \gamma_1}{M_{n-1}\left(\alpha_1\gamma_n - \alpha_n\gamma_1\right)}, & i = 1, j = 2, \\ 0, & i \in \{2,3\}, j = 1, \\ -\frac{M_{n-i}}{M_{n-1}\beta}, & i \in \{2,3\}, j = 2, \\ 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2} + \frac{1}{\beta}, & i \in \{2,3\}, j = i+1, \\ -\frac{\gamma_1}{\gamma_n}\breve{a}_{i,1}, & 1 \leq i \leq n-1, j = n, \\ -\frac{\alpha_n\breve{a}_{1,j} + \beta\breve{a}_{n-1,j}}{\gamma_n}, & i = n, j \in \{1,2\}, \\ \frac{3\breve{a}_{i-1,j} - \breve{a}_{i-2,j}}{2}, & \begin{cases} 4 \leq i \leq n-1, j \in \{1,2\}; \\ 4 \leq i < j \leq n-1. \end{cases} \\ 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, & \begin{cases} i \in \{1,2,3\}, i+2 \leq j \leq n-1; \\ 3 \leq j \leq i \leq n, \end{cases} \end{cases} \tag{3.9}$$

and M_i $(i = n-3, n-2, n-1)$ is the ith Mersenne number.

Proof. Let $A^{-1} = (\breve{a}_{i,j})_{i,j=1}^n$ and the identity matrix $I = (e_{i,j})_{i,j=1}^n$, where

$$e_{i,j} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases} \tag{3.10}$$

For nonsingular A, we get from $A^{-1}A = I$ that

$$e_{i,j} = \begin{cases} 2\breve{a}_{i,j-1}\beta - 3\breve{a}_{i,j}\beta + \breve{a}_{i,j+1}\beta, & 1 \leq i \leq n,\ 2 \leq j \leq n-1, \\ \breve{a}_{i,1}\gamma_1 + \breve{a}_{i,n}\gamma_n, & 1 \leq i \leq n,\ j = n. \end{cases} \tag{3.11}$$

Similarly, according to $AA^{-1} = I$, we get

$$e_{i,j} = \begin{cases} \breve{a}_{i-1,j}\beta - 3\breve{a}_{i,j}\beta + 2\breve{a}_{i+1,j}\beta, & 1 \le j \le n,\ 3 \le i \le n-2, \\ \breve{a}_{1,j}\alpha_n + \breve{a}_{n-1,j}\beta + \breve{a}_{n,j}\gamma_n, & 1 \le j \le n,\ i = n. \end{cases} \tag{3.12}$$

Based on (3.10), we get from (3.11) that

$$\breve{a}_{i,j} = \begin{cases} 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, & \begin{cases} 3 \le j \le i \le n; \\ i \in \{1,2,3\}, i+2 \le j \le n-1, \end{cases} \\ -\frac{\gamma_1}{\gamma_n}\breve{a}_{i,1}, & 1 \le i \le n-1, j = n \end{cases} \tag{3.13}$$

and $\breve{a}_{2,3} = 3\breve{a}_{2,2} - 2\breve{a}_{2,1} + \frac{1}{\beta},\ \breve{a}_{3,4} = 3\breve{a}_{3,3} - 2\breve{a}_{3,2} + \frac{1}{\beta}$.

Similarly, from (3.12), we get that

$$\breve{a}_{i,j} = \begin{cases} -\frac{\alpha_n\breve{a}_{1,j} + \beta\breve{a}_{n-1,j}}{\gamma_n}, & i = n,\ j \in \{1,2\}, \\ \frac{3\breve{a}_{i-1,j} - \breve{a}_{i-2,j}}{2}, & \begin{cases} 4 \le i \le n-1, j \in \{1,2\}; \\ 4 \le i < j \le n-1. \end{cases} \end{cases} \tag{3.14}$$

Based on the above analysis, we need to determine six initial values, that is, $\breve{a}_{i,j}$ ($i \in \{1,2,3\}, j \in \{1,2\}$), for the recurrence relations (3.13) and (3.14) in order to compute the inverse of A. The rest of the proof is devoted to evaluating these particular entries of A^{-1}.

We decompose A as follows

$$A = \beta\Delta + LK, \tag{3.15}$$

where

$$\Delta = \begin{pmatrix} -\frac{2M_n}{M_{n+1}} & 2 & 0 & \cdots & 0 & \frac{2^n}{M_{n+1}} \\ 1 & -3 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -3 & 2 \\ \frac{2}{M_{n+1}} & 0 & \cdots & 0 & 1 & -\frac{2M_n}{M_{n+1}} \end{pmatrix}_{n\times n},$$

$L = (l_1^T, l_2^T),\ K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ with

$$\begin{aligned} l_1 &= (\alpha_1 + \frac{2M_n\beta}{M_{n+1}}, -\beta, 0, \cdots, 0, \alpha_n - \frac{2\beta}{M_{n+1}})_{1\times n}, \\ l_2 &= (\gamma_1 - \frac{(M_n+1)\beta}{M_{n+1}}, 0, \cdots, 0, -2\beta, \gamma_n + \frac{2M_n\beta}{M_{n+1}})_{1\times n}, \\ k_1 &= (1, 0, \cdots, 0)_{1\times n},\ k_2 = (0, \cdots, 0, 1)_{1\times n}, \end{aligned}$$

and M_i the ith Mersenne number as before.

It could be verified that $\Delta^{-1} = \frac{1}{3}(t_{ij})_{i,j=1}^{n}$, where

$$t_{ij} = \begin{cases} M_{j-i+1}, & 1 \le i \le j \le n, \\ -2M_{j-i-1}, & 1 \le j < i \le n, \end{cases}$$

and M_{-m} is given in (2.2) for $m = 1, 2, \ldots$.

Applying the Sherman-Morrison-Woodbury formula [12, p.50] to (3.15) gives

$$A^{-1} = (\beta\Delta + LK)^{-1} = \frac{1}{\beta}\Delta^{-1} - \frac{1}{\beta^2}\Delta^{-1}L(I + \frac{1}{\beta}K\Delta^{-1}L)^{-1}K\Delta^{-1}. \tag{3.16}$$

Now we compute each component on the right side of (3.16).

Multiplying respectively Δ^{-1} by K and L from left and right,

$$K\Delta^{-1} = \frac{1}{3}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \tag{3.17}$$

$$\Delta^{-1}L = \frac{1}{3}\begin{pmatrix} \xi_1^T & \xi_2^T \end{pmatrix}, \tag{3.18}$$

where

$$\begin{aligned}
&\mu_1,\ \mu_2,\ \xi_1 \text{ and } \xi_2 \text{ are row vectors,} \\
&\mu_1 = (M_j)_{j=1}^{n},\ \mu_2 = (-2M_{j-n-1})_{j=1}^{n}, \\
&\xi_1 = \big(\xi_{1,1} - 3\beta, \xi_{2,1}, \cdots, \xi_{i,1}, \cdots, \xi_{n,1}\big), \\
&\xi_2 = \big(\xi_{1,2}, \cdots, \xi_{i,2}, \cdots, \xi_{n-1,2}, \xi_{n,2} - 3\beta\big), \\
&\xi_{i,1} = M_{n-i+1}\alpha_n - 2M_{-i}\alpha_1, \quad i = 1, 2, \cdots, n, \\
&\xi_{i,2} = M_{n-i+1}\gamma_n - 2M_{-i}\gamma_1, \quad i = 1, 2, \cdots, n.
\end{aligned}$$

Then multiplying (3.18) by $\frac{K}{\beta}$ from the left and further adding I, we have

$$I + \frac{1}{\beta}K\Delta^{-1}L = \frac{1}{3\beta}\begin{pmatrix} \alpha_1 + M_n\alpha_n & \gamma_1 + M_n\gamma_n \\ -2M_{-n}\alpha_1 + \alpha_n & -2M_{-n}\gamma_1 + \gamma_n \end{pmatrix}. \tag{3.19}$$

Computing the inverse of the matrices on both sides of (3.19), we obtain

$$\big(I + \frac{1}{\beta}K\Delta^{-1}L\big)^{-1} = \frac{3\beta\begin{pmatrix} -2M_{-n}\gamma_1 + \gamma_n & -\gamma_1 - M_n\gamma_n \\ 2M_{-n}\alpha_1 - \alpha_n & \alpha_1 + M_n\alpha_n \end{pmatrix}}{M_{n+1}M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1)}.$$

Multiplying the previous formula $\big(I + \frac{1}{\beta}K\Delta^{-1}L\big)^{-1}$ by $\Delta^{-1}L$ from the left and by $K\Delta^{-1}$ from the right, respectively, yields

$$G = \Delta^{-1}L\big(I + \frac{1}{\beta}K\Delta^{-1}L\big)^{-1}K\Delta^{-1} = (g_{ij})_{i,j=1}^{n}, \tag{3.20}$$

where

$$g_{ij}=\begin{cases}\frac{\left(4M_{j-1}M_{-(n+1)}\gamma_1-M_{j-n}M_{n+1}\gamma_n\right)\beta^2}{M_{n+1}M_{1-n}\left(\alpha_1\gamma_n-\alpha_n\gamma_1\right)}+\frac{M_j\beta}{3}, & i=1,1\le j\le n,\\ \frac{2\left(2M_{n-i}M_jM_{-(n+1)}-M_{1-i}M_{j-n-1}M_{n+1}\right)\beta}{3M_{n+1}M_{1-n}}, & 2\le i\le n-1,1\le j\le n,\\ \frac{\left(M_{j-n}M_{n+1}\alpha_n-4M_{j-1}M_{-(n+1)}\alpha_1\right)\beta^2}{M_{n+1}M_{1-n}\left(\alpha_1\gamma_n-\alpha_n\gamma_1\right)}-\frac{2M_{j-n-1}\beta}{3}, & i=n,1\le j\le n.\end{cases}$$

From (3.16) and (3.20), we have

$$(\breve{a}_{i,j})_{i,j=1}^n=\frac{1}{\beta}\Delta^{-1}-\frac{1}{\beta^2}(g_{ij})_{i,j=1}^n, \tag{3.21}$$

where

$$\breve{a}_{i,j}=\frac{M_{j-i+1}}{3\beta}-\frac{g_{i,j}}{\beta^2}, \qquad 1\le i\le j\le n, \tag{3.22}$$

$$\breve{a}_{i,j}=-\frac{2M_{j-i-1}}{3\beta}-\frac{g_{i,j}}{\beta^2}, \qquad 1\le j<i\le n. \tag{3.23}$$

By (3.22), we compute,

$$\breve{a}_{1,1}=\frac{\gamma_n}{\alpha_1\gamma_n-\alpha_n\gamma_1},\ \breve{a}_{1,2}=\frac{2M_{n-2}\gamma_n-\gamma_1}{M_{n-1}\left(\alpha_1\gamma_n-\alpha_n\gamma_1\right)},\ \breve{a}_{2,2}=-\frac{M_{n-2}}{M_{n-1}\beta}.$$

By (3.23), we compute,

$$\breve{a}_{2,1}=\breve{a}_{3,1}=0,\breve{a}_{3,2}=-\frac{M_{n-3}}{M_{n-1}\beta},\breve{a}_{n,1}=\frac{\alpha_n}{\alpha_n\gamma_1-\alpha_1\gamma_n},\breve{a}_{n,2}=\frac{2M_{n-2}\alpha_n-\alpha_1}{M_{n-1}\left(\alpha_n\gamma_1-\alpha_1\gamma_n\right)}.$$

This completes the proof. □

Remark 3.1. The formulas (3.22) and (3.23) would give an analytic formula for A^{-1}. However, there is a big advantage of (3.9) from computational consideration as we shall see from Section 3.

Theorem 3.3. *Let $A=(a_{i,j})_{i,j=1}^n$ $(n\ge 3)$ be given as in (1.1). The eigenpairs (λ_j,ϑ_j), $j=1,\dots,n$, of A are determined by the following formulas*

$$\lambda_j=\begin{cases}\frac{\gamma_n+\alpha_1+\sqrt{(\gamma_n-\alpha_1)^2+4\alpha_n\gamma_1}}{2}, & j=1,\\ \frac{\gamma_n+\alpha_1-\sqrt{(\gamma_n-\alpha_1)^2+4\alpha_n\gamma_1}}{2}, & j=2,\\ -3\beta+2\sqrt{2}\beta\cos\frac{(j-2)\pi}{n-1}, & j=3,4,\cdots,n,\end{cases} \tag{3.24}$$

and

$$\vartheta_j=\rho\phi\eta_j, \qquad j=1,2,\cdots,n, \tag{3.25}$$

where ρ is the same as in (3.2),

$$\phi=\operatorname{diag}\left(1,1,1,2^{-\frac{1}{2}},2^{-\frac{2}{2}},\cdots,2^{-\frac{n-3}{2}}\right),$$
$$\eta_j=(\eta_{1,j},\cdots,\eta_{n,j})^T,$$

$$\eta_1 = \left(\frac{\gamma_n - \alpha_1 + \sqrt{(\gamma_n - \alpha_1)^2 + 4\alpha_n\gamma_1}}{2\gamma_1}, 1, 0, \cdots, 0 \right)^T,$$

$$\eta_2 = \left(\frac{\gamma_n - \alpha_1 - \sqrt{(\gamma_n - \alpha_1)^2 + 4\alpha_n\gamma_1}}{2\gamma_1}, 1, 0, \cdots, 0 \right)^T,$$

$$\eta_{k,j} = \begin{cases} \sqrt{\frac{2}{n-1}} \sin \frac{(k-2)(j-2)\pi}{n-1}, & 3 \le k, j \le n, \\ \frac{2\alpha_n\beta\eta_{3,j} + 2^{-\frac{n-3}{2}}(\lambda_j - \alpha_1)\beta\eta_{n,j}}{(\lambda_j-\alpha_1)(\lambda_j-\gamma_n)-\alpha_n\gamma_1}, & k = 2,\ 3 \le j \le n, \\ \frac{2(\lambda_j-\gamma_n)\beta\eta_{3,j} + 2^{-\frac{n-3}{2}}\gamma_1\beta\eta_{n,j}}{(\lambda_j-\alpha_1)(\lambda_j-\gamma_n)-\alpha_n\gamma_1}, & k = 1,\ 3 \le j \le n. \end{cases}$$

Proof. Let $\phi = \operatorname{diag}\left(1, 1, 1, 2^{-\frac{1}{2}}, 2^{-\frac{2}{2}}, \cdots, 2^{-\frac{n-3}{2}}\right)$ and consider the similarity transformations of A,

$$\phi^{-1}(\rho^{-1}A\rho)\phi = \left(\begin{array}{cc|cccccc} \gamma_n & \alpha_n & 0 & \cdots & \cdots & 0 & 2^{-\frac{n-3}{2}}\beta \\ \gamma_1 & \alpha_1 & 2\beta & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & -3\beta & \sqrt{2}\beta & 0 & \cdots & 0 \\ \vdots & \vdots & \sqrt{2}\beta & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \sqrt{2}\beta \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{2}\beta & -3\beta \end{array}\right) = \left(\begin{array}{c|c} A'_{11} & A'_{12} \\ \hline A'_{21} & A'_{22} \end{array}\right) = \tau, \tag{3.26}$$

where ρ is the same as in (3.2).

Since similarity transformation preserves eigenvalues, it suffices to work out the eigenvalues of τ. It is easy to see the eigenvalues of τ are the union of the eigenvalues of A'_{11} and A'_{22}.

Upon simple calculation, we find the eigenvalues of A'_{11} are

$$\lambda_1 = \frac{\gamma_n + \alpha_1 + \sqrt{(\gamma_n - \alpha_1)^2 + 4\alpha_n\gamma_1}}{2},$$
$$\lambda_2 = \frac{\gamma_n + \alpha_1 - \sqrt{(\gamma_n - \alpha_1)^2 + 4\alpha_n\gamma_1}}{2}.$$

By Lemma 2.1 in [24], the eigenvalues of the symmetric tridiagonal Toeplitz matrix A'_{22} are

$$\lambda_j = -3\beta + 2\sqrt{2}\beta \cos \frac{(j-2)\pi}{n-1},\ j = 3, 4, \cdots, n.$$

Thus we have determined all the eigenvalues of A.

Next, we compute the corresponding eigenvectors η_j such that $(\lambda_j I - \tau)\,\eta_j = 0$, $j = 1, \ldots, n$.

For $j = 1$, we solve $(\lambda_1 I - \tau)\,\eta_1 = 0$ to get

$$\eta_1 = \left(\frac{\gamma_n - \alpha_1 + \sqrt{(\gamma_n - \alpha_1)^2 + 4\alpha_n\gamma_1}}{2\gamma_1}, 1, 0, \cdots, 0 \right)^T.$$

Similarly, for $j=2$, we solve $(\lambda_2 I-\tau)\,\eta_2=0$ to get

$$\eta_2=\left(\frac{\gamma_n-\alpha_1-\sqrt{(\gamma_n-\alpha_1)^2+4\alpha_n\gamma_1}}{2\gamma_1},1,0,\cdots,0\right)^T.$$

Denote $\eta_j=(\eta_{1,j},\cdots,\eta_{n,j})^T$ for $j=3,4,\cdots,n$. By Lemma 2.1 in [24], the eigenvectors of the symmetric tridiagonal Toeplitz matrix A'_{22} are

$$\eta_{k,j}=\sqrt{\frac{2}{n-1}}\sin\frac{(k-2)(j-2)\pi}{n-1},\qquad 3\le k,j\le n. \tag{3.27}$$

Now we calculate $\eta_{k,j}$ $(k=1,2)$ based on (3.27) that

$$\eta_{k,j}=\begin{cases}\frac{2(\lambda_j-\gamma_n)\beta\eta_{3,j}+2^{-\frac{n-3}{2}}\gamma_1\beta\eta_{n,j}}{(\lambda_j-\alpha_1)(\lambda_j-\gamma_n)-\alpha_n\gamma_1}, & k=1,\ 3\le j\le n,\\ \frac{2\alpha_n\beta\eta_{3,j}+2^{-\frac{n-3}{2}}(\lambda_j-\alpha_1)\beta\eta_{n,j}}{(\lambda_j-\alpha_1)(\lambda_j-\gamma_n)-\alpha_n\gamma_1}, & k=2,\ 3\le j\le n.\end{cases}$$

Next, by $\phi^{-1}(\rho^{-1}A\rho)\phi=\tau$, we have

$$A(\rho\phi\eta_j)=\lambda_j(\rho\phi\eta_j),\quad j=1,2,\cdots,n.$$

Therefore, the corresponding eigenvector of λ_j for A is $\vartheta_j=\rho\phi\eta_j$ $(j=1,2,\ldots,n)$, which completes the proof. $\square$

As the determinant of a complex matrix is equal to the product of its eigenvalues, we have the following corollary.

Corollary 3.1. *Let $n\ge 3$ and M_n be the nth Mersenne number. Then*

$$M_{n-1}=\prod_{j=3}^{n}\left(3-2\sqrt{2}\cos\frac{(j-2)\pi}{n-1}\right).$$

The next three theorems are parallel results of type I matrices.

Theorem 3.4. *Let A be given as in (1.1). If B is a periodic tridiagonal Toeplitz matrix with perturbed corners of type II, which is induced by A, then*

$$\det B=(-\beta)^{n-2}\big(\alpha_1\gamma_n-\alpha_n\gamma_1\big)M_{n-1}.$$

Proof. Since $\det B=\det\hat{I}_n\det A\det\hat{I}_n$, we obtain this conclusion by using Theorem 4 and $\det\hat{I}_n=(-1)^{\frac{n(n-1)}{2}}$. $\square$

Theorem 3.5. *Let A be given as in (1.1) and let B be a periodic tridiagonal Toeplitz matrix with perturbed corners of type II, which is induced by A. Then*

$$B^{-1}=(\breve{a}_{n+1-i,n+1-j})_{i,j=1}^{n},$$

where $\breve{a}_{i,j}$ is the same as (3.9).

Proof. It follows immediately from $B^{-1}=\hat{I}_n^{-1}A^{-1}\hat{I}_n^{-1}=\hat{I}_nA^{-1}\hat{I}_n$ and Theorem 3.2. $\square$

Theorem 3.6. *Let A be given as in (1.1) and let B be a periodic tridiagonal Toeplitz matrix with perturbed corners of type II, which is induced by A. Assume the eigenpairs of B are $(\lambda_j,\breve{\vartheta}_j)$, $j=1,\ldots,n$. Then λ_j is the same as in (3.24) and $\breve{\vartheta}_j=\hat{I}_n\vartheta_j$, where ϑ_j is the same as in (3.25).*

Proof. Since B is similar to A, B and A have the same eigenvalues λ_j $(j=1,2,\cdots,n)$, where λ_j is same with (3.24). The other claim is straightforward. $\square$

4. Algorithms

In this section, we give three algorithms for finding the determinant, inverse and eigenpairs of periodic tridiagonal Toeplitz matrix with perturbed corners of type I, which is called A. Besides, we make some analysis about these algorithms to illustrate our theoretical results.

Table 1. Comparison of the total number operations for determinant of A.

Algorithms	Number operations
LU decomposition algorithm	$14n-12$
Algorithm 1	$2n+1$[1]

[1] The operation for the determinant of A in our algorithm is $2n+1$, which can be reduced to $O(logn)$ (see, [33], p.226–227).

Firstly, based on Theorem 3.1, we give an algorithm for computing determinant of A as following:

Algorithm 1.
Step 1: Input $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \beta$, order n and generate Mersenne number M_{n-1} by (2.1).
Step 2: Calculate and output the determinant of A by (3.1).

Based on Algorithm 1, we make a comparison of the total number operations for determinant of A between LU decomposition and Algorithm 1 in Table 1. Specifically, we get that the total number operation for the determinant of A is $2n+1$. What's more, this number can be reduced to $O(logn)$ (see, [33], p.226–227).

Next, based on Theorem 3.2, we give an algorithm for computing inverse of A as following:

Algorithm 2.
Step 1: Input $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \beta$, order n and generate Mersenne numbers by (2.1);
Step 2: By the formula (3.9), compute respectively the six elements $\breve{a}_{1,1}, \breve{a}_{1,2}, \breve{a}_{2,1}, \breve{a}_{2,2}, \breve{a}_{3,1}, \breve{a}_{3,2}$ and then compute the remaining elements of the inverse A^{-1};
Step 3: Output the inverse $A^{-1} = (\breve{a}_{i,j})_{i,j=1}^{n}$.

Table 2. Comparison of the total number operations for inverse of A.

Algorithms	Number operations
LU decomposition algorithm	$\frac{5n^3}{6}+3n^2+\frac{91n}{6}-21$
Algorithm 2	$3n^2+n+20$

To test the effectiveness of Algorithm 2, we firstly compare the total number operation for the inverse of A between LU decomposition and Algorithm 2 in Table 2. The total number operation of LU decomposition is $\frac{5n^3}{6}+3n^2+\frac{91n}{6}-21$, while that of Algorithm 2 is $3n^2+n+20$.

Lastly, Algorithm 3 gives the eigenpairs of A based on Theorem 3.3. The total number operation for computing eigenpairs of A is $5n^2+20n-23$.

Algorithm 3.
Step 1: Input $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \beta, n$, generate $\phi = \operatorname{diag}\left(1, 1, 1, 2^{-\frac{1}{2}}, 2^{-\frac{2}{2}}, \cdots, 2^{-\frac{n-3}{2}}\right)$

and ρ is the same as in (3.2).
Step 2: Calculate the eigenvalues $\lambda_j,\ j = 1, 2, \cdots, n$ by (3.24).
Step 3: Calculate η_j and the eigenvectors $\vartheta_j,\ j = 1, 2, \cdots, n$ by (3.25).
Step 4: Output the eigenpairs (λ_j, ϑ_j), $j = 1, 2, \cdots, n$.

5. Conclusions

In this paper, we present the explicit formulas for determinants, inverses and eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners. The representation of the determinant in the form of products of Mersenne number and some initial values from matrix transformations. For inverse, our main approach includes a clever use of matrix decomposition with the Sherman-Morrison-Woodbury formula. Besides, we calculate the eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners based on the eigenpairs of the symmetric tridiagonal Toeplitz matrix. To test our method's effectiveness, we propose three algorithms for finding the determinants, inverses and eigenpairs of periodic tridiagonal Toeplitz matrices with perturbed corners as well as compare the total number operation for these basic quantities between different algorithms. After comparison, we draw a conclusion that our algorithms are superior to other algorithms to some extent.

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References

[1] J. Bender, M. Müller, M. Otaduy, T. Matthias and M. Miles, *A survey on position-based simulation methods in computer graphics*, Comput. Graph. Forum., 2014, 33, 228–251.

[2] R. H. Chan, X. Q. Jin, *Circulant and skew-circulant preconditioners for skew-Hermitian type Toeplitz systems*, BIT 31, 1991, 632–646.

[3] S. S. Cheng, *Partial difference equations*, Taylor and Francis, London, 2003.

[4] M. Dow, *Explicit inverses of Toeplitz and associated matrices*, ANZIAM J., 2008, 44, 185–215.

[5] M. El-Mikkawy and F. Atlan, *A novel algorithm for inverting a general k-tridiagonal matrix*, Appl. Math. Lett., 2014, 32, 41–47.

[6] M. El-Mikkawy and F. Atlan, *A new recursive algorithm for inverting general k-tridiagonal matrices*, Appl. Math. Lett., 2015, 44, 34–39.

[7] M. El-Mikkawy, *A new computational algorithm for solving periodic tri-diagonal linear systems*, Appl. Math. Comput., 2005, 161, 691–696.

[8] M. El-Shehawey, G. El-Shreef and A. ShAl-Henawy, *Analytical inversion of general periodic tridiagonal matrices*, J. Math. Anal. Appl., 2008, 345, 123–134.

[9] Q. H. Feng and F. W. Meng, *Explicit solutions for space-time fractional partial differential equations in mathematical physics by a new generalized fractional Jacobi elliptic equation-based sub-equation method*, Optik, 2016, 127, 7450–7458.

[10] C. M. da Fonseca and J. Petronilho, *Explicit inverse of a tridiagonal k-Toeplitz matrix*, Numerische Mathematik, 2005, 100, 457–482.

[11] C. M. da Fonseca, *On the eigenvalues of some tridiagonal matrices*, J. Comput. Appl. Math., 2007, 200, 283–286.

[12] G. H. Golub and C. F. Van Loan, *Matrix Computations third ed.*, The John Hopkins University Press, Baltimore, 1996.

[13] S. Holmgren, K. Otto, *Iterative solution methods and preconditioners for non-symmetric non-diagonally dominant block-tridiagonaI systems of equations*, Dept. of Computer Sci., Uppsala Univ., Sweden, 1989.

[14] R. W. Hockney and C.R. Jesshope, *Parallel Computers*, Adam Hilger, Bristol, 1981.

[15] Y. Huang and W. F. McColl, *Analytical inversion of general tridiagonal matrices*, J. Phys. A: Math. Gen., 1997, 30, 7919.

[16] X. Y. Jiang and K. Hong, *Skew cyclic displacements and inversions of two innovative patterned matrices*, Appl. Math. Comput., 2017, 308, 174–184.

[17] X. Y. Jiang, K. Hong and Z. W. Fu, *Skew cyclic displacements and decompositions of inverse matrix for an innovative structure matrix*, J. Nonlinear Sci. Appl., 2017, 10, 4058–4070.

[18] Z. L. Jiang and D. D. Wang, *Explicit group inverse of an innovative patterned matrix*, Appl. Math. Comput., 2016, 274, 220–228.

[19] Z. L. Jiang, X. T. Chen and J. M. Wang, *The explicit inverses of CUPL-Toeplitz and CUPL-Hankel matrices*, E. Asian J. Appl. Math., 2017, 7, 38–54.

[20] J. T. Jia and S. M. Li, *Symbolic algorithms for the inverses of general k-tridiagonal matrices*, Comput. Math. Appl., 2015, 70, 3032–3042.

[21] J. T. Jia and S. M. Li, *On the inverse and determinant of general bordered tridiagonal matrices*, Comput. Math. Appl., 2015, 69, 503–509.

[22] J. T. Jia and Q. X. Kong, *A symbolic algorithm for periodic tridiagonal systems of equations*, J. Math. Chem., 2014, 52, 2222–2233.

[23] J. T. Jia, Tomohiro Sogabe and Moawwad El-Mikkawy, *Inversion of k-tridiagonal matrices with Toeplitz structure*, Comput. Math. Appl., 2013, 65, 116–125.

[24] J. T. Jia, *Tomohiro Sogabe and Moawwad El-Mikkawy, Inversion of k-tridiagonal matrices with Toeplitz structure*, Comput. Math. Appl., 2013, 65, 116–125.

[25] M. Krizek, F. Luca and L. Somer, *17 Lectures on Fermat Numbers: From Number Theory to Geometry*, Springer Science & Business Media, 2013.

[26] M. Myllykoski, R. Glowinski, T. Kärkkäinen and T. Rossi, *A GPU-accelerated augmented Lagrangian based L^1-mean curvature image denoising algorithm implementation*, WSCG 2015 Conference on Computer Graphics, Visualization and Computer Vision, Plzen, Czech Republic, Union Agency, 2015.

[27] M. Myllykoski, T. Rossi and J. Toivanen, *On solving separable block tridiagonal linear systems using a GPU implementation of radix-4 PSCR method*, J. Parallel Distrib. Comput., 2018, 115, 56–66.

[28] H. J. Nussbaumer, *Fast Fourier Transform and convolution algorithms*, Springer Science & Business Media, 1981.

[29] H. J. Nussbaumer, *Digital filtering using complex Mersenne transforms*, IBM J. Res. Dev., 1976, 20, 498–504.

[30] H. J. Nussbaumer, *Digital filtering using pseudo Fermat number transform*, IEEE Trans. Acoust. Speech, Signal Processing, 1977, 25, 79–83.

[31] C. M. Rader, *Discrete convolution via Mersenne transforms*, IEEE Trans. Comput C., 1972, 21, 1269–1273.

[32] R. M. Robinson, *Mersenne and Fermat numbers*, P. Am. Math. Soc., 1954, 5, 842–846.

[33] K. H. Rosen, *Discrete mathematics and its applications*, McGraw-Hill, New York, 2011.

[34] J. Shao, Z. W. Zheng and F. W. Meng, *Oscillation criteria for fractional differential equations with mixed nonlinearities*, Adv. Differ. Equ-ny., 2013, 2013, 323.

[35] Y. G. Sun and F. W. Meng, *Interval criteria for oscillation of second-order differential equations with mixed nonlinearities*, Appl. Math. Comput., 2008, 198, 375–381.

[36] W. C. Siu and A. G. Constantinides, *Fast mersenne number transforms for the computation of discrete fourier transforms*, Signal Processing, 1985, 9, 125–131.

[37] H. Tim and K. Emrah, *An analytical approach: Explicit inverses of periodic tridiagonal matrices*, J. Comput. Appl. Math., 2018, 335, 207–226.

[38] R. A. Usmani, *Inversion of a tridiagonal Jacobi matrix*, Linear Algebra Appl., 1994, 212, 413–414.

[39] H. H. Wang, *A parallel method for tridiagonal equations*, ACM Trans. Math. Softw., 1981, 7, 170–183.

[40] R. Xu and F. W. Meng, *Some new weakly singular integral inequalities and their applications to fractional differential equations*, J. Inequal. Appl., 2016, 2016, 78.

[41] W. D. Yang, K. L. Li and K. Q. Li, *A parallel solving method for block-tridiagonal equations on CPU-GPU heterogeneous computing systems*, J. Supercomput., 2017, 73, 1760–1781.

[42] W. C. Yueh and S. S. Cheng, *Explicit eigenvalues and inverses of tridiagonal Toeplitz matrices with four perturbed corners*, ANZIAM J., 2008, 49, 361–387.

[43] Y. P. Zheng, S. Shon and J. Kim, *Cyclic displacements and decompositions of inverse matrices for CUPL Toeplitz matrices*, J. Math. Anal. Appl., 2017, 455, 727–741.

[44] F. Z. Zhang, *The Schur Complement and Its Applications*, Springer Science & Business Media, New York, 2006.

[45] B. S. Zuo, Z. L. Jiang and D. Q. Fu, *Determinants and inverses of Ppoeplitz and Ppankel matrices*, Special Matrices, 2018, 6, 201–215.