UPPER BOUNDS ON THE NUMBER OF DETERMINING MODES, NODES, AND VOLUME ELEMENTS FOR A 3D MAGNETOHYDRODYNAMIC-α MODEL

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Abstract In this paper we give upper bounds on the number of determining Fourier modes, determining nodes, and determining volume elements for a 3D MHD-α model. Here the bounds are estimated explicitly in terms of flow parameters, such as viscosity, magnetic diffusivity, smoothing length, forcing and domain size.

Keywords MHD-α model, determining modes, determining nodes, determining volume elements.

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1. Introduction

Let Ω = (0, L)³, L > 0, be a periodic box in R³. We consider the following 3D MHD-α model which was introduced by Linshiz and Titi in [27]

\[ \partial_t v - \nu \Delta v - u \times (\nabla \times v) - (B \cdot \nabla)B + \nabla p + \frac{1}{2} \nabla |B|^2 = f \text{ in } \Omega \times (0, \infty), \quad (1.1) \]
\[ \partial_t B - \eta \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u = 0 \text{ in } \Omega \times (0, \infty), \quad (1.2) \]
\[ v = u - \alpha^2 \Delta u \text{ in } \Omega \times (0, \infty), \quad (1.3) \]
\[ \nabla \cdot u = \nabla \cdot v = \nabla \cdot B = 0 \text{ in } \Omega \times (0, \infty), \quad (1.4) \]

subject to the periodic boundary conditions and the initial conditions

\[ u(0) = u^0, B(0) = B^0 \text{ in } \Omega. \quad (1.5) \]

Here u = u(x, t) is the unknown velocity, B = B(x, t) is the unknown magnetic field and p = p(x, t) is the unknown pressure, ν > 0 is the kinematic viscosity coefficient, η > 0 is the constant magnetic diffusivity and α is a length scale parameter. When α = 0 we formally recover the 3D classical MHD equations in [30]. Notice that here we only filter the velocity field but not the magnetic field, and it contrasts with the so-called Lagrangian-averaged magnetohydrodynamic-α (LAMHD-α) model (also called hyperbolic MHD equations or MHD-Voigt model) in [15].

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A 3D MHD-$\alpha$ model

The MHD-$\alpha$ model (1.1)-(1.4) involves coupling Maxwell’s equations governing the magnetic field and the Navier-Stokes-$\alpha$ equations (sometimes called the viscous Camassa-Holm equations). In recent years, the existence and long-time behavior of solutions to this MHD-$\alpha$ model has attracted the attention of many mathematicians. In [27], Linshiz and Titi proved the existence, uniqueness and regularity of solutions with periodic boundary conditions, while Fan and Ozawa [9] and Liu [28] achieved the same result in the whole space $\mathbb{R}^2$ for both cases ($\nu = 1, \eta = 0$) and ($\nu = 0, \eta = 1$). More recently, Zhou and Fan [34] also established the regularity criteria to guarantee the existence of smooth solutions for higher dimensional case. For the long-time behavior of solutions, the existence of a finite-dimensional global attractor was proved by Catania in [5] in the case of three-dimensional periodic box, and the time decay rate in $L^2(\mathbb{R}^3)$ of solutions was proved by Jiang and Fan in [19]. The Sobolev regularity and the Gevrey regularity of the global attractor was proved recently in [2]. When $B = 0$, the above MHD-$\alpha$ model reduces to the well-known Navier-Stokes-$\alpha$ equations, where the existence of a finite-dimensional global attractor was proved in [10, 18] in the case of periodic boundary conditions, and the decay rate of solutions on the whole space was proved by Bjorland and Schonbek in [4] and improved recently in [3] by using the theory of decay characters. We also refer the interested reader to [6, 7, 16, 24, 26, 35, 36] for results related to other MHD-$\alpha$ models.

The conventional theory of turbulence asserts that turbulent flows are monitored by a finite number of degrees of freedom. The notions and results for the case of 2D Navier-Stokes equations on determining modes [11, 12], determining nodes [13, 14, 20] and determining volume elements [14, 21] are rigorous attempts to identify those parameters that control turbulent flows. We refer the interested reader to [8] for a general unified framework for this issue of determining parameters and [17, 29] for some recent related results. In recent years, upper bounds on the number of determining modes and nodes were established for some $\alpha$-models, which were suggested as regularization models for the 3D Navier-Stokes equations when $\alpha$ is a small regularization parameter. More precisely, for the 3D Navier-Stokes-Voigt equations, Kalantarov and Titi in [23] proved a result on the determining modes. The results on the determining modes and determining nodes for some regularization models of 3D Navier-Stokes equations such as 3D Navier-Stokes-$\alpha$, 3D Leray-$\alpha$ and 3D Navier-Stokes-$\omega$ equations were proved by Korn in [25]. For MHD-$\alpha$ models, to the best of our knowledge, there is only a result on the determining modes for the 3D MHD-Voigt equations in [6].

In this paper, we study the number of determining modes, determining nodes and determining volume elements for the MHD-$\alpha$ model (1.1)-(1.4). To do this, we follow the general lines of the approach used in [22] for 2D Navier-Stokes equations. We first estimate the large time asymptotics for the solutions. Then, we establish some inequalities related to the nodal in the three-dimensional case which are extension of that in the two-dimensional case in [22], and hence we can get an upper bound on the number of determining nodes. The determining volume elements is proved in a similar manner with the help of our new inequalities related to the finite volume elements. To obtain the bound on the number of determining modes, we need some technical estimates which are similar to that used to prove the determining nodes and determining volume elements. It is worthy noticing that in the present paper the number of determining nodes, modes and finite volume elements is estimated explicitly in terms of flow parameters, such as viscosity, magnetic diffu-
sivity, smoothing length, forcing and domain size, and these estimates are global as they do not depend on an individual solution. It is also noticed that our arguments in the paper and our new technical estimates in three-dimensional case given in Lemmas 3.1 and 4.1 below can be used to study the degrees of freedom for some other 3D MHD-α models in [7, 16, 27].

The paper is organized as follows. In Section 2, we recall the functional setting of the 3D MHD-α model. Section 3 gives an upper bound on the number of determining nodes. The number of determining volume elements is studied in Section 4. We prove an upper bound on the number of determining modes in Section 5. For clarity of the presentation, the proof of some technical results used in the proof of main results is given in the Appendix.

2. Functional setting and preliminaries

Let \( \mathcal{V} \) be the set of all vector valued trigonometric polynomials \( u \) defined in \( \Omega \) such that \( \nabla \cdot u = 0 \) and \( \int_\Omega u(x)dx = 0 \). Denote by \( H \) and \( V \) the closures of \( \mathcal{V} \) in \( L^2(\Omega)^3 \) and in \( H^1(\Omega)^3 \), respectively. We denote by \( (\cdot, \cdot) \) and \( \| \cdot \| \) the inner product and norm in \( H \), and by \( ((\cdot, \cdot)) = (\nabla \cdot, \nabla \cdot) \) and \( \| \cdot \| = |\nabla \cdot| \) the inner product and norm in \( V \).

Let \( P \) be the Helmholtz-Leray orthogonal projection in \( L^2(\Omega)^3 \) onto the space \( H \). Following the notations for the MHD-α equations, we denote
\[
B(u, v) = P(u \cdot \nabla)v \quad \text{and} \quad \widetilde{B}(u, v) := -P(u \times (\nabla \times v)), \quad \forall u, v \in V.
\]

Using the identity
\[
(b \cdot \nabla)a + \sum_{j=1}^3 a_j \nabla b_j = -b \times (\nabla \times a) + \nabla (a \cdot b),
\]
one can easily show that
\[
\left( \widetilde{B}(u, v), w \right)_{V', V} = (B(u, v), w)_{V', V} - (B(w, v), u)_{V', V}.
\]

Here, for a Banach space \( X \), we have used the notation \( \langle \cdot, \cdot \rangle_{X', X} \) to denote the dual pairing between \( X \) and its dual space \( X' \).

We denote by \( A = -P\Delta \) the Stokes operator with domain \( D(A) = H^2(\Omega)^3 \cap V \). Notice the fact that in the case of periodic boundary conditions, \( A = -\Delta \) is a self-adjoint positive operator with compact inverse. Hence there exists a complete set of eigenfunctions \( \{w_j\}_{j=1}^\infty \) which is orthonormal in \( H \), and orthogonal in both \( V \) and \( D(A) \) such that \( Aw_j = \lambda_j w_j \) with
\[
(2\pi/L)^2 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \sim j^{2/3}L^{-2} \leq \cdots
\]

We have the following Poincaré type inequalities
\[
\|u\|^2 \geq \lambda_1 |u|^2 \quad \text{for all } u \in V, \tag{2.2}
\]
\[
|Au|^2 \geq \lambda_1 \|u\|^2 \quad \text{for all } u \in D(A). \tag{2.3}
\]

Notice that
\[
|u + \alpha^2 Au|^2 = |u|^2 + 2\alpha^2 \|u\|^2 + \alpha^4 |Au|^2,
\]
so
\[ |u + \alpha^2 Au|^2 \geq 2^{3/2}\alpha^3 \|u\| \|Au\|, \] \tag{2.4}
and
\[ |Au|^2 \leq \frac{1}{\alpha^4} |u + \alpha^2 Au|^2. \] \tag{2.5}

From the definitions of $B$ and $\tilde{B}$, we have
\[ \langle B(u, v), w \rangle_{V', V} = -\langle B(u, w), v \rangle_{V', V} \quad \text{for all } u, v, w \in V, \] \tag{2.6}
and in particular,
\[ \langle B(u, v), v \rangle_{V', V} = 0 \quad \text{for all } u, v \in V. \] \tag{2.7}

Also, since (2.1) we have
\[ \langle \tilde{B}(u, v), u \rangle_{V', V} = 0 \quad \text{for all } u, v \in V. \] \tag{2.8}

We have the following estimates (see e.g. [31]):
\[ |\langle B(u, v), w \rangle| \leq c_1 \|u\|^{1/2} |Au|^{1/2} \|v\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H, \] \tag{2.9}
\[ |\langle \tilde{B}(u, v), w \rangle| \leq c_1 \|u\|^{1/2} |Au|^{1/2} \|v\| \|w\|, \quad \forall u \in D(A), v \in V, w \in H, \] \tag{2.10}
\[ |\langle B(u, v), w \rangle| \leq c_2 \|u\| \|v\| \|w\|^{1/2} |Au|^{1/2} \|w\|, \quad \forall u \in V, v \in D(A), w \in H, \] \tag{2.11}
\[ |\langle \tilde{B}(u, v), w \rangle| \leq c_3 \|u\| \|v\| \|w\|^{1/2} \|w\|^{1/2}, \quad \forall u, v, w \in V, \] \tag{2.12}
\[ |\langle \tilde{B}(u, v), w \rangle| \leq c_4 \|u\| \|v\| \|w\|^{1/2} |w|^{1/2}, \quad \forall u, v, w \in V, \] \tag{2.13}

for some positive constants $c_i, i = 1, \ldots, 4$.

We apply the projection $P$ to (1.1)-(1.5) to obtain the equivalent system of equations
\begin{align*}
\frac{dv}{dt} + \nu Av + \tilde{B}(u, v) - B(B, B) &= Pf, \tag{2.14} \\
v &= u + \alpha^2 Au, \tag{2.15} \\
\frac{dB}{dt} + \eta AB + B(u, B) - B(B, u) &= 0, \tag{2.16}
\end{align*}
with the initial datum
\[ u(0) = u^0, \quad B(0) = B^0. \] \tag{2.17}

**Definition 2.1** ([27]). Let $T > 0$ and given $(u^0, B^0) \in V \times H$ and $f \in L^\infty(0, T; H)$.

A weak solution of (2.14)-(2.17) on the interval $[0, T]$ is a pair of functions $(u, B)$ such that
\[ u \in C([0, T]; V) \cap L^2(0, T; D(A)) \text{ with } \frac{du}{dt} \in L^2(0, T; H) \]
(or equivalently $v \in C([0, T]; V') \cap L^2(0, T; H)$ with $dv/dt \in L^2(0, T; D(A'))$ and
\[ B \in C([0, T]; H) \cap L^2(0, T; V) \text{ with } \frac{dB}{dt} \in L^2(0, T; V'), \]
satisfying $u(0) = u^0$, $B(0) = B^0$, and
\[
\langle \frac{dv}{dt}(t), w \rangle_{D(A)'D(A)} + \nu(v(t), Aw) + \langle \bar{B}(u(t), v(t)), w \rangle_{D(A)'D(A)} - \langle B(B(t), B(t)), w \rangle_{V',V} = (f(t), w),
\]
and
\[
\langle \frac{dB}{dt}(t), \varphi \rangle_{V',V} + \eta(B(t), A\varphi) + \langle B(u(t), B(t)), \varphi \rangle_{V',V} - \langle B(B(t), u(t)), \varphi \rangle_{V',V} = 0,
\]
for every $w \in D(A), \varphi \in V$ and for almost every $t \in [0,T]$.

When $(u^0, B^0) \in D(A) \times V$, we call a strong solution of (2.14)-(2.17) in the interval $[0,T]$ the solution that satisfies
\[
B \in C([0,T]; V) \cap L^2(0,T; D(A)), \quad u \in C([0,T]; D(A)) \cap L^2(0,T; D(A^{3/2})
\]
(or equivalently $v \in C([0,T]; H) \cap L^2(0,T; V)$).

We define the generalized three-dimensional Grashof number $Gr$ as follows
\[
Gr = \frac{1}{\mu^2 \lambda_1^{3/4}} \limsup_{t \to \infty} |f(t)|, \text{ for } f \in L^\infty(0,\infty; H),
\]
where $\mu = \min\{\nu, \eta\}$.

**Remark 2.1.** It is noticing that that for any $f_1$ and $f_2$ belonging to $L^\infty(0,\infty; H)$ such that $\lim_{t \to \infty} |f_1(t) - f_2(t)| = 0$, then the generalized Grashof number $Gr$ which defines on $f_1$ is equivalent to the one which defines on $f_2$.

To prove our main results, we will use the following well-posedness and large-time asymptotic result, whose proof will be postponed in the Appendix.

**Theorem 2.1.** Let $(u^0, B^0) \in V \times H$ and $f \in L^\infty(0,\infty; H)$, then problem (2.14)-(2.17) has a unique global weak solution $(u, B)$ such that
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |u(\tau)|^2 + \alpha^2 \|u(\tau)\|^2 + |B(\tau)|^2 \right) \, d\tau \leq \frac{\mu^3 Gr^2}{\nu \lambda_1^{1/2}},
\]
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|u(\tau)\|^4 \, d\tau \leq \frac{\mu^3 Gr^4}{\nu^2 \lambda_1^{3/4}},
\]
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |u(\tau)|^2 + \alpha^2 |Au(\tau)|^2 + \|B(\tau)|^2 \right) \, d\tau \leq \frac{2 \mu^3 \lambda_1^{1/2} Gr^2}{\nu},
\]
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|B(\tau)\|^4 \, d\tau \leq \frac{32 \mu^6 \lambda_1^4 Gr^4}{\nu^2} + 2 \left( \frac{216 c_1^4 (\lambda_1^{-1} + \alpha^2)^4}{\nu^6 \lambda_1^{10}} + \frac{16 c_1^4}{\nu^2} + \frac{27 (c_1 + c_2)^4}{\nu^4 \eta^2} \right)^2 \frac{\nu^6 Gr^{12}}{\lambda_1^{1/2} \lambda_1^{8}}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |Au(\tau)|^2 + \alpha^2 |Au(\tau)|^2 + |AB(\tau)|^2 \right) \, d\tau \leq \frac{6 \mu^3 \lambda_1^2 Gr^2}{\nu} + \left( \frac{432 c_1^4 (\lambda_1^{-1} + \alpha^2)^4}{\nu^6 \lambda_1^{10}} + \frac{32 c_1^4}{\nu^2} + \frac{54 (c_1 + c_2)^4}{\nu^4 \eta^2} \right) \frac{\nu^6 Gr^6}{\lambda_1^{1/2} \alpha^4}.
\]
where $T = (\mu \lambda_1)^{-1}$.

We also need the following generalized Gronwall inequality.

**Lemma 2.1** ([11, 20]). Suppose that $\phi(t)$ is an absolutely continuous non-negative function on $[0, \infty)$ that satisfies the following inequality

$$\frac{d\phi}{dt} + \beta \phi \leq \gamma,$$

where $\beta$ and $\gamma$ are locally integrable real-valued functions on $[0, \infty)$ that satisfy the following conditions for some $T > 0$

$$\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta(\tau) d\tau > 0,$$

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^-(\tau) d\tau < \infty,$$

$$\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \gamma^+(\tau) d\tau = 0,$$

with $\beta^- := \max\{-\beta, 0\}$, $\gamma^+ := \max\{\gamma, 0\}$. Then it follows that $\lim_{t \to \infty} \phi(t) = 0$.

3. Determining nodes

A finite set of points $N := \{x_1, x_2, \ldots, x_N\} \subset \Omega$ is called a set of determining nodes if for any two solutions $(u_1, B_1)$ and $(u_2, B_2)$ of (2.14)-(2.17) with the initial data $(u_1(0), B_1(0)), (u_2(0), B_2(0))$ and forcings $f_1, f_2 \in L^\infty(0, \infty; H)$, respectively, the assumptions

$$\lim_{t \to \infty} |(u_1, B_1)(x_j, t) - (u_2, B_2)(x_j, t)| = 0 \text{ for } x_j \in N, \ j = 1, \ldots, N,$$

and

$$\lim_{t \to \infty} |f_1(t) - f_2(t)| = 0,$$

imply that

$$\lim_{t \to \infty} (|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2) = 0.$$

We divide the domain $\Omega$ into $N$ equal squares $\Omega_j, j = 1, \ldots, N$, where $\Omega_j$ is the $j$-th cubic with edge $h = L/\sqrt{N}$. Furthermore, we place the point $x_j \in \Omega_j, j = 1, \ldots, N$.

To estimate the number of determining nodes, we need the following lemma whose proof is given in the Appendix.

**Lemma 3.1.** For every $w \in D(A)$, there exist some positive constants $c_5, c_6, c_7, c_8$ such that

$$|w|^2 \leq 4L^2 \vartheta^2(w) + \frac{c_5 L^4}{N^{4/3}} |Aw|^2,$$

$$\|w\|^2 \leq c_6 LN^{2/3} \vartheta^2(w) + \frac{c_7 L^2}{N^{2/3}} |Aw|^2,$$

$$\|w\|^2_{L^\infty(\Omega)} \leq c_7 LN \vartheta^2(w) + \frac{c_8 L}{N^{1/3}} |Aw|^2,$$

where

$$\vartheta(w) = \max_{1 \leq j \leq N} |w(x_j)|.$$
The main result in this section is the following theorem.

**Theorem 3.1.** Let $\Omega$ be divided into $N$ squares with the points $\mathcal{N} := \{x_1, x_2, \ldots, x_N\}$ distributed one in each square. Then $\mathcal{N}$ is a set of determining nodes provided

$$N \geq L^3 \left( \frac{4c_7}{3\mu} \right)^{3/2} \left( \rho^{MHD-\alpha} \right)^{3/2},$$

where

$$\rho^{MHD-\alpha} = \left( \frac{\sqrt{2}}{\nu^2 \lambda_1^{3/2} \alpha^3} + \frac{6\sqrt{2}\lambda_1^{1/2}}{\nu^2 \alpha^3} + \frac{\sqrt{2}}{\nu^2 \lambda_1^{1/2} \alpha^3} + \frac{2\sqrt{2}\lambda_1^{1/2}}{\nu^2} + \frac{2\sqrt{2}\lambda_1^{1/2}}{\nu^2 \eta \alpha} \right)^{3/2}$$

Then 

$$\mathcal{N} = \{\mathcal{N} \setminus \{x_i\} \mid x_i \in \Omega, i = 1, 2, \ldots, N\}$$

is a set of determining nodes.

**Proof.** Let $\tilde{u} = u_1 - u_2$, $\tilde{B} = B_1 - B_2$. Then $\tilde{u}$ and $\tilde{B}$ satisfy

$$\frac{d\tilde{u}}{dt} + \nu A\tilde{v} + B(\tilde{u}, \tilde{v}) + \tilde{B}(u_1, \tilde{v}) - B(B_1, \tilde{B}) = P(f_1 - f_2), \quad (3.4)$$

where

$$\frac{d\tilde{B}}{dt} + \eta A\tilde{B} + B(u_1, \tilde{B}) + \tilde{B}(u_2, \tilde{B}) - B(B_1, \tilde{B}) = 0. \quad (3.5)$$

Multiplying (3.4) by $A\tilde{v}$ and (3.5) by $A\tilde{B}$, we get

$$\frac{1}{2} \frac{d}{dt} \left( \|\tilde{v}\|^2 + \|\tilde{B}\|^2 \right) + \nu |A\tilde{v}|^2 + \eta |A\tilde{B}|^2$$

Then

$$= - \left( \tilde{B}(u_1, \tilde{v}), A\tilde{v} \right) - \left( \tilde{B}(u_2, \tilde{v}), A\tilde{v} \right) + \left( \tilde{B}(B_2, \tilde{B}), A\tilde{v} \right) + \left( B(B_1, \tilde{B}), A\tilde{v} \right) + (f_1 - f_2, A\tilde{v})$$

We now estimate the terms on the right-hand side. First, we have

$$|(f_1 - f_2, A\tilde{v})| \leq \frac{2}{\nu} \|f_1 - f_2\|^2 + \frac{\nu}{8} |A\tilde{B}|^2. \quad (3.6)$$

Using (2.10), the Cauchy inequality and the Poincaré inequality (2.2), we have

$$\left| \left( \tilde{B}(u_1, \tilde{v}), A\tilde{v} \right) \right| \leq c_1 \|\tilde{u}\|^2 + c_2 \|\tilde{v}\|^2$$

$$\leq \frac{c_1^2 \lambda_1^{-1}}{\sqrt{2\nu \alpha^3}} \|\tilde{v}\|^2 \|\tilde{v}\|^2 + \frac{\nu}{8} |A\tilde{v}|^2, \quad (3.7)$$
where we have used (2.4). Analogously, we derive

\[
\left| \left( \mathcal{B}(u_1, \bar{v}), A\bar{v} \right) \right| \leq c_1 \| u_1 \|^{1/2} |Au_1|^{1/2} \| \bar{v} \| \| A\bar{v} \| \\
\leq \frac{c_1^2}{\sqrt{2\nu \alpha^3}} |v_1|^2 \| \bar{v} \|^2 + \frac{\nu}{8} |A\bar{v}|^2. \tag{3.8}
\]

Now, using (2.9) and the Cauchy inequality, we obtain

\[
\left| \left( \mathcal{B}(\bar{B}, B_2), A\bar{v} \right) \right| \leq c_1 \| \bar{B} \|^{1/2} |A\bar{B}|^{1/2} \| B_2 \| \| A\bar{v} \| \\
\leq \frac{2c_1^2}{\nu} \| \bar{B} \| \| A\bar{B} \| \| B_2 \|^2 + \frac{\nu}{8} |A\bar{v}|^2 \tag{3.9}
\]

By (2.9), the Cauchy inequality and the Poincaré inequality (2.3), we have

\[
\left| \left( \mathcal{B}(B_1, \bar{B}), A\bar{v} \right) \right| \leq c_1 \| B_1 \|^{1/2} |AB_1|^{1/2} \| \bar{B} \| \| A\bar{v} \| \\
\leq \frac{2c_1^2\lambda^{-1}}{\nu} |AB_1|^2 \| \bar{B} \|^2 + \frac{\nu}{8} |A\bar{v}|^2. \tag{3.10}
\]

Using (2.9) and the Cauchy inequality, we deduce that

\[
\left| \left( \mathcal{B}(u_1, \bar{B}), AB \right) \right| \leq c_1 \| u_1 \|^{1/2} |Au_1|^{1/2} \| \bar{B} \| \| AB \| \\
\leq \frac{c_1^2}{\sqrt{2\nu \alpha^3}} |v_1|^2 \| \bar{B} \|^2 + \frac{\eta}{8} |AB|^2, \tag{3.11}
\]

where we have used (2.4). Analogously with using the Poincaré inequality (2.2), we arrive at

\[
\left| \left( \mathcal{B}(\bar{u}, B_2), AB \right) \right| \leq c_1 \| \bar{u} \|^{1/2} |A\bar{u}|^{1/2} \| B_2 \| \| AB \| \\
\leq \frac{c_1^2\lambda^{-1}}{\sqrt{2\eta \alpha^3}} \| B_2 \| \| \bar{u} \|^2 + \frac{\eta}{8} |AB|^2. \tag{3.12}
\]

Using (2.9), the Cauchy inequality and the Poincaré inequality (2.3) yields

\[
\left| \left( \mathcal{B}(B_1, \bar{u}), AB \right) \right| \leq c_1 \| B_1 \|^{1/2} |AB_1|^{1/2} \| \bar{u} \| \| AB \| \\
\leq \frac{2c_1^2\lambda^{-3/2}}{\eta} |AB_1|^2 \| \bar{u} \|^2 + \frac{\eta}{8} |AB|^2 \tag{3.13}
\]

\[
\leq \frac{2c_1^2\lambda^{-5/2}}{\eta^2 \alpha^3} |AB_1|^2 \| \bar{v} \|^2 + \frac{\eta}{8} |AB|^2,
\]

where we have used (2.5). Using (2.9) and the Young inequality, we have

\[
\left| \left( \mathcal{B}(\bar{B}, u_2), AB \right) \right| \leq c_1 \| \bar{B} \|^{1/2} |u_2| \| A\bar{B} \|^{3/2} \\
\leq \frac{54c_1^4}{\eta^2} \| u_2 \|^4 \| \bar{B} \|^3 + \frac{\eta}{8} |AB|^2. \tag{3.14}
\]
From (3.6)-(3.14), we get
\[
\frac{d}{dt} \left( \| \dot{v} \|^2 + \| \dot{B} \|^2 \right) + \frac{3}{4} \left( \nu |\dot{A}v|^2 + \eta |A\dot{B}|^2 \right) \leq K_1(t) \| \dot{v} \|^2 + K_2(t) \| \dot{B} \|^2 + \frac{4}{\nu} |f_1 - f_2|^2, \tag{3.15}
\]
where
\[
K_1(t) = 2 \left( \frac{c_1^2 \lambda_1^{-1}}{\nu^2 \alpha^3} \| v_2 \|^2 + \frac{c_1}{\nu^2 \alpha^3} |v_1|^2 + \frac{c_1^2 \lambda_1^{-1}}{2 \eta \alpha^3} \| B_2 \|^2 + \frac{2 c_1^2 \lambda_1^{-5/2}}{\eta \alpha^4} |AB_1|^2 \right),
\]
\[
K_2(t) = 2 \left( \frac{8 c_1^4}{\nu^2 \eta^2} \| B_2 \|^4 + \frac{2 c_1^2 \lambda_1^{-1}}{\nu} |AB_1|^2 + \frac{c_1^2}{\nu^2 \eta \alpha} |v_1|^2 + \frac{54 c_1^4}{\eta^3} \| u_2 \|^4 \right). \tag{3.16}
\]
Using (3.2) then (3.15) becomes
\[
\frac{d}{dt} \left( \| \dot{v} \|^2 + \| \dot{B} \|^2 \right) + \left( \frac{3 \mu N^{2/3}}{4 c_7 L} - (K_1(t) + K_2(t)) \right) \left( \| \dot{v} \|^2 + \| \dot{B} \|^2 \right) \leq \beta(t) \left( \| \dot{v} \|^2 + \| \dot{B} \|^2 \right) \leq \gamma(t),
\]
with
\[
\beta(t) = \frac{3 \mu N^{2/3}}{4 c_7 L} - (K_1(t) + K_2(t)),
\]
and
\[
\gamma(t) = \frac{3 \mu N^{2/3}}{4 c_7 L} \left( \nu \vartheta^2(\tilde{v}) + \eta \vartheta^2(\tilde{B}) \right) + \frac{4}{\nu} |f_1(t) - f_2(t)|^2.
\]
Hence, to complete the proof, we will show that \( \beta(t) \) and \( \gamma(t) \) fulfill the requirements of Lemma 2.1.

First, since the assumption of \( f_1, f_2 \) and \( \vartheta \) one sees that
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \gamma^+(\tau) d\tau = 0. \tag{3.17}
\]
Now, from the definitions of \( K_1(t) \) and \( K_2(t) \) we have
\[
\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} (K_1(\tau) + K_2(\tau)) d\tau \leq \frac{\sqrt{2} c_1^2}{\nu \lambda_1 \alpha^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |v_2(\tau)|^2 d\tau + \frac{\sqrt{2} c_1^2}{\nu \alpha^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |v_1(\tau)|^2 d\tau
\]
\[
+ \frac{\sqrt{2} c_1^2}{\eta \lambda_1 \alpha^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \| B_2(\tau) \|^2 d\tau + \frac{16 c_1^4}{\nu^2 \eta} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \| B_2(\tau) \|^4 d\tau
\]
\[
+ 4 c_1^2 \left( \frac{1}{\nu \lambda_1} + \frac{1}{\eta \lambda_1^{-5/2} \alpha^4} \right) \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |AB_1(\tau)|^2 d\tau
\]
\[
+ \frac{108 c_1^4}{\eta^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \| u_2(\tau) \|^4 d\tau. \tag{3.18}
\]
By using the facts that
\[ |v_1|^2 = |u_1|^2 + \alpha^2 |u_1|^2 + \alpha^4 |Au_1|^2, \]
and
\[ \|v_2\|^2 = \|u_2\|^2 + \alpha^2 \|Au_2\|^2 + \alpha^4 \|Au_2\|^2, \]
then (3.18) becomes
\[
\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} (K_1(\tau) + K_2(\tau)) \, d\tau 
\leq \frac{\sqrt{2}c^2}{\nu \lambda_1 \alpha^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |u_2(\tau)|^2 + \alpha^2 \left[ |Au_2(\tau)|^2 + \alpha^2 \|Au_2(\tau)\|^2 \right] \right) \, d\tau 
+ \frac{\sqrt{2}c^2 (\nu + \eta)}{\nu \eta \lambda_1 \alpha^3} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |u_1(\tau)|^2 + \alpha^2 \|u_1(\tau)\|^2 + \alpha^4 \|Au_1(\tau)\|^2 \right) \, d\tau 
+ \frac{\sqrt{2}c^2}{\eta \lambda_1 \alpha} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|B_2(\tau)\|^2 \, d\tau 
+ \frac{16c^4}{\nu \eta} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|B_2(\tau)\|^4 \, d\tau 
+ 4\frac{c^2}{\nu \lambda_1} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |AB_1(\tau)|^2 \, d\tau 
+ \frac{108c^4}{\nu \eta \lambda_1 \alpha} \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|u_2(\tau)\|^4 \, d\tau.
\]
So, using the large-time asymptotic estimates (2.18)-(2.22) together Remark 2.1, we deduce with the choice $T = (\mu \lambda_1)^{-1}$ that
\[
\liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} (K_1(\tau) + K_2(\tau)) \, d\tau 
\leq \left\{ \frac{\sqrt{2}}{\nu \lambda_1 \alpha^2} + \frac{6\sqrt{2} \lambda_1^{1/2}}{\nu^2 \alpha^3} + \frac{\sqrt{2}}{\nu \eta \lambda_1^{1/2} \alpha^3} + \frac{2\sqrt{2} \lambda_1^{1/2}}{\nu \eta \lambda_1 \alpha} \right\} c_1^4 \mu^6 Gr^2 
+ \left\{ \frac{108}{\nu^2 \eta \lambda_1 \alpha^4} + \frac{512 \lambda_1^3}{\nu \eta} \right\} c_1^4 \mu^6 Gr^4 
+ \left( \frac{\sqrt{2}}{\nu \alpha^3} + \frac{4}{\nu \eta \lambda_1 \alpha^4} \right) \left( \frac{432 \lambda_1^{4}(\lambda_1^{-1} + \alpha^2)^4}{\nu^6 \alpha^{10}} + \frac{32 \lambda_3^4}{\nu^7 \eta^3} + \frac{54(c_1 + c_2)^4}{\nu^7 \eta^3} \right) c_1^4 \mu^8 Gr^6 
+ \left( \frac{216 \lambda_1^{4}(\lambda_1^{-1} + \alpha^2)^4}{\nu \beta^6 \alpha^{10}} + \frac{16c^4}{\nu \beta^7 \eta^3} + \frac{27(c_1 + c_2)^4}{\nu \beta^7 \eta^3} \right)^2 \frac{32c^4 \mu^6 Gr^12}{\nu^2 \eta \lambda_1 \alpha^8}.
\]
So,
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta(\tau) \, d\tau < \infty \quad \text{and} \quad \liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta(\tau) \, d\tau > 0
\]
hold provided $N \geq L^3 \left( \frac{4c^4}{3\mu} \right)^{3/2} \left( \rho^{MHD-\alpha} \right)^{3/2}$. From (3.17) and (3.20), applying Lemma 2.1, we get
\[
\lim_{t \to \infty} \left( \|\bar{v}(t)\|^2 + \|\bar{B}(t)\|^2 \right) = 0.
\]
4. Determining volume elements

We divide the domain $\Omega$ into $N$ equal squares $\Omega_j$, $j = 1, \ldots, N$, where $\Omega_j$ is the $j$-th cubic with edge $h = L/\sqrt[3]{N}$, and so the volume of $\Omega_j$ is $|\Omega_j| = L^3/N$. The local average of $\varphi$ in $\Omega_j$ defined by

$$\langle \varphi \rangle_{\Omega_j} = \frac{1}{|\Omega_j|} \int_{\Omega_j} \varphi(x) dx.$$  

Let $f_1, f_2$ belong to $L^\infty(0, \infty; H)$ satisfying

$$\lim_{t \to \infty} |f_1(t) - f_2(t)| = 0.$$  

A set of volume elements is said to be determining if for any two solutions $(u_1, B_1)$ and $(u_2, B_2)$ corresponding to external forces $f_1$ and $f_2$ satisfying

$$\lim_{t \to \infty} \langle u_1 \rangle_{\Omega_j} - \langle u_2 \rangle_{\Omega_j} = \lim_{t \to \infty} \langle B_1 \rangle_{\Omega_j} - \langle B_2 \rangle_{\Omega_j} = 0,$$

we have

$$\lim_{t \to \infty} (|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2) = 0.$$  

To establish the result on the number of determining volume elements, we need the following lemma, which will be proved in the Appendix.

**Lemma 4.1.** For every $w \in D(A)$, we have the following estimates for some positive constants $c_9, c_{10}, c_{11}, c_{12}, c_{13}$,

$$|w|^2 \leq L^3 \chi^2(w) + \frac{L^2}{6N^{2/3}} \|w\|^2, \quad (4.1)$$

$$|w|^2 \leq 4L^3 \chi^2(w) + \frac{c_9 L^4}{N^{4/3}} |Aw|^2, \quad (4.2)$$

$$\|w\|^2 \leq c_{10} L N^{2/3} \chi^2(w) + \frac{c_{11} L^2}{N^{2/3}} |Aw|^2, \quad (4.3)$$

$$\|w\|^2_{L^\infty(\Omega)} \leq c_{12} L N \chi^2(w) + \frac{c_{13} L}{N^{1/3}} |Aw|^2, \quad (4.4)$$

where

$$\chi(w) = \max_{1 \leq j \leq N} |\langle w \rangle_{\Omega_j}|.$$  

**Theorem 4.1.** Let $\Omega$ be divided into $N$ squares $\Omega_j$. Suppose

$$\lim_{t \to \infty} \langle (u_1)_j \rangle - \langle (u_2)_j \rangle = \lim_{t \to \infty} \langle (B_1)_j \rangle - \langle (B_2)_j \rangle = 0,$$

for $j = 1, \ldots, N$. Then the volume elements are determining, that is,

$$\lim_{t \to \infty} (|u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2) = 0,$$

provided that

$$N \geq L^3 \left( \frac{4c_{11}}{3\mu} \right)^{3/2} \left( \rho^{MHD-\alpha} \right)^{3/2},$$

where $\rho^{MHD-\alpha}$ is defined in Theorem 3.1.
Proof. Let \( \bar{u} = u_1 - u_2, \bar{B} = B_1 - B_2 \). Similarly to the proof of inequality (3.15) in Theorem 3.1, we have the following estimate

\[
\frac{d}{dt} \left( \|\bar{v}\|^2 + \|\bar{B}\|^2 \right) + \frac{3}{4} \left( \nu |A\bar{v}|^2 + \eta |A\bar{B}|^2 \right) \leq K_1(t)\|\bar{v}\|^2 + K_2(t)\|\bar{B}\|^2 + \frac{4}{\nu} |f_1 - f_2|^2,
\]

where \( K_1(t) \) and \( K_2(t) \) are as in (3.16). Using (4.3) we have

\[
\frac{d}{dt} \left( \|\bar{v}\|^2 + \|\bar{B}\|^2 \right) + \left( 3\mu N^{2/3} \frac{4}{4c_{11}L^2} - (K_1(t) + K_2(t)) \right) \left( \|\bar{v}\|^2 + \|\bar{B}\|^2 \right) \leq \frac{3c_{10}N^{4/3}}{4c_{11}L} \left( \nu \chi^2(\bar{v}) + \eta \chi^2(\bar{B}) \right) + \frac{4}{\nu} |f_1 - f_2|^2.
\]

Hence, to complete the proof, we rewrite (4.5) in the form

\[
\frac{d}{dt} \left( \|\bar{v}\|^2 + \|\bar{B}\|^2 \right) + \beta(t) \left( \|\bar{v}\|^2 + \|\bar{B}\|^2 \right) \leq \gamma(t),
\]

with

\[
\beta(t) = \frac{3\mu N^{2/3}}{4c_{11}L^2} - (K_1(t) + K_2(t)),
\]

and

\[
\gamma(t) = \frac{3c_{10}N^{4/3}}{4c_{11}L} \left( \nu \chi^2(\bar{v}) + \eta \chi^2(\bar{B}) \right) + \frac{4}{\nu} |f_1(t) - f_2(t)|^2.
\]

Firstly, since the assumptions of \( f_1, f_2 \) and \( \chi \) one has

\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \gamma^-(\tau) d\tau = 0.
\]

Using (3.19) then

\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^-(\tau) d\tau < \infty \quad \text{and} \quad \liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta(\tau) d\tau > 0
\]

hold provided \( N \geq L^3 \left( \frac{4c_{11}}{3\mu} \right)^{3/2} \left( \rho^{MHD-\alpha} \right)^{3/2} \). Hence, applying Lemma 2.1, we get

\[
\lim_{t \to \infty} \left( \|\bar{v}(t)\|^2 + \|\bar{B}(t)\|^2 \right) = 0.
\]

This completes the proof. \( \square \)

5. Determining modes

Let \( \{w_1, \ldots, w_m\} \) be the first \( m \) eigenfunctions of the Stokes operator \( A \). We denote by \( P_m \) the orthogonal projection onto \( \text{span}\{w_1, \ldots, w_m\} \), and \( Q_m = I - P_m \). Let \( (u_1, B_1) \) and \( (u_2, B_2) \) be two solutions of (2.14)-(2.17) with the forcings \( f_1 \) and \( f_2 \) given in \( L^\infty(0, \infty; H) \), respectively.

A set modes \( \{w_j\}_{j=1}^m \) is called determining if we have

\[
\lim_{t \to \infty} \left( |u_1(t) - u_2(t)|^2 + |B_1(t) - B_2(t)|^2 \right) = 0
\]
whenever
\[
\lim_{t \to \infty} |f_1(t) - f_2(t)| = 0,
\]
and
\[
\lim_{t \to \infty} (|P_m(u_1(t) - u_2(t))|^2 + |P_m(B_1(t) - B_2(t))|^2) = 0.
\]

**Theorem 5.1.** With $\rho^{\text{MHD-} \alpha}$ as in Theorem 3.1, suppose that $m$ satisfies
\[
\lambda_{m+1} \geq \frac{4}{3\mu} \rho^{\text{MHD-} \alpha}.
\]
Then the number of determining modes is not larger than $m$.

**Proof.** Let $\vec{u} = u_1 - u_2, \vec{B} = B_1 - B_2$. Then $\vec{u}$ and $\vec{B}$ satisfy
\[
\frac{d\vec{v}}{dt} + \nu A\vec{v} + \vec{B}(\vec{u}, v_2) + \vec{B}(u_1, \vec{v}) - \vec{B}(\vec{B}, B_2) - \vec{B}(B_1, \vec{B}) = P(f_1 - f_2),
\]
\[
\frac{d\vec{B}}{dt} + \eta A\vec{B} + B(u_1, \vec{B}) + B(\vec{u}, B_2) - B(B_1, \vec{B}) - B(\vec{B}, u_2) = 0.
\]
Multiplying (5.1) by $Q_m A\vec{v}$ and (5.2) by $Q_m A\vec{B}$, we get
\[
\frac{1}{2} \frac{d}{dt} \left( |Q_m \vec{v}|^2 + |Q_m \vec{B}|^2 \right) + \nu |Q_m A\vec{v}|^2 + \eta |Q_m A\vec{B}|^2 = (f_1 - f_2, Q_m A\vec{v}) - \left( \vec{B}(\vec{u}, v_2), Q_m A\vec{v} \right) - \left( \vec{B}(u_1, \vec{v}), Q_m A\vec{v} \right) + \left( \vec{B}(\vec{B}, B_2), Q_m A\vec{v} \right) + \left( \vec{B}(B_1, \vec{B}), Q_m A\vec{v} \right) + \left( \vec{B}(\vec{B}, u_2), Q_m A\vec{B} \right).
\]
We now estimate the terms on the right-hand side. First, we have
\[
|\langle f_1 - f_2, Q_m A\vec{v} \rangle| \leq \frac{2}{\nu} |f_1 - f_2|^2 + \frac{\nu}{8} |Q_m A\vec{v}|^2.
\]
Similarly to (3.7)-(3.15) with note that $\vec{v} = P_m \vec{v} + Q_m \vec{v}$ and $\vec{B} = P_m \vec{B} + Q_m \vec{B}$, we have the following estimates
\[
\left| \left( \vec{B}(\vec{u}, v_2), Q_m A\vec{v} \right) \right| \leq \frac{\nu^2 \lambda_1^{-1}}{2\nu^3} \|v_2\|^2 \left( |P_m \vec{v}|^2 + |Q_m \vec{v}|^2 \right) + \frac{\nu}{8} |Q_m A\vec{v}|^2,
\]
\[
\left| \left( \vec{B}(u_1, \vec{v}), Q_m A\vec{v} \right) \right| \leq \frac{\nu^2 \lambda_1^{-1}}{2\nu^3} \|v_2\|^2 \left( |P_m \vec{v}|^2 + |Q_m \vec{v}|^2 \right) + \frac{\nu}{8} |Q_m A\vec{v}|^2,
\]
\[
\left| \left( \vec{B}(\vec{B}, B_2), Q_m A\vec{v} \right) \right| \leq \frac{8e^4}{\nu^2 \eta} \|B_2\|^2 \left( |P_m \vec{B}|^2 + |Q_m \vec{B}|^2 \right) + \frac{\nu}{8} |Q_m A\vec{B}|^2 + \frac{\nu}{8} |Q_m A\vec{v}|^2,
\]
\[
\left| \left( \vec{B}(\vec{B}, u_2), Q_m A\vec{v} \right) \right| \leq \frac{2\nu^2 \lambda_1^{-1}}{\nu} \|A\vec{B}\|^2 \left( |P_m \vec{B}|^2 + |Q_m \vec{B}|^2 \right) + \frac{\nu}{8} |Q_m A\vec{v}|^2.
\]
where

\[ K \]

This completes the proof.

By using Theorem \( \lambda \)

Hence from the facts that \( P \)

\[ \frac{d}{dt} \left( \|Q_m \bar{v}\|^2 + \|Q_m \bar{B}\|^2 \right) + \frac{3}{4} \left( \nu \|Q_m A \bar{v}\|^2 + \eta \|Q_m \bar{A} \bar{B}\|^2 \right) \]

\[ \leq K_1(t) \left( \|Q_m \bar{v}\|^2 + \|P_m \bar{v}\|^2 \right) + K_2(t) \left( \|Q_m \bar{B}\|^2 + \|P_m \bar{B}\|^2 \right) \]

(5.12)

where \( K_1(t) \) and \( K_2(t) \) are as in (3.16).

Hence from the facts that \( \|Q_m Aw\|^2 \geq \lambda_{m+1} \|Q_m w\|^2, \|P_m Aw\|^2 \leq \lambda_m \|P_m w\|^2 \leq \lambda_m^2 \|P_m w\|^2 \), and using \( \bar{v} = \bar{u} + \alpha^2 \bar{A} \bar{u} \), we have from (5.12) that

\[ \frac{d}{dt} \left( \|Q_m \bar{v}\|^2 + \|Q_m \bar{B}\|^2 \right) + \beta(t) \left( \|Q_m \bar{v}\|^2 + \|Q_m \bar{B}\|^2 \right) \leq \gamma(t), \]

where

\[ \beta(t) = \frac{3\mu}{4} \lambda_{m+1} - K_1(t) - K_2(t), \]

and

\[ \gamma(t) = K_1(t) \lambda_m (1 + \alpha^2 \lambda_m) \|P_m \bar{u}\|^2 + K_2(t) \lambda_m \|P_m \bar{B}\|^2 + \frac{\eta}{4} \gamma_m \|P_m \bar{B}\|^2 + \frac{4}{\nu} |f_1 - f_2|^2. \]

By using Theorem 2.1 and assumptions on \( |P_m \bar{v}(t)|, |P_m \bar{B}(t)| \) and \( |f_1(t) - f_2(t)| \), we get

\[ \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \gamma^+(\tau) d\tau = 0. \]

Using (3.19) then

\[ \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^-(\tau) d\tau < \infty \text{ and } \liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau > 0 \]

hold provided \( \lambda_{m+1} \geq \frac{4}{3\mu} \rho^{MHD-\alpha} \). Hence, applying Lemma 2.1, we get

\[ \lim_{t \to \infty} \left( \|\bar{v}(t)\|^2 + \|\bar{B}(t)\|^2 \right) = 0. \]

This completes the proof.
6. Appendix

We now give the proofs of some technical results that have been used in the previous sections.

6.1. Proof of Theorem 2.1

The existence and uniqueness of a weak solution to (2.14)-(2.17) can be proved similarly as in the case the external force \( f \equiv 0 \) in [27]. We only prove the estimates (2.18)-(2.22). Multiplying (2.14) by \( u \) and (2.16) by \( B \), then integrating over \( \Omega \) and using (2.6), (2.7), (2.8), we have

\[
\frac{1}{2} \frac{d}{dt} (|u|^2 + \alpha^2 ||u||^2) + \nu (||u||^2 + \alpha^2 |Au|^2) = (B(B, B), u) + (f, u),
\]

and

\[
\frac{1}{2} \frac{d}{dt} |B|^2 + \eta ||B||^2 = -(B(B, B), u).
\]

Summing up these equalities and using the Cauchy inequality we get

\[
\frac{1}{2} \frac{d}{dt} (|u|^2 + \alpha^2 ||u||^2 + |B|^2) + \nu (||u||^2 + \alpha^2 |Au|^2) + \eta ||B||^2 = (f, u) \leq \frac{1}{2 \nu \lambda_1} |f|^2 + \frac{\nu \lambda_1}{2} |u|^2.
\]

By using the Poincaré inequality, we have

\[
\frac{d}{dt} (|u|^2 + \alpha^2 ||u||^2 + |B|^2) + \mu (||u||^2 + \alpha^2 |Au|^2 + ||B||^2) \leq \frac{1}{\nu \lambda_1} |f|^2, \quad (6.1)
\]

where \( \mu = \min\{\nu, \eta\} \). By using the Poincaré inequality once again, we deduce from (6.1) that

\[
\frac{d}{dt} (|u|^2 + \alpha^2 ||u||^2 + |B|^2) + \lambda_1 \mu (||u||^2 + \alpha^2 ||u||^2 + |B|^2) \leq \frac{1}{\nu \lambda_1} |f|^2.
\]

Applying the Gronwall inequality, we infer that

\[
|u(t)|^2 + \alpha^2 ||u(t)||^2 + |B(t)|^2 \leq e^{-\lambda_1 \mu t} I_0 + \frac{e^{-\lambda_1 \mu t}}{\nu \lambda_1} \int_0^t e^{\lambda_1 \mu s} |f(s)|^2 ds, \quad \forall t \geq 0,
\]

where

\[
I_0 := |u^0|^2 + \alpha^2 ||u^0||^2 + |B^0|^2.
\]

Hence,

\[
\limsup_{t \to \infty} (|u(t)|^2 + \alpha^2 ||u(t)||^2 + |B(t)|^2) \leq \frac{1}{\mu \nu \lambda_1^2} \limsup_{t \to \infty} |f(t)|^2.
\]

Using the definition of \( Gr \), we get

\[
\limsup_{t \to \infty} (|u(t)|^2 + \alpha^2 ||u(t)||^2 + |B(t)|^2) \leq \frac{\mu^3 Gr^2}{\nu \lambda_1^2}. \quad (6.2)
\]
So we get (2.18) and (2.19) from (6.2) for some $T > 0$. From now on, we choose $T = (\lambda_1\mu)^{-1}$.

Integrating (6.1) over $(t, t + T)$ we have
\[
\mu \int_t^{t+T} \left( \|u(\tau)\|^2 + \alpha^2 \|Au(\tau)\|^2 + \|B(\tau)\|^2 \right) d\tau \leq \|u(t)\|^2 + \alpha^2 \|u(t)\|^2 + \|B(t)\|^2 + \frac{1}{\nu \lambda_1} \int_t^{t+T} |f(\tau)|^2 d\tau.
\]

Hence
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( \|u(\tau)\|^2 + \alpha^2 \|Au(\tau)\|^2 + \|B(\tau)\|^2 \right) d\tau \\
\leq \lambda_1 \limsup_{t \to \infty} \left( \|u(t)\|^2 + \alpha^2 \|u(t)\|^2 + \|B(t)\|^2 \right) + \frac{1}{\nu \lambda_1} \limsup_{t \to \infty} |f(t)|^2.
\]

Substituting (6.2) into (6.3) and using the definition of $Gr$ we deduce (2.20).

Taking the inner product of (2.14) with $Au$, the inner product of (2.16) with $AB$ and summing up, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|^2 + \alpha^2 \|Au\|^2 + \|B\|^2 \right) + \nu \left( \|Au\|^2 + \alpha^2 \|Au\|^2 \right) + \eta |AB|^2 \\
= (f, Au) + (B(B, B), Au) - \left( \tilde{B}(u, v), Au \right) + (B(B, u), AB) - (B(u, B), AB).
\]

By the Cauchy inequality, we have
\[
|\langle f, Au \rangle| \leq \frac{|f|^2}{\nu} + \frac{\nu}{\lambda} |Au|^2.
\]

Using (2.12) and the Cauchy inequality, we have
\[
|\langle B(B, B), Au \rangle| \leq c_3 \|B\|^{1/2} |AB|^{1/2} |B| \|Au\| \\
\leq \frac{4c_3^4 \|B\|^2 |B|^4 + \nu \alpha^2}{8 \nu \lambda} |Au|^2 + \frac{\eta \lambda}{4} |AB|^2.
\]

Now, by using (2.13) and the Young inequality, we get
\[
\left| \left( \tilde{B}(u, v), Au \right) \right| \leq c_4 \|u\| \|v\| \|Au\|^{1/2} |Au|^{1/2} \\
\leq c_4 (\lambda_1^{-1} + \alpha^2) \|u\| |Au|^{1/2} |Au|^{1/2} \\
\leq \frac{54c_4^3 (\lambda_1^{-1} + \alpha^2)}{4 \nu \lambda} \|u\|^4 |Au|^2 + \frac{\nu \alpha^2}{8} |Au|^2.
\]

Using (2.9), (2.11) and the Young inequality yields
\[
|\langle B(B, u), AB \rangle| + |\langle B(u, B), AB \rangle| \leq (c_1 + c_2) \|B\|^{1/2} \|u\| |AB|^{3/2} \\
\leq \frac{27 (c_1 + c_2)}{4 \eta} \|B\|^2 |u|^4 + \frac{\eta}{4} |AB|^2.
\]
Substituting (6.5), (6.6), (6.7) and (6.8) into (6.4) we deduce that
\[
\frac{d}{dt} (\|u\|^2 + \alpha^2 |Au|^2 + \|B\|^2) + \mu \left( |Au|^2 + \alpha^2 \|Au\|^2 + |AB|^2 \right) \\
\leq \frac{2}{\nu} |f|^2 + \frac{8c_4^4}{\mu^2 \eta \lambda_1^4} \|B\|^2 |B|^4 + \frac{54c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{(\nu \alpha^2)^3} \|u\|^4 |Au|^2 + \frac{27(c_1 + c_2)^4}{2 \eta^3} \|B\|^2 \|u\|^4.
\] (6.9)

Integrating the inequality (6.9) from \( t \) to \( t + T \) we get
\[
\frac{1}{\mu} \left( \|u(t)\|^2 + \alpha^2 |Au(t)|^2 + \|B(t)\|^2 \right) + \int_t^T \left( |Au(\tau)|^2 + \alpha^2 \|Au(\tau)\|^2 + |AB(\tau)|^2 \right) d\tau \\
\leq \frac{1}{\mu} \left( \|u(s)\|^2 + \alpha^2 |Au(s)|^2 + \|B(s)\|^2 \right) \\
+ \frac{2}{\mu \nu} \int_s^t |f(\tau)|^2 d\tau + \frac{8c_4^4}{\mu^2 \eta \lambda_1^4} \int_s^t \|B(\tau)\|^2 |B(\tau)|^4 d\tau \\
+ \frac{108c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{\mu (\nu \alpha^2)^3} \int_s^t \|u(\tau)\|^4 |Au(\tau)|^2 d\tau + \frac{27(c_1 + c_2)^4}{2 \mu \eta^3} \int_s^T \|B(\tau)\|^2 \|u(\tau)\|^4 d\tau.
\] (6.10)

From (6.2), we have
\[
\limsup_{t \to \infty} \|B(t)\|^4 \leq \frac{\mu^6 Gr^4}{\nu^2 \lambda_1^4} \text{ and } \limsup_{t \to \infty} \|u(t)\|^4 \leq \frac{\mu^6 Gr^4}{\nu^2 \lambda_1^4}.
\] (6.11)

Moreover we get from (6.10) that
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |Au(\tau)|^2 + \alpha^2 \|Au(\tau)\|^2 + |AB(\tau)|^2 \right) d\tau \\
\leq \frac{1}{T \mu} \limsup_{t \to \infty} \left( \|u(t)\|^2 + \alpha^2 |Au(t)|^2 + \|B(t)\|^2 \right) + \frac{2}{\mu \nu} \limsup_{t \to \infty} |f(t)|^2 \\
+ \frac{108c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{\mu (\nu \alpha^2)^3} \limsup_{t \to \infty} \|u(t)\|^4 \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} |Au(\tau)|^2 d\tau \\
+ \frac{8c_4^4}{\mu^2 \eta \lambda_1^4} \limsup_{t \to \infty} \|B(t)\|^4 \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|B(\tau)\|^2 d\tau \\
+ \frac{27(c_1 + c_2)^4}{2 \mu \eta^3} \limsup_{t \to \infty} \|u(t)\|^4 \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \|B(\tau)\|^2 d\tau.
\] (6.12)

So, using (2.20), (6.11) and the definition of \( Gr \) then (6.12) becomes
\[
\limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \left( |Au(\tau)|^2 + \alpha^2 \|Au(\tau)\|^2 + |AB(\tau)|^2 \right) d\tau \\
\leq \frac{\lambda_1}{T \mu} \limsup_{t \to \infty} \left( \|u(t)\|^2 + \alpha^2 |Au(t)|^2 + \|B(t)\|^2 \right) + \frac{2 \mu^3 \lambda_1^{3/2} Gr^2}{\nu} \\
+ \left( \frac{216c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{\mu^6 \lambda_1^{10}} + \frac{16c_4^4}{\nu^6 \eta} + \frac{27(c_1 + c_2)^4}{\nu^3 \eta^3} \right) \frac{\mu^6 Gr^6}{\lambda_1^{1/2} \alpha^4}.
\] (6.13)

Now, we consider
\[
\lambda_1 \limsup_{t \to \infty} \left( \|u(t)\|^2 + \alpha^2 |Au(t)|^2 + \|B(t)\|^2 \right).
\]
For any $0 < s < t$, integrating the inequality (6.9) from $s$ to $t$ we get
\[
||u(t)||^2 + \alpha^2 |Au(t)|^2 + ||B(t)||^2 \leq ||u(s)||^2 + \alpha^2 |Au(s)|^2 + ||B(s)||^2
\]
\[
+ \frac{2}{\nu} \int_s^t |f(\tau)|^4 d\tau + \frac{8c_1^2}{\nu^2 \eta \lambda_1^3} \int_s^t ||B(\tau)||^2 |B(\tau)|^4 d\tau
\]
\[
+ \frac{108c_1^2(\lambda_1^1 + \alpha^2)}{(\nu \alpha^2)^3} \int_s^t \int_s^t |u(\tau)||^4 |Au(\tau)||^2 d\tau ds
\]
\[
+ \frac{27(c_1 + c_2)^4}{2\eta^3} \int_s^t \int_s^t ||B(\tau)||^2 |u(\tau)||^4 d\tau ds.
\]

Integrating the last inequality with respect to $s$ over the interval $(t - \frac{1}{\mu \lambda_1}, t)$ we get
\[
\frac{1}{\mu \lambda_1} (||u(t)||^2 + \alpha^2 |Au(t)|^2 + ||B(t)||^2)
\]
\[
\leq \int_{t - \frac{1}{\mu \lambda_1}}^t (||u(s)||^2 + \alpha^2 |Au(s)|^2 + ||B(s)||^2) ds
\]
\[
+ \frac{8c_1^2}{\nu^2 \eta \lambda_1^3} \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t ||B(\tau)||^2 |B(\tau)|^4 d\tau ds
\]
\[
+ \frac{108c_1^2(\lambda_1^1 + \alpha^2)}{(\nu \alpha^2)^3} \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t |u(\tau)||^4 |Au(\tau)||^2 d\tau ds
\]
\[
+ \frac{27(c_1 + c_2)^4}{2\eta^3} \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t ||B(\tau)||^2 |u(\tau)||^4 d\tau ds.
\]

Using (2.20), we have
\[
\lim_{t \to \infty} \int_{t - \frac{1}{\mu \lambda_1}}^t (||u(s)||^2 + \alpha^2 |Au(s)|^2 + ||B(s)||^2) ds
\]
\[
=T \lim_{t \to \infty} \frac{1}{T} \int_t^{t+T} (||u(\tau)||^2 + \alpha^2 |Au(\tau)|^2 + ||B(\tau)||^2) d\tau
\]
\[
\leq \frac{2\nu^2 Gr^2}{\mu \lambda_1^{1/2}}.
\]

Using the fact that
\[
\int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t |f(\tau)|^2 d\tau d\sigma \leq \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t |f(\tau)|^2 d\tau d\sigma = \frac{1}{\mu \lambda_1} \int_{t - \frac{1}{\mu \lambda_1}}^t |f(\tau)|^2 d\tau,
\]
then
\[
\limsup_{t \to \infty} \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t |f(\tau)|^2 d\tau d\sigma \leq \frac{1}{\mu^2 \lambda_1^2} \limsup_{t \to \infty} |f(t)|^2 = \frac{\mu^2 Gr^2}{\lambda_1^{1/2}}.
\]

We also have the fact that
\[
\int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t ||B(\tau)||^2 |B(\tau)|^4 d\tau d\sigma \leq \int_{t - \frac{1}{\mu \lambda_1}}^t \int_{t - \frac{1}{\mu \lambda_1}}^t ||B(\tau)||^2 |B(\tau)|^4 d\tau d\sigma
\]
\[
= \frac{1}{\mu \lambda_1} \int_{t - \frac{1}{\mu \lambda_1}}^t ||B(\tau)||^2 |B(\tau)|^4 d\sigma.
\]
then
\[
\limsup_{t \to \infty} \int_{t-\frac{T}{\mu \lambda_1}}^{t} \int_{s}^{t} \|B(\tau)\|^2 |B(\tau)|^4 \, d\tau \, ds \\
\leq \limsup_{t \to \infty} \frac{1}{\mu \lambda_1} \int_{t-\frac{T}{\mu \lambda_1}}^{t} \|B(\tau)\|^2 |B(\tau)|^4 \, d\tau \\
\leq \frac{1}{\mu \lambda_1} \limsup_{t \to \infty} |B(t)|^4 \limsup_{t \to \infty} \int_{t-\frac{T}{\mu \lambda_1}}^{t} \|B(\tau)\|^2 \, d\tau \\
= \frac{1}{\mu^2 \lambda_1^2} \limsup_{t \to \infty} |B(t)|^4 \limsup_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \|B(\tau)\|^2 \, d\tau. \quad (6.17)
\]

So, by using (6.2) and (2.20) we get from (6.17) that
\[
\limsup_{t \to \infty} \int_{t-\frac{T}{\mu \lambda_1}}^{t} \int_{s}^{t} \|B(\tau)\|^2 |B(\tau)|^4 \, d\tau \, ds \leq \frac{2 \mu^7 Gr^6}{\nu^3 \lambda_1^5/2}. \quad (6.18)
\]

By the same as (6.18), we deduce that
\[
\limsup_{t \to \infty} \int_{t-\frac{T}{\mu \lambda_1}}^{t} \int_{s}^{t} \|u(\tau)\|^4 |Au(\tau)|^2 \, d\tau \, ds \leq \frac{2 \mu^7 Gr^6}{\nu^3 \lambda_1^5/2 \alpha^4}, \quad (6.19)
\]
and
\[
\int_{t-\frac{T}{\mu \lambda_1}}^{t} \int_{s}^{t} \|B(\tau)\|^2 \|u(\tau)\|^4 \, d\tau \, ds \leq \frac{2 \mu^7 Gr^6}{\nu^3 \lambda_1^5/2 \alpha^4}. \quad (6.20)
\]
Substituting (6.15), (6.16), (6.18), (6.19) and (6.20) into (6.14), we conclude that
\[
\lambda_1 \limsup_{t \to \infty} \left( \|u(t)\|^2 + \alpha^2 |Au(t)|^2 + \|B(t)\|^2 \right) \\
\leq \frac{4 \mu^3 \lambda_1^3/2 Gr^2}{\nu} \left( \frac{216 c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{\nu^3 \alpha^{10}} \left( \frac{16 c_3^4}{\nu^5 \eta} + \frac{27 (c_1 + c_2)^4}{\nu^3 \eta^3} \right) \right) \frac{\mu^8 Gr^6}{\lambda_1^{1/2} \alpha^4}. \quad (6.21)
\]
Combining (6.21) and (6.13), we deduce that
\[
\limsup_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \left( |Au(\tau)|^2 + \alpha^2 \|u(\tau)\|^2 + |AB(\tau)|^2 \right) \, d\tau \\
\leq \frac{6 \mu^3 \lambda_1^3/2 Gr^2}{\nu} \left( \frac{432 c_4^4 (\lambda_1^{-1} + \alpha^2)^4}{\nu^3 \alpha^{10}} \left( \frac{32 c_3^4}{\nu^5 \eta} + \frac{54 (c_1 + c_2)^4}{\nu^3 \eta^3} \right) \right) \frac{\mu^8 Gr^6}{\lambda_1^{1/2} \alpha^4}.
\]
This is the estimate (2.22). Moreover, we get (2.21) by using (6.21) with noting that
\[
\limsup_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \|B(\tau)\|^4 \, d\tau \leq \limsup_{t \to \infty} \|B(t)\|^4 \leq \left( \limsup_{t \to \infty} \|B(t)\|^2 \right)^2.
\]

### 6.2. Proof of Lemma 3.1

We have
\[
|w|^2 = \sum_{j=1}^{N} \int_{\Omega_j} |w(x)|^2 \, dx = \sum_{j=1}^{N} \int_{\Omega_j} |w(x)|^2 1_{\Omega_j}(x) \, dx.
\]
where $1_{\Omega_j}$ is the characteristic function of $\Omega_j$. Consider $\Omega_j$ for $j$ fixed, but arbitrary. Choose $z \in \Omega_j$ such that $z$ is in the line of the intersection of two planes: the plane which contains the point $x$ and is parallel to $xy$-plane and the plane which contains the point $x_j$ and is parallel to the $xz$-plane in three dimensions. In other words, if $x$ and $x_j$ are such that $x = (\xi_1, \xi_2, \xi_3)$ and $x_j = (\eta_1, \eta_2, \eta_3)$, then $z = (\tau_1, \eta_2, \xi_3)$. Therefore,

$$|w(x) - w(x_j)|^2 \leq (|w(x) - w(z)| + |w(z) - w(x_j)|)^2 \leq 2(|w(x) - w(z)|^2 + |w(z) - w(x_j)|^2).$$

Following the results which were proved in [1, Lemma 4], then for every $w \in H^2(\Omega_j)$, we have

$$|w(x) - w(z)|^2 \leq \frac{2}{h} \left( 4\|\nabla w\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega_j)} \right),$$

and

$$|w(z) - w(x_j)|^2 \leq \frac{2}{h} \left( 4\|\nabla w\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega_j)} \right).$$

Hence

$$|w(x) - w(x_j)|^2 \leq \frac{4}{h} \left( 8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega_j)} \right).$$

This implies that

$$|w(x)|^2 \leq 2|w(x_j)|^2 + 2|w(x) - w(x_j)|^2 \leq 2|w(x_j)|^2 + \frac{8}{h} \left( 8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega_j)} \right).$$

Hence,

$$|w|^2 \leq 2\sum_{j=1}^N \int_{\Omega_j} |w(x_j)|^2 1_{\Omega_j}(x) dx + \frac{8}{h} \sum_{j=1}^N \left( 8\|\nabla w\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega_j)} + h^2 \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega_j)} \right) \int_{\Omega_j} 1_{\Omega_j}(x) dx.$$

$$\leq 2\partial^2 (w) \sum_{j=1}^N \int_{\Omega_j} 1_{\Omega_j}(x) dx + \frac{64}{h} \|\nabla w\|^2_{L^2(\Omega_j)} \int_{\Omega} 1_{\Omega_j}(x) dx + 8h \sum_{j=1}^N \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega_j)} \int_{\Omega_j} 1_{\Omega_j}(x) dx + 8h \sum_{j=1}^N \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega_j)} \int_{\Omega_j} 1_{\Omega_j}(x) dx.$$

$$= 2\partial^2 (w) N h^3 + 64h^2 \|\nabla w\|^2_{L^2(\Omega)} + 8h^4 \left\| \frac{\partial^2 w}{\partial x \partial y} \right\|^2_{L^2(\Omega)} + 8h^4 \left\| \frac{\partial^2 w}{\partial x \partial z} \right\|^2_{L^2(\Omega)},$$

where we have used the fact that $\int_{\Omega_j} 1_{\Omega_j}(x) dx = |\Omega_j| = h^3$ for all $j = 1, \ldots, N$. So, we obtain

$$|w|^2 \leq 2L^3 \partial^2 (w) + 64 \frac{L^2}{N^{2/3}} \|w\|^2 + 8c \frac{L^4}{N^{4/3}} |A w|^2. \quad (6.22)$$
Since 

\[ \|w\|^2 \leq |w| |Aw|, \]
then by the Cauchy inequality, we get

\[ 64 \frac{L^2}{N^{2/3}} \|w\|^2 \leq 64 \frac{L^2}{N^{2/3}} |w| |Aw| \leq \frac{1}{2} |w|^2 + \frac{64^2}{2} \frac{L^4}{N^{4/3}} |Aw|^2. \] (6.23)

Substituting (6.23) into (6.22) we get

\[ |w|^2 \leq 4L^3 \vartheta^2(w) + (64^2 + 16c) \frac{L^4}{N^{4/3}} |Aw|^2. \]

So we get (3.1) with \( c_5 = 64^2 + 16c \).

By the inequality (6.2), using (3.1) and the Cauchy inequality, we get (3.2). Moreover, by the Agmon inequality in the case of three dimensions:

\[ \|w\|_{L^\infty(\Omega)} \leq c\|w\| |Aw|, \]
then using (3.2) and the Cauchy inequality, we deduce (3.3). The proof of Lemma 3.1 is complete.

6.3. Proof of Lemma 4.1

We first prove the estimates for the domain \((0, \ell)^3 := (0, \ell) \times (0, \ell) \times (0, \ell)\) for any \( \ell > 0 \). Following the proof in the one-dimensional and two-dimensional cases in [20, Appendix], we have the following estimates:

- In the case of one dimension: for all \( w \in C_0^\infty(\mathbb{R}) \),

  \[ \int_0^\ell |w(x)|^2 dx \leq \ell |\langle w \rangle|^2 + \frac{\ell^2}{6} \int_0^\ell |w_x|^2 dx, \] (6.24)

  where

  \[ \langle w \rangle = \frac{1}{\ell} \int_0^\ell w(x) dx. \]

- In the case of two dimensions: for all \( w(x_1, x_2) \in C_0^\infty(\mathbb{R}^2) \),

  \[ \int_0^\ell \int_0^\ell |w(x_1, x_2)|^2 dx_1 dx_2 \leq \ell^2 |\langle w \rangle|^2 + \frac{\ell^2}{6} \int_0^\ell \int_0^\ell \left( \frac{\partial w}{\partial x_1} \right)^2 + \left( \frac{\partial w}{\partial x_2} \right)^2 dx_1 dx_2, \] (6.25)

  where

  \[ \langle w \rangle = \frac{1}{\ell^2} \int_0^\ell \int_0^\ell w(x_1, x_2) dx_1 dx_2. \]

Applying the two-dimensional estimate (6.25) to \( w(x_1, x_2, x_3) \) holding \( x_3 \) fixed, we have

\[ \int_0^\ell \int_0^\ell |w(x_1, x_2, x_3)|^2 dx_1 dx_2 \leq \frac{1}{\ell^2} \left( \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2 \right)^2 + \frac{\ell^2}{6} \int_0^\ell \int_0^\ell \left( \left| \frac{\partial w}{\partial x_1} \right| + \left| \frac{\partial w}{\partial x_2} \right| \right)^2 dx_1 dx_2. \]
Integrating this inequality with respect to $x_3$ from 0 to $\ell$ to obtain
\[
\int_0^\ell \int_0^\ell |w(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3 \leq \frac{1}{\ell^2} \int_0^\ell \left( \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2 \right)^2 dx_3 \\
+ \frac{\ell^2}{6} \int_0^\ell \int_0^\ell \left( \int_0^\ell \frac{\partial w}{\partial x_1} \right)^2 + \left( \int_0^\ell \frac{\partial w}{\partial x_2} \right)^2 dx_1 dx_2 dx_3.
\] (6.26)

Now, applying the estimate (6.24) to $\varphi(x_3) = \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2$, we have
\[
\int_0^\ell \left( \int_0^\ell \int_0^\ell \frac{\partial w}{\partial x_3} dx_1 dx_2 \right)^2 dx_3 \leq \frac{1}{\ell} \left( \int_0^\ell \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right)^2 \\
+ \frac{\ell^2}{6} \int_0^\ell \left( \int_0^\ell \int_0^\ell \frac{\partial w}{\partial x_3} dx_1 dx_2 \right)^2 dx_3.
\]

Using the Buniakovsky-Schwarz inequality twice, we have
\[
\int_0^\ell \left( \int_0^\ell \int_0^\ell \frac{\partial w}{\partial x_3} dx_1 dx_2 \right)^2 dx_3 \leq \ell^2 \int_0^\ell \int_0^\ell \left( \int_0^\ell \frac{\partial w}{\partial x_3} \right)^2 dx_1 dx_2 dx_3.
\]

So
\[
\int_0^\ell \left( \int_0^\ell \int_0^\ell \frac{\partial w}{\partial x_3} dx_1 dx_2 \right)^2 dx_3 \leq \frac{1}{\ell} \left( \int_0^\ell \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right)^2 \\
+ \frac{\ell^2}{6} \int_0^\ell \int_0^\ell \left( \int_0^\ell \frac{\partial w}{\partial x_3} \right)^2 dx_1 dx_2 dx_3.
\] (6.27)

Substituting (6.27) into (6.26), we deduce that
\[
\int_0^\ell \int_0^\ell \int_0^\ell |w(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3 \\
\leq \ell^3 \left( \frac{1}{\ell^3} \int_0^\ell \int_0^\ell \int_0^\ell w(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right)^2 \\
+ \frac{\ell^2}{6} \int_0^\ell \int_0^\ell \int_0^\ell \left( \int_0^\ell \frac{\partial w}{\partial x_1} \right)^2 + \left( \int_0^\ell \frac{\partial w}{\partial x_2} \right)^2 + \left( \int_0^\ell \frac{\partial w}{\partial x_3} \right)^2 dx_1 dx_2 dx_3.
\]

Hence, in any box $\Omega_j$, we have
\[
\int_{\Omega_j} |w(x)|^2 dx \leq \frac{L^3}{N} |\langle w \rangle_{\Omega_j}|^2 + \frac{L^2}{6N^{2/3}} \int_{\Omega_j} |\nabla w(x)|^2 dx.
\]
Summing in $j$ from 1 to $N$ we get

$$\|w\|^2 \leq L^3 \chi^2(w) + \frac{L^2}{6N^{2/3}} \|w\|^2.$$  

This is the inequality (4.1).

Using the following interpolation inequality

$$\|w\|^2 \leq |w| |Aw|$$

(6.28)

and the Cauchy inequality, we get (4.2). Moreover, by using (4.1) and (6.28) once again and the Cauchy inequality, we get (4.3). Finally, by using the Agmon inequality in three dimensions, namely

$$\|w\|^2_{L^\infty(\Omega)} \leq c \|w\| |Aw|,$$

we get (4.4).

References


A 3D MHD-α model


