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INFINITELY MANY SOLUTIONS FOR CRITICAL FRACTIONAL EQUATION WITH SIGN-CHANGING WEIGHT FUNCTION

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Abstract In this work, we consider the fractional Schrödinger type equations with critical exponent, concave nonlinearity and sign-changing weight function on $\mathbb{R}^N$. With the aid of the symmetric Mountain Pass Theorem, we prove this problem has infinitely many small energy solutions.

Keywords Fractional Schrödinger equations, critical exponent, sign-changing weight function, symmetric Mountain Pass Theorem.

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1. Introduction and main result

In this paper, we are concerned the infinitely many solutions to the following fractional Schrödinger equation with critical exponent

$$\begin{cases} (-\triangle)^s u + u = \mu h(x)|u|^{q-1}u + |u|^{2_s^*-2}u, \, x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\mu > 0$ is a parameter, $1 < q < 2$, $0 < s < 1$, $N > 2s$ and $2_s^* = \frac{2N}{N-2s}$ is a non-local fractional Sobolev exponent. Here, $(-\triangle)^s$ is the fractional Laplace operator (see [2]), which is defined as

$$(-\triangle)^s u(x) := C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \, x \in \mathbb{R}^N.$$

The symbol P.V. represents the Cauchy principal value, $C_{N,s}$ is a normalization constant that depends on N and s. The weight function $h(x)$ satisfies the following condition:

(H) $h \in L^{q^*}(\mathbb{R}^N)$, where $q^* = \frac{2_s^*}{2_s^*-q}$ and $h^+ = \max\{h, 0\} \neq 0$.

The fractional Schrödinger equation is a class of fundamental equation in fractional quantum mechanics. It reflects the stable diffusion of particles of Lévy processes, which was first discovered by Laskin [15, 16]. Through the equivalent definition of fractional operator, the authors obtained the corresponding variational principle and proved the existence of solutions in [3, 11, 19–21]. Specially, for the

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concave-convex nonlinearity, this type of problems has currently been actively studied, see [5, 8, 13, 22] and their references. Here, we are interested in the case of the fractional Schrödinger equations involving concave-convex nonlinearities with critical exponent.

For Schrödinger equations, Chabrowski and Drabek [10] studied the following nonlinear elliptic problem:

$$-\triangle u + u = \varepsilon h(x)|u|^{q-1}u + |u|^{2_s^*-2}u,\ x \in \mathbb{R}^N, \tag{1.2}$$

where ε is a positive constant, $1 < q < 2$ and $N \geq 3$. Under the condition that h is a nonnegative and nonzero function in $L^{\frac{2^*}{2^*-q-1}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, they obtained infinitely many solutions of equation (1.2) for ε small.

For fractional Schrödinger equations, if the weight functions $h(x) = 1$, Barrios etc [4] dealt with the following problem:

$$\begin{cases} (-\triangle)^s u = \lambda |u|^{q-2}u + |u|^{2_s^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda > 0$, $0 < s < 1$, $N > 2s$ and $0 < q < 1$. They obtained that there exist at least two positive solutions for every $0 < \lambda < \Lambda$, at least one positive solution if $\lambda = \Lambda$, no positive solution if $\lambda > \Lambda$. In [23], Zhang etc proved the existence of a nontrivial radially symmetric weak solution to the following problem:

$$(-\triangle)^s u + V(x)u = k(x)f(u) + \lambda|u|^{2_s^*-2}u,\ x \in \mathbb{R}^N, \tag{1.4}$$

where $N \geq 2$, λ is a positive real parameter, $V(x)$ and $k(x)$ are positive and bounded functions satisfying some suitable conditions.

Motivated by above papers, we consider the fractional problem (1.1) with critical exponent on $\mathbb{R}^N$. The main difficulty is how to recover the compactness. To the best of our knowledge, few papers deal with this problem with sign-changing weight function up to now. Inspired by [9], the main purpose of this paper is to study the existence of infinitely many small energy solutions of (1.1) for μ sufficiently small via a new version of the symmetric Mountain Pass Theorem due to Kajikiya [14].

The main result of this paper is the following.

Theorem 1.1. *Assume* $1 < q < 2$ *and the condition* (H) *is fulfilled. There exists* $\mu_* > 0$ *such that for all* $\mu \in (0, \mu_*)$, *problem* (1.1) *possesses infinitely many nontrivial solutions* $\{u_k\}_{k=1}^{\infty}$ *satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^N}\left(|(-\triangle)^{\frac{s}{2}}u_k|^2+|u_k|^2\right)dx-\frac{\mu}{q}\int_{\mathbb{R}^N}h(x)|u_k|^q dx-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u_k|^{2_s^*}dx \to 0^-\,(k\to\infty).$$

The rest of this paper is organized as follows. In Section 2 we will introduce some knowledge of dealing with the fractional Laplacian operator and get some helpful results. We will finish the proof of Theorem 1.1 in Section 3.

2. Preliminaries

In this part we first recall some results on Sobolev spaces of fractional order. For a deeper introduction to fractional Sobolev spaces can be found in [6, 18] and references therein.

Consider the fractional order Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : [u]_{H^s(\mathbb{R}^N)} < \infty \right\},$$

while

$$[u]_{H^s(\mathbb{R}^N)} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{1/2}$$

is the Gallardo semi-norm. Observe Proposition 3.6 in [18], we can get

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy = \int_{\mathbb{R}^N} |(-\triangle)^{\frac{s}{2}} u(x)|^2 dx.$$

Thus, it follows that norm $\|u\|_{H^s(\mathbb{R}^N)}$ is equivalent to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left[|(-\triangle)^{\frac{s}{2}} u(x)|^2 + |u(x)|^2 \right] dx \right)^{\frac{1}{2}} \tag{2.1}$$

and the corresponding inner product is

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \left[(-\triangle)^{\frac{s}{2}} u(x) (-\triangle)^{\frac{s}{2}} v(x) + u(x) v(x) \right] dx. \tag{2.2}$$

Throughout this paper, we will use $\|\cdot\|$ to represent the norm of $H^s(\mathbb{R}^N)$. As usual, for $1 \leq \nu < \infty$, we let

$$|u|_\nu = \left(\int_{\mathbb{R}^N} |u|^\nu dx \right)^{\frac{1}{\nu}}, \quad u \in L^\nu(\mathbb{R}^N).$$

In order to discuss the weak solutions of (1.1), we need to find the critical points of the energy functional $I : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[|(-\triangle)^{\frac{s}{2}} u|^2 dx + |u|^2 \right] dx - \frac{\mu}{q} \int_{\mathbb{R}^N} h(x) |u|^q dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} dx. \tag{2.3}$$

Under the condition (H), we can get the energy functional I is well-defined by the Sobolev embedding theorem. It's not hard to prove $I \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$ and its derivative is given by

$$\langle I'(u), v \rangle = \langle u, v \rangle - \mu \int_{\mathbb{R}^N} h(x) |u|^{q-2} uv dx - \int_{\mathbb{R}^N} |u|^{2_s^*-2} uv dx.$$

Lemma 2.1 ([18]). *Let $0 < s < 1$ such that $N > 2s$. The embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$ and the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p_{loc}(\mathbb{R}^N)$ is compact for any $p \in [2, 2_s^*)$.*

We define the best Sobolev constant

$$S_* = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\triangle)^{\frac{s}{2}} u(x)|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2_s^*} dx)^{2/2_s^*}}. \tag{2.4}$$

Lemma 2.2. *Assume $1 < q < 2$ and the condition* (H) *holds. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} h(x) |u_n|^q dx \to \int_{\mathbb{R}^N} h(x) |u|^q dx.$$

Proof. The proof is similar to Lemma 3.4 in [12], we omit it. □

Lemma 2.3. *If there is a convergent subsequence for any sequence* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *satisfying* $I(u_n) \to c$ *and* $I'(u_n) \to 0$, *we say that* I *satisfies the* $(\mathrm{PS})_c$ *condition. Assume the condition* (H) *holds, the functional* I *for any* $\lambda > 0$ *satisfies the* $(\mathrm{PS})_c$ *condition with*

$$c \in \left(-\infty, \frac{s}{N} S_*^{\frac{N}{2s}} - C_0 \mu^{q^*}\right),$$

where $C_0 = \frac{1}{q^*}\left[\left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{Nq}{s2_s^*}\right)^{\frac{q}{2_s^*}} |h^+|_{q^*}\right]^{q^*}$.

Proof. Let $\{u_n\}$ be a sequence in $H^s(\mathbb{R}^N)$ and satisfy

$$I(u_n) \to c, \quad I'(u_n) \to 0. \tag{2.5}$$

First we prove that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Arguing by contradiction, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. By the Hölder inequality and (2.5) for sufficiently large $n \in \mathbb{N}$, we obtain

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{2_s^*}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|u_n\|^2 + \mu\left(\frac{1}{2_s^*} - \frac{1}{q}\right)\int_{\mathbb{R}^N} h(x)|u_n|^q dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|u_n\|^2 - \mu\left(\frac{1}{q} - \frac{1}{2_s^*}\right)|h|_{q^*}|u_n|_{2_s^*}^q \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|u_n\|^2 - \mu S_*^{-\frac{q}{2}}\left(\frac{1}{q} - \frac{1}{2_s^*}\right)|h|_{q^*}\|u_n\|^q \\ &\to \infty. \end{aligned}$$

This implies $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Therefore, up to a subsequence, for some $u \in H^s(\mathbb{R}^N)$, we get

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^s(\mathbb{R}^N), \\ u_n &\rightharpoonup u \quad \text{in } L^p(\mathbb{R}^N),\ 2 \leq p < 2_s^*, \\ u_n(x) &\to u(x) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Since I is C^1, we have $\langle I'(u), u\rangle = 0$, which implies that

$$\|u\|^2 = \mu\int_{\mathbb{R}^N} h(x)|u|^q dx + \int_{\mathbb{R}^N} |u|^{2_s^*} dx \tag{2.6}$$

and

$$\begin{aligned} I(u) =& I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ =& \mu\left(\frac{1}{2} - \frac{1}{q}\right)\int_{\mathbb{R}^N} h(x)|u|^q dx + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} |u|^{2_s^*} dx \\ =& \mu\left(\frac{1}{q} - \frac{1}{2}\right)\int_{\mathbb{R}^N} \left[h^-(x) - h^+(x)\right]|u|^q dx + \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_{\mathbb{R}^N} |u|^{2_s^*} dx \\ \geq& \frac{s}{N}\int_{\mathbb{R}^N} |u|^{2_s^*} dx - \mu\left(\frac{1}{q} - \frac{1}{2}\right)\int_{\mathbb{R}^N} h^+(x)|u|^q dx. \end{aligned} \tag{2.7}$$

By the Hölder and Young inequalities, we have

$$\begin{aligned}
&\mu\left(\frac{1}{q}-\frac{1}{2}\right)\int_{\mathbb{R}^N} h^+(x)|u|^q dx\\
\leq&\mu\left(\frac{1}{q}-\frac{1}{2}\right)\left(\int_{\mathbb{R}^N}(h^+)^{q^*}dx\right)^{\frac{1}{q^*}}\left(\int_{\mathbb{R}^N}|u|^{2_s^*}dx\right)^{\frac{q}{2_s^*}}\\
\leq&\frac{s}{N}\int_{\mathbb{R}^N}|u|^{2_s^*}dx+C_0\mu^{q^*}.
\end{aligned}\tag{2.8}$$

From (2.7) and (2.8), we obtain

$$I(u)\geq -C_0\mu^{q^*}.\tag{2.9}$$

Taking $w_n=u_n-u$, by the Brezis-Lieb lemma (see [7]) yields

$$\|u_n\|^2=\|w_n\|^2+\|u\|^2+o(1)\tag{2.10}$$

and

$$\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx=\int_{\mathbb{R}^N}|w_n|^{2_s^*}dx+\int_{\mathbb{R}^N}|u|^{2_s^*}dx+o(1).\tag{2.11}$$

From (2.5), we have

$$\frac{1}{2}\|u_n\|^2-\frac{\mu}{q}\int_{\mathbb{R}^N}h(x)|u_n|^q dx-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx=c+o(1).\tag{2.12}$$

Hence by (2.10), (2.11), (2.12) and Lemma 2.2, one has

$$\frac{1}{2}\|w_n\|^2-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|w_n|^{2_s^*}dx=c-I(u)+o(1).\tag{2.13}$$

In view of $\langle I'(u_n),u_n\rangle=o(1)$ and $\langle I'(u),u\rangle=0$, we derive that

$$\|w_n\|^2=\int_{\mathbb{R}^N}|w_n|^{2_s^*}dx+o(1).\tag{2.14}$$

Now, we may assume that $\|w_n\|^2\to L\geq 0$. By (2.14), it follows that $\int_{\mathbb{R}^N}|w_n|^{2_s^*}dx\to L$.

Let us suppose that $L>0$. By applying the Sobolev inequality we know that

$$|w_n|_{2_s^*}^2 S_*\leq [w_n]_{H^s(\mathbb{R}^N)}^2\leq\|w_n\|^2.$$

Hence we can deduce that $L\geq S_*^{\frac{N}{2s}}$.

This fact combining with (2.9) and (2.13) yield

$$c\geq\frac{s}{N}S_*^{\frac{N}{2s}}-C_0\mu^{q^*},$$

this contradicts the definition of c. Hence, $\|w_n\|\to 0$, i.e. $\|u_n-u\|\to 0$. Thus, we prove that $\{u_n\}$ converges strongly to u in $H^s(\mathbb{R}^N)$. □

3. Proof of the main results

In this section, we will use some knowledge of genus, but it will not be listed here. Detailed definition and properties of genus can be seen in [1]. In [17], the author first established the following new version of the symmetric mountain-pass lemma based on R. Kajikiya [14].

Theorem 3.1. *Let X be infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfy* (B1) *and* (B2) *below.*

(B1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the* $(\mathrm{PS})_c$ *condition.*

(B2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$, where Γ_k signifies a family of closed symmetric subsets A of X with $0 \notin A$ and $\gamma(A) \geq k$.*

Then I admits a sequence of critical points $\{u_k\}$ such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need the following arguments. Like (2.8), by the Hölder and Young inequalities, we have

$$\begin{aligned}
&\frac{\mu}{q}\int_{\mathbb{R}^N} h^+(x)|u|^q dx \\
\leq &\frac{\mu}{q}\left(\int_{\mathbb{R}^N}(h^+)^{q^*}dx\right)^{\frac{1}{q^*}}\left(\int_{\mathbb{R}^N}u^{2_s^*}dx\right)^{\frac{q}{2_s^*}} \\
\leq &\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx + C\mu^{q^*},
\end{aligned} \tag{3.1}$$

where $C = \frac{1}{qq^*}|h^+|_{q^*}^{q^*} > 0$. From (3.1) and the Sobolev embedding, we can infer that

$$\begin{aligned}
I(u) =&\frac{1}{2}\|u\|^2 - \frac{\mu}{q}\int_{\mathbb{R}^N} h(x)|u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx \\
\geq&\frac{1}{2}\|u\|^2 - \frac{\mu}{q}\int_{\mathbb{R}^N} h^+(x)|u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx \\
\geq&\frac{1}{2}\|u\|^2 - \frac{2}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}dx - C\mu^{q^*} \\
\geq&\frac{1}{2}\|u\|^2 - \frac{2}{2_s^* S_*^{2_s^*/2}}\|u\|^{2_s^*} - C\mu^{q^*} \\
:=&A\|u\|^2 - B\|u\|^{2_s^*} - C\mu^{q^*},
\end{aligned} \tag{3.2}$$

where A, B, C are some positive constants.

Let $Q(t) = At^2 - Bt^{2_s^*} - C\mu^{q^*}$. Then

$$I(u) \geq Q(\|u\|)$$

and

$$Q'(t) = t\left(2A - 2_s^* B t^{2_s^*-2}\right).$$

By simple mathematical analysis, there exists $\mu_1 = \left(\frac{2sA}{NC}\right)^{\frac{1}{q^*}} \left(\frac{2A}{2_s^* B}\right)^{\frac{N-2s}{2sq^*}}$ such that for $\mu \in (0, \mu_1)$, $Q(t)$ attains its positive maximum. That means, there exists

$$t_* = \left(\frac{2A}{2_s^* B}\right)^{\frac{N-2s}{4s}}$$

such that

$$e = \max_{t \geq 0} Q(t) = Q(t_*) = \frac{2sA}{N} \left(\frac{2A}{2_s^* B}\right)^{\frac{N-2s}{2s}} - C\mu^{q^*} > 0.$$

Therefore, for $e_0 \in (0, e)$, we may find $t_0 \in (0, t_*)$ such that $Q(t_0) = e_0$. Now we choose a cut-off function $\chi(t) \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \chi(t) \leq 1$. Set

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq t_0, \\ \frac{At^2 - C\mu^{q^*} - e}{Bt^{2_s^*}}, & t > t_*. \end{cases}$$

We consider the perturbation of $I(u)$:

$$G(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{q}\chi(\|u\|) \int_{\mathbb{R}^N} h(x)|u|^q dx - \frac{1}{2_s^*}\chi(\|u\|) \int_{\mathbb{R}^N} |u|^{2_s^*} dx. \tag{3.3}$$

Therefore

$$G(u) \geq A\|u\|^2 - B\chi(\|u\|)\|u\|^{2_s^*} - C\mu^{q^*} := \bar{Q}(\|u\|),$$

where $\bar{Q}(t) = At^2 - B\chi(t)t^{2_s^*} - C\mu^{q^*}$ and

$$\bar{Q}(t) = \begin{cases} Q(t), & 0 \leq t \leq t_0, \\ e, & t > t_*. \end{cases}$$

It is easy to see that G is even, $G(0) = 0$ and bounded from below. In Lemma 2.3, let $\mu_2 = \left(\frac{s}{C_0 N} S_*^{\frac{N}{2s}}\right)^{1/q^*}$ such that for $\mu \in (0, \mu_2)$, we have $\frac{s}{N} S_*^{\frac{N}{2s}} - C_0\mu^{q^*} > 0$. Hence there exists $\mu^* = \min\{\mu_1, \mu_2\}$, for $\mu \in (0, \mu^*)$, G satisfies a $(\mathrm{PS})_c$ condition with

$$c < e_0 < \min\{e, \frac{s}{N} S_*^{\frac{N}{2s}} - C_0\mu^{q^*}\}.$$

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. If we can prove that $G(u)$ has a sequence of nontrivial weak solutions $\{u_n\}$ satisfying $u_n \to 0$ as $n \to \infty$ in $H^s(\mathbb{R}^N)$, Theorem 1.1 holds. In fact, in this case, it is clear that $I(u) = G(u)$ for $\|u\| < t_0$. Next we just need to verify that $G(u)$ satisfies the conditions of Theorem 3.1.

Let E_k be a k-dimensional subspace of $H^s(\mathbb{R}^N)$. By the equivalence of any norm in finite dimensional space, we obtain

$$\alpha_k = \inf_{u \in E_k, \|u\|=1} \int_{\mathbb{R}^N} h^+(x)|u|^q dx > 0. \tag{3.4}$$

We take $u \in E_k$ with norm $\|u\| = 1$ and $\rho > 0$ small enough, we get

$$G(\rho u) = I(\rho u)$$

$$=\frac{1}{2}\|\rho u\|^2-\frac{\mu}{q}\int_{\mathbb{R}^N}h(x)|\rho u|^q dx-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|\rho u|^{2_s^*}dx$$
$$=\frac{1}{2}\|\rho u\|^2-\frac{\mu}{q}\int_{\mathbb{R}^N}h^+(x)|\rho u|^q dx+\frac{\mu}{q}\int_{\mathbb{R}^N}h^-(x)|\rho u|^q dx-\frac{1}{2_s^*}\int_{\mathbb{R}^N}|\rho u|^{2_s^*}dx.$$

By the Hölder and Young inequalities, we have

$$\frac{\mu}{q}\int_{\mathbb{R}^N}h^-(x)|\rho u|^q dx\le\frac{1}{2_s^*}\int_{\mathbb{R}^N}|\rho u|^{2_s^*}dx+C_2\mu^{q^*},\tag{3.5}$$

where $C_2=\frac{1}{qq^*}|h^-|_{q^*}^{q^*}\ge 0$.
From (3.4) and (3.5), we get

$$G(\rho u)\le\frac{\rho^2}{2}-\frac{\mu}{q}\alpha_k\rho^q+C_2\mu^{q^*}:=-\beta(k)<0.$$

The above inequality implies that

$$\{u\in E_k:\|u\|=\rho\}\subset\left\{u\in H^s(\mathbb{R}^N):\ G(u)\le-\beta(k)\right\}.$$

Therefore, letting $A_k=\left\{u\in H^s(\mathbb{R}^N):\ G(u)\le-\beta(k)\right\}$. By genus proposition, we have $\gamma(A_k)\ge\gamma(\{u\in E_k:\|u\|=\rho\})\ge k$. We can also see $A_k\in\Gamma_k$ and $\sup_{u\in A_k}G(u)\le-\beta(k)<0$. We define $c_k=\inf_{A\in\Gamma_k}\sup_{u\in A}G(u)$, then $-\infty<c_k<0$. This shows $G(u)$ satisfies assumptions (B1) and (B2) of Theorem 3.1. This means that G has a sequence of solutions u_k converging to zero. □

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