PERSISTENCE OF TRAVELLING WAVEFRONTS IN A GENERALIZED BURGERS-HUXLEY EQUATION WITH LONG-RANGE DIFFUSION*

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Abstract In this paper, we study the persistence of travelling wavefronts in a generalized Burgers-Huxley equation with long-range diffusion. When the influence of long-range diffusion effect is sufficiently small, we prove the persistence of these waves by using geometric singular perturbation theory. When the influence becomes large, the behavior of these waves can only be investigate numerically. In this case, we find that the solutions lose monotonicity by using Matlab program bvp4c. Some previous results are extended.

Keywords Generalized Burgers-Huxley equation, travelling wavefronts, Fenichel’s theory, persistence.

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1. Introduction

Travelling wave solutions are solutions of special type, and can be usually characterized as solutions invariant with respect to translation in space. The existence of traveling waves appears to be very common in nonlinear equations. From the physical point of view, travelling waves usually describe transition processes [23]. Transition from one equilibrium to another is a typical case, and the corresponding wave is called as travelling wavefronts. Since travelling wave solutions may provide more information for understanding the physical phenomena, its investigation plays an important role in the study of nonlinear physical phenomena. This is the reason why there are so many methods for exact travelling wave solutions, such as bifurcation method [9, 18], Lie symmetry method [19], tanh-function method [17], trigonometric function expansion [6] and so on.

In [20], the following generalized Burgers-Huxley (gBH) equation

\[ u_t + \alpha u^n u_x - u_{xx} = \beta u(1 - u^n)(u^n - \gamma), \]  

(1.1)

where \( \alpha, \beta, n > 0 \) and \( 0 < \gamma < 1 \), was used to model the interaction between reaction mechanisms, convection effects and diffusion transports. The solutions of Eq. (1.1) have been extensively studied, including numerical solutions [1, 4, 11, 13, 15] and exact travelling wave solutions [5, 10, 22].

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When $n = 1$, Eq. (1.1) becomes the Burgers-Huxley (BH) equation
\begin{equation}
  u_t + \alpha uu_x - u_{xx} = \beta u(1 - u)(u - \gamma). \tag{1.2}
\end{equation}
When $\beta = 0$, it reduces to the Burgers equation. When $\alpha = 0$, it reduces to the Huxley equation, sometimes known as the FitzHugh-Nagumo [8]. In [16], Kyrychko et al. proved the persistence of travelling wavefronts of the BH equation with a small fourth-order derivative term
\begin{equation}
  u_t + \alpha uu_x - u_{xx} + \delta u_{xxxx} = \beta u(1 - u)(u - \gamma), \tag{1.3}
\end{equation}
where $0 < \delta \ll 1$. It is worthwhile to note that, Fredholm theory in $L^2$ was used to prove the heteroclinic connection in the slow manifold. However, the travelling fronts are continuous and thus aren’t appropriately studied in $L^2$. Moreover, they didn’t investigate what would happen to the travelling wavefronts when $\delta$ becomes large.

In this paper, we study the gBH equation with long-range diffusion
\begin{equation}
  u_t + \alpha u^n u_x - u_{xx} + Du_{xxxx} = \beta u(1 - u^n)(u^n - \gamma), \tag{1.4}
\end{equation}
where $D$ is a positive parameter characterizing long-range diffusion effect [3]. When $D$ is sufficiently small, we prove the persistence of the travelling wavefronts by using geometric singular perturbation theory [14]. In order to prove the heteroclinic connection in the slow manifold, we use the implicit function theorem. When $D$ becomes large, we numerically investigate the behavior of the travelling wavefronts by using Matlab program bvp4c, and find that the solutions lose monotonicity.

2. Dynamical systems reformulation

The travelling wave solutions of Eq. (1.4) are of the form
\begin{equation}
  u(x, t) = U(\xi) \quad \text{with} \quad \xi = x - ct, \tag{2.1}
\end{equation}
where $c$ is the wave speed. Substituting (2.1) into (1.4), we get
\begin{equation}
  -cU' + \alpha U^n U' - U'' + DU''' = \beta U(1 - U^n)(U^n - \gamma). \tag{2.2}
\end{equation}
Defining new variables
\begin{equation}
  U' = v, \quad v' = w, \quad w' = z, \tag{2.3}
\end{equation}
we rewrite Eq. (2.2) as
\begin{equation}
  Y = \begin{pmatrix} U \\ v \\ w \\ z \end{pmatrix}, \quad Y' = \begin{pmatrix} U' \\ v' \\ w' \\ z' \end{pmatrix} = \begin{pmatrix} v \\ w \\ z \\ \frac{1}{\beta} [\beta U(1 - U^n)(U^n - \gamma) + cv - \alpha U^n v + w] \end{pmatrix} = F(Y). \tag{2.4}
\end{equation}
Obviously, \( Y^0 = (0, 0, 0, 0)^T \) and \( Y^1 = (1, 0, 0, 0)^T \) are two equilibria of system (2.4). The linearization matrix at \( Y^0 \) is

\[
A_0 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{\beta\gamma}{D} & \frac{c}{D} & 1 & 0
\end{pmatrix}
\]

with the corresponding characteristic equation

\[
\lambda^4 - \frac{1}{D}\lambda^2 - \frac{c}{D}\lambda + \frac{\beta\gamma}{D} = 0. \tag{2.5}
\]

Similarly the linearization matrix at \( Y^1 \) is

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{\beta n(\gamma-1)}{D} & \frac{c-\alpha}{D} & 1 & 0
\end{pmatrix}
\]

with the corresponding characteristic equation

\[
\lambda^4 - \frac{1}{D}\lambda^2 - \frac{c-\alpha}{D}\lambda + \frac{\beta n(1-\gamma)}{D} = 0. \tag{2.6}
\]

We have the following result regarding the linearization of system (2.4).

**Theorem 2.1.** In system (2.4), the unstable manifold of \( Y^0 \) and the stable manifold of \( Y^1 \) both have dimension two.

**Proof.** Our proof is based on Argument Principle. Spectrum of linearization at \( Y^0 \) is determined by the roots of Eq. (2.5), which can be written as \( m_0(\lambda) = 0 \) with

\[
m_0(\lambda) = \lambda^4 - \frac{1}{D}\lambda^2 - \frac{c}{D}\lambda + \frac{\beta\gamma}{D}. \tag{2.7}
\]

We want to show that \( m_0(\lambda) \) has only two roots in the right half complex plane. Since \( m_0(\lambda) \) is analytic, the number of roots in the right half complex plane is

\[
\frac{1}{2\pi} \lim_{R \to \infty} \Delta_{C_0} \arg m_0(\lambda), \tag{2.8}
\]

where the contour \( C_0 \) is the boundary, traversed anticlockwise, of the semicircle of radius \( R \), centered at the origin, contained in \( \text{Re}\lambda \geq 0 \), and \( \Delta_{C_0} \arg m_0(\lambda) \) denotes the total change quantity in the argument of \( m_0(\lambda) \) along \( C_0 \). The formula (2.8) equals

\[
2 + \frac{1}{2\pi} [\Delta \arg m_0(iR)]_{R=-\infty}^{R=\infty}.
\]

The quantity in the bracket denotes the change in the argument of \( m_0(iR) \) as \( R \) goes from \( \infty \) to \( -\infty \), and thus we compute the number of times the image \( m_0(iR) \) winds around the origin. Note that

\[
m_0(iR) = \left(R^4 + \frac{1}{D} R^2 + \frac{\beta\gamma}{D}\right) + i \left(-\frac{c}{D} R\right).
\]
Since $R^4 + \frac{1}{D} R^2 + \frac{\beta n}{D} > 0$, the image $m_0(iR)$ only lies on the right half complex plane. For $|R|$ sufficiently large, $m_0(iR)$ has the asymptotic behavior
\[
\text{Re}m_0(iR) \sim R^4, \quad \text{Im}m_0(iR) \sim -\frac{c}{D} R \text{ as } R \to \pm \infty.
\]
So $\Delta \arg m_0(iR)|_{R=\pm \infty} = 0$, and thus the number of roots of Eq. (2.5) in the right half complex plane is two.

Similarly, for $Y^1$ we rewrite Eq. (2.6) as $m_1(\lambda) = 0$ with
\[
m_1(\lambda) = \lambda^4 - \frac{1}{D} \lambda^2 - \frac{c - \alpha}{D} \lambda + \frac{\beta n(1 - \gamma)}{D}.
\]
Now let $C_1$ be the boundary of left half complex plane defined same as $C_0$, then the number of roots of $m_1(\lambda)$ in the left half complex plane is
\[
2 + \frac{1}{2\pi} \Delta \arg m_1(iR)|_{R=\pm \infty}.
\]
Note that
\[
m_1(iR) = \left( R^4 + \frac{1}{D} R^2 + \frac{\beta n(1 - \gamma)}{D} \right) + i \left( \frac{\alpha - c}{D} R \right).
\]
Since $R^4 + \frac{1}{D} R^2 + \frac{\beta n(1 - \gamma)}{D} > 0$, the image $m_1(iR)$ only lies on the right half complex plane. For $|R|$ sufficiently large, $m_1(iR)$ has the asymptotic behavior
\[
\text{Re}m_1(iR) \sim R^4, \quad \text{Im}m_1(iR) \sim \frac{\alpha - c}{D} R \text{ as } R \to \pm \infty.
\]
So $\Delta \arg m_1(iR)|_{R=\pm \infty} = 0$, and thus the number of roots of Eq. (2.6) in the left half complex plane is two. This completes the proof of Theorem 2.1.

From Theorem 2.1, we know that the stable manifold $W^s(Y^0)$ and the unstable manifold $W^u(Y^1)$ also both have dimension two. This is important for numerical investigation of the behavior of the travelling wavefronts, which will be discussed in Section 4. However, Theorem 1 isn’t sufficient to show the intersection of $W^u(Y^0)$ and $W^s(Y^1)$, which is a heteroclinic orbit of system (2.4) amounting to a travelling wavefront of Eq. (1.4). In order to prove rigorously the existence of this intersection, we resort to geometric singular perturbation theory.

3. Persistence of travelling wavefronts for sufficiently small long-range diffusion

In this section, we prove the persistence of travelling wavefronts of Eq. (1.4) for sufficiently small long-range diffusion.

Let $D = \varepsilon^2 \ll 1$. Redefining (2.3) as
\[
U' = v, \quad \varepsilon w' = z, \quad \varepsilon w' = z,
\]
we rewrite system (2.4) as
\[
\begin{cases}
U' = v, \\
\varepsilon w' = z, \\
\varepsilon' = \beta U(1 - U^n)(U^n - \gamma) + cv - \alpha U^n v + w,
\end{cases}
\]

\[
\varepsilon z' = \beta U(1 - U^n)(U^n - \gamma) + cv - \alpha U^n v + w,
\]
which is called the slow system. With \( \eta = \xi / \varepsilon \), the dual fast system associated with system (3.2) is

\[
\begin{align*}
\dot{U}_\eta &= \varepsilon v, \\
\dot{v}_\eta &= \varepsilon w, \\
\dot{w}_\eta &= z, \\
\dot{z}_\eta &= \beta U(1 - U^n)(U^n - \gamma) + cv - \alpha U^n v + w.
\end{align*}
\] (3.3)

If \( \varepsilon \) is set to zero in system (3.2), then \( U \) and \( v \) are governed by

\[
\begin{align*}
\dot{U} &= v, \\
\dot{v} &= \beta U(1 - U^n)(U^n - \gamma) - cv + \alpha U^n v,
\end{align*}
\] (3.4)

while \( w \) and \( z \) lie on the set

\( M_0 = \{(U, v, w, z) : z = 0, w = -\beta U(1 - U^n)(U^n - \gamma) - cv + \alpha U^n v\} \),

which is a two-dimensional submanifold of \( R^4 \). Note that system (3.4) is the dynamical systems reformulation of Eq. (1.1).

By the definition in [7], the manifold \( M_0 \) is said to be normally hyperbolic if the linearization of the fast system, restricted to \( M_0 \), has exactly \( \dim M_0 \) eigenvalues on the imaginary axis, with the remainder of the spectrums hyperbolic. The linearization of the fast system (3.3) restricted to \( M_0 \) is

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
s & c & -\alpha U^n & 1 & 0
\end{pmatrix}
\]

with \( s = \gamma - (1 + \gamma)(1 + n)U^n + (2n + 1)U^{2n} \). So the matrix \( A \) has the eigenvalues 0, 0, 1, −1, and thus \( M_0 \) is normally hyperbolic. Therefore by Fenichel’s invariant manifold theory [7], for sufficiently small \( \varepsilon > 0 \) there exists a two-dimensional submanifold \( M_\varepsilon \) of \( R^4 \) which lies within \( O(\varepsilon) \) of \( M_0 \) and is diffeomorphic to \( M_0 \). Moreover, \( M_\varepsilon \) is invariant under the flow (3.2) and \( C^r \) smooth for any \( r < \infty \).

To determine the dynamics on \( M_\varepsilon \), we write

\[
M_\varepsilon = \{(U, v, w, z) : z = g(U, v, \varepsilon), w = h(U, v, \varepsilon) - H(U, v)\},
\] (3.5)

where \( g \) and \( h \) depend smoothly on \( \varepsilon \) and satisfy \( g(U, v, 0) = h(U, v, 0) = 0 \), and \( H(U, v) = \beta U(1 - U^n)(U^n - \gamma) + cv - \alpha U^n v \). Now, we expand \( g \) and \( h \) in Taylor series in \( \varepsilon \)

\[
\begin{align*}
g(U, v, \varepsilon) &= g(U, v, 0) + \varepsilon g\varepsilon(U, v, 0) + \frac{1}{2} \varepsilon^2 g\varepsilon\varepsilon(U, v, 0) + \cdots, \\
h(U, v, \varepsilon) &= h(U, v, 0) + \varepsilon h\varepsilon(U, v, 0) + \frac{1}{2} \varepsilon^2 h\varepsilon\varepsilon(U, v, 0) + \cdots.
\end{align*}
\]

Substituting the representations of \( z \) and \( w \) in \( M_\varepsilon \) from (3.5) into the third and fourth equations of system (3.2), equating the same order in \( \varepsilon \) (up to 2 order), we
have
\[ g(U, v, 0) = h(U, v, 0) = h_c(U, v, 0) = g_{ee}(U, v, 0) = 0, \]
\[ g_e(U, v, 0) = [(2n+1)U^{2n} - (1+\gamma)(1+n)U^n + \alpha U^n c - c)H(U, v) + \alpha n U^{n-1}v, \]
\[ \frac{1}{2}h_{ee}(U, v, 0) = vg_e(U, v, 0) - \frac{\partial g_e(U, v, 0)}{\partial v}H(U, v). \]

This allows us to rewrite system (3.2) as
\[
\begin{align*}
U' &= v, \\
v' &= -H(U, v) + \frac{1}{2}v^2 h_{ee}(U, v, 0) + O(\varepsilon^3),
\end{align*}
\]
which determines the dynamics on \( M_\varepsilon \).

When \( \varepsilon = 0 \), system (3.6) reduces to system (3.4). Now we are in the position to state and prove the following persistence theorem.

**Theorem 3.1.** If Eq. (1.1) admits a strictly increasing travelling wavefront \( u_0(x, t) = U_0(\xi) \) satisfying \( \lim_{\xi \to -\infty} U_0(\xi) = 0 \) and \( \lim_{\xi \to \infty} U_0(\xi) = 1 \), then for sufficiently small \( \varepsilon > 0 \) this travelling wavefront persists in Eq. (1.4). In other words, Eq. (1.4) also admits a strictly increasing travelling wavefront \( u(x, t) = U(\xi) \) satisfying \( \lim_{\xi \to -\infty} U(\xi) = 0 \) and \( \lim_{\xi \to \infty} U(\xi) = 1 \).

**Proof.** Obviously, the travelling wavefront \( u_0(x, t) \) corresponds to a heteroclinic orbit of system (3.4) in \((U, v)\) phase plane. This heteroclinic orbit connects the two equilibria \( E_- \) and \( E_+ \), where \( E_- = (0, 0) \) and \( E_+ = (1, 0) \). Let \( c_0 \) be the wave speed of the travelling wavefront. For sufficiently small \( \varepsilon \), \( E_- \) and \( E_+ \) are still two equilibria of system (3.6). Now we prove that system (3.6) admits a heteroclinic orbit connecting \( E_- \) and \( E_+ \). We rewrite system (3.6) as
\[
\begin{align*}
U' &= v, \\
v' &= \Phi(U, v, c, \varepsilon).
\end{align*}
\]
Note that
\[ \Phi(U, v, c, 0) = -H(U, v). \]

Since \( U_0(\xi) \) is strictly increasing, it can be characterized as the graph of some function
\[ v = f(U, c_0). \]

By the stable manifold theorem, for sufficiently small \( \varepsilon \) we can also characterize the unstable manifold of \( E_- \) as the graph of some function
\[ v = f_1(U, c, \varepsilon), \]
where \( f_1(0, c, \varepsilon) = 0 \). Furthermore, by continuous dependence of solutions on parameters, this manifold must cross the line \( U = 1/2 \) somewhere.

Similarly, let \( v = f_2(U, c, \varepsilon) \) be the function for the stable manifold of \( E_+ \). Then \( f_2(1, c, \varepsilon) = 0 \), and for sufficiently small \( \varepsilon \) it must also cross the line \( U = 1/2 \) somewhere. Thus
\[ f_1(U, c_0, 0) = f_2(U, c_0, 0) = f(U, c_0). \]

(3.8)
To show that Eq. (3.7) admits a heteroclinic orbit, we prove that there exists a
unique value \( c = c(\varepsilon) \), near \( c_0 \), such that the manifolds \( f_1 \) and \( f_2 \) cross the line
\( U = 1/2 \) at a same point. Define

\[
G(c, \varepsilon) = f_1 \left( \frac{1}{2}, c, \varepsilon \right) - f_2 \left( \frac{1}{2}, c, \varepsilon \right). \tag{3.9}
\]

Noticing that \( v = f_1(U, c, \varepsilon) \) and \( v = f_2(U, c, \varepsilon) \) both satisfy the equation

\[
\frac{dv}{dU} = \frac{\Phi(U, v, c, \varepsilon)}{v}, \tag{3.10}
\]
we have

\[
\frac{d}{dU} \left( \frac{\partial f_1(U, c_0, 0)}{\partial c} \right) = \frac{\partial}{\partial c} \left( \frac{df_1(U, c, 0)}{dU} \right) \bigg|_{c=c_0} = \partial \left( \frac{\Phi(U, f_1(U, c, 0), c, 0)}{f_1(U, c, 0)} \right) \bigg|_{c=c_0} = \partial \left( \frac{-\beta U(1-U^n)(U^n-\gamma) - c f_1(U, c, 0) + \alpha U^n f_1(U, c, 0)}{f_1(U, c, 0)} \right) \bigg|_{c=c_0} = \partial \left( -c + \alpha U^n - \frac{\beta U(1-U^n)(U^n-\gamma)}{f_1(U, c, 0)} \right) \bigg|_{c=c_0} = \frac{\beta U(1-U^n)(U^n-\gamma)}{f^2(U, c_0)}. \tag{3.11}
\]

Let

\[
P(U) = \frac{\beta U(1-U^n)(U^n-\gamma)}{f^2(U, c_0)}.
\]

Since

\[
\frac{\partial f_1(0, c, \varepsilon)}{\partial c} = 0,
\]
we solved Eq. (3.11) and get

\[
\frac{\partial f_1(U, c_0, 0)}{\partial c} = -e^{\int_0^U P(\xi)d\xi} \int_0^U e^{-\int_0^\xi P(\xi)d\xi} ds. \tag{3.12}
\]

It follows that

\[
\frac{\partial f_1(1/2, c_0, 0)}{\partial c} = -\int_0^{1/2} e^{-\int_0^\xi P(\xi)d\xi} ds. \tag{3.13}
\]

Similarly, we have

\[
\frac{\partial f_2(1/2, c_0, 0)}{\partial c} = -\int_1^{1/2} e^{-\int_0^\xi P(\xi)d\xi} ds. \tag{3.14}
\]

Therefore

\[
\frac{\partial G(c_0, 0)}{\partial c} = \frac{\partial f_1(1/2, c_0, 0)}{\partial c} - \frac{\partial f_2(1/2, c_0, 0)}{\partial c} = -\int_0^1 e^{-\int_0^\xi P(\xi)d\xi} ds < 0.
\]

By the implicit function theorem, for sufficiently small \( \varepsilon, G(c, \varepsilon) = 0 \) has a unique
root \( c = c(\varepsilon) \) near \( c_0 \). This implies that the manifolds \( f_1 \) and \( f_2 \) cross the line
\( U = 1/2 \) at a same point, that is, system (3.6) admits a heteroclinic orbit connect-
ing \( E_- \) and \( E_+ \). So Eq. (1.4) also admits a travelling wavefront \( u(x, t) = U(\xi) \)
satisfying \( \lim_{\xi \to -\infty} U(\xi) = 0 \) and \( \lim_{\xi \to \infty} U(\xi) = 1 \). Moreover for sufficiently small
\( \varepsilon \), the strict monotonicity of \( U_0(\xi) \) guarantees the strict monotonicity of \( U(\xi) \). This
completes the proof of Theorem 2.

\[ \square \]
4. Numerical investigation of travelling wavefronts for large long-range diffusion

In this section, we numerically investigate the behavior of the travelling wavefronts for large long-range diffusion.

![Figure 1](image1.png)

**Figure 1.** Heteroclinic orbits for (2.4) shown in $\xi - U$ plane, where $n = \alpha = \beta = 1$ and $\gamma = c = 0.5$.

![Figure 2](image2.png)

**Figure 2.** Truncations of the three charts in Figure 1 on the region $[-50, 50] \times [-0.03, 0.03]$.

For our purpose, we look for a solution $U(\xi)$ of system (2.4) satisfying the boundary conditions

$$U(-\infty) = 0, \quad U(\infty) = 1.$$  \hspace{1cm} (4.1)

We consider the boundary value problem (BVP) consisting of (2.4) and (4.1) on a finite interval $[L_1, L_2]$, with the approximate solution converging to a correct solution as $L_1 \to -\infty$ and $L_2 \to \infty$ [2, 12]. For this reason, we require the solution to have no projection on the stable manifold of $Y^0$ at $\xi = L_1$ and no projection on the unstable manifold of $Y^1$ at $\xi = L_2$. From Theorem 1, $W^s(Y^0)$ and $W^u(Y^1)$ both have dimension two, and thus constitute four boundary conditions for system (2.4). As regards numerical approximate solutions of BVP of ordinary differential equations, Matlab program bvp4c is an effective solver [21]. bvp4c implements a collocation method and requires users to supply a guess for the desired solution. In order to solve our problem, we use the solution [5]

$$u(x, t) = \left[\frac{1}{2} + \frac{1}{2} \tanh \left(\frac{n(\rho - \alpha)}{4(n + 1)} \left[x - \frac{\alpha - \rho + (\alpha + \rho)(n + 1)\gamma}{2(n + 1)} t + x_0\right]\right)\right]^\frac{1}{n}$$

as the guess solution. With the help of bvp4c, numerical simulations for some particular values of parameters are shown in Fig. 1 and Fig. 2, where the RelTol is $10^{-3}$ and AbsTol is $10^{-6}$. As shown in Fig. 1 and Fig. 2, when $D$ is small, the
shape of the perturbed travelling wavefront is close to the unperturbed one; when \(D\) becomes large, the perturbed travelling wavefront loses monotonicity.

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**References**


