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SUCCESSIVE ITERATIONS FOR UNIQUE POSITIVE SOLUTION OF A NONLINEAR FRACTIONAL Q-INTEGRAL BOUNDARY VALUE PROBLEM*

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Abstract In this paper, under certain nonlinear growth conditions, we investigate the existence and successive iterations for the unique positive solution of a nonlinear fractional q-integral boundary problem by employing hybrid monotone method, which is a novel approach to nonlinear fractional q-difference equation. This paper not only proves the existence of the unique positive solution, but also gives some computable explicit hybrid iterative sequences approximating to the unique positive solution.

Keywords Fractional q-difference equation, q-integral condition, explicit iterative sequence, hybrid monotone method.

MSC(2010) 34A08, 34B10, 34B18, 39A10.

1. Introduction

Since the fractional q-calculus theory was founded by Al-Salam [6] and Agarwal [2], its theory and application research has made great progress [1, 4, 7–9, 12–15, 17, 22, 23, 25, 26, 41]. Nonlinear equations, as a powerful tool for describing nonlinear phenomena in nature, are used and studied by more and more people. At present, with the development of fractional q-calculus, many researchers are concentrating their attention on the study of nonlinear fractional q-difference equations, and some excellent results have been obtained. Readers can find them in the literatures [3, 5, 18–21, 34, 35] and references therein.

In 2011, by employing a fixed point theorem in cones, Ferreira [12] considered a nonlinear q-fractional difference boundary value problem

$$\begin{cases} (D_q^\alpha y)(x) = -f(x, y(x)), & x \in (0,1), \\ y(0) = (D_q y)(0) = 0, & (D_q y)(1) = \beta \geq 0, \end{cases}$$

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where D_q^α is the Riemann-Liouville q-derivative of order α. Under some sufficient conditions, the author proved the existence of positive solutions for the above problem.

In 2013, Graef and Kong [15] considered the singular boundary value problem with fractional q-derivatives

$$\begin{cases} (D_q^\nu u)(t) = f(t, u(t)), \qquad t \in (0,1), \\ (D_q^i u)(0) = 0,\ i = 0, \cdots, n-2, \\ (D_q u)(1) = \displaystyle\sum_{j=1}^{m} a_j (D_q u)(t_j) + \lambda, \end{cases}$$

where $m, n \in \mathbb{N},\ m \geq 1,\ n \geq 2,\ n-1 < \nu \leq n,\ \lambda \geq 0$ is a parameter, $q, t_i \in (0,1)$ for $i = 1, \cdots, m$. $f \in C((0,1] \times (0,\infty), [0,\infty)),\ a_j \geq 0$ and D_q^ν is the Riemann-Liouville type q-derivative of order ν. By means of a nonlinear alternative of Leray-Schauder, the authors gave some sufficient conditions for the existence of positive solutions.

In 2014, Yang [34] investigated the integral boundary value problem for systems of nonlinear q-fractional difference equations as follows

$$\begin{cases} (D_q^\alpha u)(t) + \lambda f(t, u(t), v(t)) = 0, \\ (D_q^\beta v)(t) + \lambda g(t, u(t), v(t)) = 0,\ t \in (0,1),\ \lambda > 0, \\ (D_q^j u)(0) = (D_q^j v)(0) = 0, \quad 0 \leq j \leq n-2, \\ u(1) = \mu \displaystyle\int_0^1 u(s) d_q s,\ v(1) = \nu \displaystyle\int_0^1 v(s) d_q s, \end{cases}$$

where $\alpha, \beta \in (n-1, n]$ are real numbers and $n \geq 3$ is an integer, $D_q^{(\cdot)}$ denotes the Riemann-Lioville type fractional q-derivative, λ, μ, ν are three parameters with $0 < \mu < [\beta]_q, 0 < \nu < [\alpha]_q$, and $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to \mathbb{R}$ are continuous.

In 2015, Li et al. [21] investigated a class of fractional q-difference Schrölnger equations precisely the time-independent.

$$D_q^\alpha u(x) + \frac{m}{\hbar}(E - v(x))u(x) = 0,$$

where $v(x)$ is the trapping potential, m is the mass of a particle, $\hbar$ is the Planck constant, E is the energy of a particle. In order to study the fractional q-difference Schrölnger equations, the authors considered a general fractional q-difference equation

$$(D_q^\alpha u)(x) + \lambda h(x) f(u(x)) = 0, \qquad 0 < x < 1,$$

subject to the boundary conditions

$$u(0) = D_q u(0) = D_q u(1) = 0,$$

where $0 < q < 1,\ 2 < \alpha < 3,\ f \in C([0,\infty), (0,\infty)),\ h \in ((0,1), (0,\infty))$. The authors obtained some existence results for the above general fractional q-difference equation with the assistance of a fixed point theorem in cones.

It should be pointed out that in the study of nonlinear fractional q-difference equations, much effort focused on the existence of solutions. *"How to discover the solution?"* is a very challenging and meaningful subject. In our current work, we are going to challenge this topic by applying the hybrid monotone method. Precisely, we will study the following nonlinear fractional q-difference equation with q-integral boundary condition

$$\begin{cases} D_q^\alpha z(t) + f(z(t), (I_q^\beta z)(t)) = 0, \quad t \in (0,1), \\ D_q^j z(0) = 0, \quad 0 \le j \le n-2, \quad z(1) = \mu \int_0^1 z(s) d_q s, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$ and $n \ge 3$, μ is a parameter with $0 < \mu < [\alpha]_q$, $(I_q^\beta z)(t) = \frac{1}{\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} z(s) d_q s$ is the Riemann-Liouville type fractional q-integral of order $\beta > 0$, D_q^α denotes the Riemann-Lioville type fractional q-derivative of order α, and $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is continuous with $f(t, 0) \not\equiv 0$ on $[0, 1]$.

Different from the methods used in the existing literature [1, 3, 5, 12–15, 17–23, 34, 35, 41], in this article, we employ the hybrid monotone iterative method, which is a novel approach to the nonlinear fractional q-difference equation with q-integral boundary condition (1.1). We establish the existence and uniqueness of the positive solution of the q-integral boundary problem. Meanwhile, two hybrid monotone iterative algorithms for approximating the positive solution are also derived. For the details and recent developments of monotone iterative methods, the readers can refer to [10, 11, 24, 27–33, 36–40].

2. Preliminaries and Lemmas

For the readers' convenience, we present some background materials of fractional q-calculus theory and lemmas.

Definition 2.1 (Annaby et al. [8]). For $\alpha \ge 0$, the Riemann-Liouville fractional q-integral of order α of a function f is $I_q^0 f(t) = f(t)$ and

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s, \quad \alpha > 0, t \in [0,1],$$

where $\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}$ and satisfies the relation $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$,

here

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, \quad (a-b)^{(0)} = 1 \ \ \textit{and} \ \ (a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{1-(b/a)q^n}{1-(b/a)q^{n+\alpha}}, a, b, \alpha \in \mathbb{R}.$$

Note that if $a = 1$ and $b = q$, then $(1-q)^{(\alpha)} = \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{1+\alpha+n}}$; if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

The well-known q-derivative and q-integral of the function $f(t)$ are defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad I_q f(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f(tq^n),$$

where $(D_q f)(0) = \lim_{t \to 0} (D_q f)(t)$ and q-derivative and q-integral of higher order by

$$(D_q^0 f)(t) = f(t) \quad \text{and} \quad (D_q^n f)(t) = D_q(D_q^{n-1} f)(t), \quad n \in \mathbb{N},$$

$$(I_q^0 f)(t) = f(t) \quad \text{and} \quad (I_q^n f)(t) = I_q(I_q^{n-1} f)(t), \quad n \in \mathbb{N}.$$

Definition 2.2 (Annaby et al. [8]). For $\alpha \geq 0$, the Riemann-Liouville fractional q-derivative of order α of a function f is $D_q^0 f(t) = f(t)$ and

$$(D_q^\alpha f)(t) = (D_q^m I_q^{m-\alpha} f)(t), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 (Rajkovic et al. [26]). *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the following relations hold:*
(1) $(I_q^\beta I_q^\alpha f)(t) = I_q^{\alpha+\beta} f(t)$, (2) $(D_q^\alpha I_q^\alpha f)(t) = f(t)$.

Lemma 2.2 (Ferreira [12]). *Assume that $\alpha > 0$ and p is a positive integer. Then*

$$(I_q^\alpha D_q^p f)(t) = (D_q^p I_q^\alpha f)(t) - \sum_{k=0}^{p-1} \frac{t^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Remark 2.1 (Rajkovic et al. [26]). Let $\alpha \geq 0$ and $\lambda > -1$. Then we have

$$I_q^\alpha((t-a)^{(\lambda)}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(t-a)^{(\alpha+\lambda)}, \quad 0 < a < t.$$

Lemma 2.3 (Yang [34]). *For any $y \in C[0,1]$, the q-integral boundary value problem*

$$\begin{cases} D_q^\alpha u(t) + y(t) = 0, \quad t \in (0,1), \\ D_q^j u(0) = 0, \quad 0 \leq j \leq n-2, \quad u(1) = \mu \int_0^1 u(s) d_q s, \end{cases} \tag{2.1}$$

has a unique solution given by

$$u(t) = \int_0^1 G(t, qs) y(s) d_q s, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} \left(1 + \dfrac{\mu q^{\alpha-1} s}{[\alpha]_q - \mu}\right) t^{\alpha-1}(1-s)^{(\alpha-1)} - (t-s)^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ \left(1 + \dfrac{\mu q^{\alpha-1} s}{[\alpha]_q - \mu}\right) t^{\alpha-1}(1-s)^{(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Lemma 2.4 (Yang [34]). *The function $G(t,s)$ defined by (2.3) has the following properties:*

$$\begin{aligned} &(a): G(t,s) \text{ is continous and } G(t,qs) \geq 0, \quad \forall t, s \in [0,1] \\ &(b): \rho(s) t^{\alpha-1} \leq G(t,qs) \leq \Delta t^{\alpha-1}, \quad \forall t, s \in [0,1], \end{aligned} \tag{2.4}$$

where

$$\rho(s) = \frac{\mu q^\alpha (1-qs)^{(\alpha-1)} s}{([\alpha]_q - \mu)\Gamma_q(\alpha)}, \quad \Delta = \frac{q[\alpha-1]_q([\alpha]_q - \mu) + \mu q^\alpha}{([\alpha]_q - \mu)\Gamma_q(\alpha)}, \quad t, s \in [0,1].$$

Let $\mathring{P}=\{x \in P | x$ is an interior point of cone $P\}$. A cone P is called to be a solid cone if its interior $\mathring{P}$ is nonempty.

Lemma 2.5 (Guo [16]). *Let P be a normal, solid cone of Banach space E, and $T : \mathring{P} \to \mathring{P}$ be a mixed monotone operator. Suppose that there exists $0 < \sigma < 1$ such that*

$$T(cx, \frac{1}{c}y) \geq c^{\sigma} T(x, y), \quad x, y \in \mathring{P}, \quad 0 < c < 1. \tag{2.5}$$

Then the operator T has a unique fixed point $x^* \in \mathring{P}$. Moreover, for any initial $x_0, y_0 \in \mathring{P}$, by constructing successively the sequences $x_n = T(x_{n-1}, y_{n-1}), y_n = T(y_{n-1}, x_{n-1}), n = 1, 2, ...,$ we have $\|x_n - x^*\| \to 0$, and $\|y_n - x^*\| \to 0$ as $n \to +\infty$.

3. Main Results

In this section, we formulate our main results on the uniqueness of positive solution for nonlinear fractional q-difference equation with q-integral boundary condition (1.1).

Let $E = C[0, 1]$, then E is a Banach space endowed with the norm $\|z\| = \max_{t \in [0,1]} |z(t)|$.

Define a cone P in E by $P = \{z \mid z \in E : z(t) \geq 0, t \in [0, 1]\}$ and a cone

$$Q_\omega = \{z \in P : \frac{1}{M}\omega(t) \leq z(t) \leq M\omega(t), t \in [0, 1]\} \tag{3.1}$$

where

$$\begin{aligned} M > \max \Big\{ & a^{-1}, 2b, \Big(2^{\sigma} a^{\sigma} \Delta\phi(1, 1) + \frac{1-q}{1-q^{1-\sigma(\alpha+\beta-1)}} b^{-\sigma} \Delta\psi(1, 1)\Big)^{\frac{1}{1-\sigma}}, 1, \\ & \Big(\frac{[\alpha]_q - \mu}{\mu q^{\alpha}}\Big)^{\frac{1}{1-\sigma}} \Big(\frac{2^{-\sigma} a^{-\sigma} \psi(1, 1)}{\Gamma_q(\alpha + 2)} + \frac{b^{\sigma} \Gamma_q(2 + \sigma(\alpha + \beta - 1))\phi(1, 1)}{\Gamma_q(\alpha + 2 + \sigma(\alpha + \beta - 1))}\Big)^{-\frac{1}{1-\sigma}} \Big\}, \end{aligned} \tag{3.2}$$

$$a = \max\Big\{\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha + \beta)}, 1\Big\}, \quad b = \min\Big\{\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha + \beta)}, 1\Big\}, \quad \omega(t) = t^{\alpha-1}. \tag{3.3}$$

Then the cone Q_ω is solid.

Now, we present some assumptions to be used throughout the rest of the paper.
$(H1)$ $f(u, v) = \phi(u, v) + \psi(u, v)$, where $\phi, \psi \in C([0, +\infty)^2 \to [0, +\infty))$, and $\phi(u, v)$ is non-decreasing and $\psi(u, v)$ is non-increasing for $u, v > 0$, respectively.
$(H2)$ There exists $0 < \sigma < \frac{1}{\alpha+\beta-1}$, for $u, v > 0$ and any $c \in (0, 1)$, such that

$$\phi(cu, cv) \geq c^{\sigma}\phi(u, v), \quad \psi(c^{-1}u, c^{-1}v) \geq c^{\sigma}\psi(u, v). \tag{3.4}$$

Remark 3.1. By the condition $(H2)$, for $u, v > 0$ and any $c \geq 1$, the following inequalities hold

$$\phi(cu, cv) \leq c^{\sigma}\phi(u, v), \quad \psi(c^{-1}u, c^{-1}v) \leq c^{\sigma}\psi(u, v). \tag{3.5}$$

In order to investigate the uniqueness of positive solution for nonlinear fractional q-difference equation (1.1), we first study an auxiliary problem for q-integral boundary value problems (1.1):

$$\begin{cases} D_q^{\alpha} z(t) + f(z(t) + \frac{1}{n}, (I_q^{\beta} z)(t) + \frac{1}{n}) = 0, \quad t \in (0,1), \\ D_q^j z(0) = 0, \quad 0 \le j \le n-2, \quad z(1) = \mu \int_0^1 z(s) d_q s, \end{cases} \tag{3.6}$$

where $n \in 2, 3, ...$ Assume that $f : (\mathbb{R}\backslash\{0\})^2 \to \mathbb{R}$ is continuous. It is obvious z is a solution of q-integral boundary value problems (3.6) if $z \in C[0,1]$ fulfills the following nonlinear q-integral equation:

$$z(t) = \int_0^1 G(t, qs) f(z(s) + \frac{1}{n}, (I_q^{\beta} z)(s) + \frac{1}{n}) d_q s. \tag{3.7}$$

Theorem 3.1. *Assume the conditions* $(H1)$ *and* $(H2)$ *are fulfilled. Then the q-integral boundary value problems* (1.1) *has a unique positive solution* z^*, *and there exist two positive constants* $\lambda, \rho (\lambda < \rho)$ *such that*

$$\lambda t^{\alpha-1} \le z^* \le \rho t^{\alpha-1}. \tag{3.8}$$

Moreover, for any initial values $u_0, v_0 \in Q_{\omega}$, *one can construct successively two explicit mixed iterative sequences*

$$\begin{aligned} u_m(t) &= \int_0^1 G(t,qs)[\phi(u_{m-1}(s), (I_q^{\beta} u_{m-1})(s)) + \psi(v_{m-1}(s), (I_q^{\beta} v_{m-1})(s))] d_q s, \\ v_m(t) &= \int_0^1 G(t,qs)[\phi(v_{m-1}(s), (I_q^{\beta} v_{m-1})(s)) + \psi(u_{m-1}(s), (I_q^{\beta} u_{m-1})(s))] d_q s, \end{aligned} \tag{3.9}$$

such that u_m, v_m *converge uniformly to* z^* *as* $m \to \infty$ *on* $[0,1]$, *i.e.,* $\|u_m - z^*\| \to 0, \|v_m - z^*\| \to 0$ *as* $m \to \infty$.

Proof. First, in order to prove the uniqueness of positive solution of the auxiliary q-integral boundary value problems (3.6), we define an operator $H : P \times P \to P$ by

$$\begin{aligned} H(u,v)(t) = \int_0^1 G(t,qs) \Big[&\phi\Big(u(s) + \frac{1}{n}, (I_q^{\beta} u)(s) + \frac{1}{n}\Big) \\ &+ \psi\Big(v(s) + \frac{1}{n}, (I_q^{\beta} v)(s) + \frac{1}{n}\Big)\Big] d_q s. \end{aligned} \tag{3.10}$$

Now, we show that the operator H maps $Q_{\omega} \times Q_{\omega}$ into Q_{ω}. From the definition of Riemann-Liouville fractional q-integral, we know that

$$\begin{aligned} (I_q^{\beta} \omega)(t) =& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{\beta-1} \omega(s) d_q s \\ =& \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{\beta-1} s^{\alpha-1} d_q s \\ =& \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)} t^{\alpha+\beta-1}. \end{aligned} \tag{3.11}$$

On one hand, for any $u, v \in Q_\omega$, we have

$$\begin{aligned}\phi\Big(u(t)+\frac{1}{n},(I_q^\beta u)(t)+\frac{1}{n}\Big) \leq&\phi\Big(Mt^{\alpha-1}+1,\frac{M\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}t^{\alpha+\beta-1}+1\Big)\\ \leq&\phi(Ma+1,Ma+1)\leq(Ma+1)^\sigma\phi(1,1)\\ \leq&2^\sigma a^\sigma M^\sigma\phi(1,1),\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\psi\Big(v(t)+\frac{1}{n},(I_q^\beta v)(t)+\frac{1}{n}\Big) \leq&\psi\Big(\frac{1}{M}t^{\alpha-1}+\frac{1}{n},\frac{\Gamma_q(\alpha)}{M\Gamma_q(\alpha+\beta)}t^{\alpha+\beta-1}+\frac{1}{n}\Big)\\ \leq&\psi\Big(\frac{b}{M}t^{\alpha-1}+\frac{1}{n},\frac{b}{M}t^{\alpha+\beta-1}+\frac{1}{n}\Big)\\ \leq&\Big(\frac{b}{M}t^{\alpha+\beta-1}+\frac{1}{n}\Big)^{-\sigma}\psi(1,1)\\ \leq&b^{-\sigma}M^\sigma t^{-\sigma(\alpha+\beta-1)}\psi(1,1),\end{aligned} \tag{3.13}$$

in which, $a,\ b$ are defined in (3.3).

It follows from $0<\sigma<\frac{1}{\alpha+\beta-1}$ that

$$\begin{aligned}H(u,v)(t)=&\int_0^1 G(t,qs)\Big[\phi\Big(u(s)+\frac{1}{n},(I_q^\beta u)(s)+\frac{1}{n}\Big)+\psi\Big(v(s)+\frac{1}{n},(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs\\ \leq&\Delta\int_0^1 t^{\alpha-1}(2^\sigma a^\sigma M^\sigma\phi(1,1)+b^{-\sigma}M^\sigma s^{-\sigma(\alpha+\beta-1)}\psi(1,1))d_qs\\ \leq&2^\sigma a^\sigma M^\sigma\phi(1,1)\Delta t^{\alpha-1}+\frac{1-q}{1-q^{1-\sigma(\alpha+\beta-1)}}b^{-\sigma}M^\sigma\psi(1,1)\Delta t^{\alpha-1}\\ \leq&Mt^{\alpha-1}.\end{aligned} \tag{3.14}$$

On the other hand, in view of the conditions (H_1) and (H_2), we get

$$\begin{aligned}\psi\Big(v(t)+\frac{1}{n},(I_q^\beta v)(t)+\frac{1}{n}\Big) \geq&\psi\Big(Mt^{\alpha-1}+\frac{1}{n},\frac{M\Gamma_q(\alpha)}{\Gamma_q(\alpha+\beta)}t^{\alpha+\beta-1}+\frac{1}{n}\Big)\\ \geq&\psi\Big(Ma+\frac{1}{n},Ma+\frac{1}{n}\Big)\geq\psi(Ma+1,Ma+1)\\ \geq&(Ma+1)^{-\sigma}\psi(1,1)\geq2^{-\sigma}a^{-\sigma}M^{-\sigma}\psi(1,1),\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\phi\Big(u(t)+\frac{1}{n},(I_q^\beta u)(t)+\frac{1}{n}\Big) \geq&\phi\Big(\frac{1}{M}t^{\alpha-1}+\frac{1}{n},\frac{\Gamma_q(\alpha)}{M\Gamma_q(\alpha+\beta)}t^{\alpha+\beta-1}+\frac{1}{n}\Big)\\ \geq&\phi\Big(\frac{b}{M}t^{\alpha-1}+\frac{1}{n},\frac{b}{M}t^{\alpha+\beta-1}+\frac{1}{n}\Big)\\ \geq&\Big(\frac{b}{M}t^{\alpha+\beta-1}+\frac{1}{n}\Big)^{\sigma}\phi(1,1)\\ \geq&b^\sigma M^{-\sigma}t^{\sigma(\alpha+\beta-1)}\phi(1,1),\end{aligned} \tag{3.16}$$

In view of (3.15), (3.16) and Lemma 2.4, we can obtain that

$$
\begin{aligned}
H(u,v)(t) =& \int_0^1 G(t,qs)\Big[\phi\Big(u(s)+\frac{1}{n},(I_q^\beta u)(s)+\frac{1}{n}\Big)\\
&+\psi\Big(v(s)+\frac{1}{n},(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs\\
\geq&\frac{\mu q^\alpha t^{\alpha-1}}{([\alpha]_q-\mu)}\int_0^1\frac{(1-qs)^{(\alpha-1)}s}{\Gamma_q(\alpha)}(2^{-\sigma}a^{-\sigma}M^{-\sigma}\psi(1,1)\\
&+b^\sigma M^{-\sigma}s^{\sigma(\alpha+\beta-1)}\phi(1,1))d_qs\\
=&\frac{\mu q^\alpha t^{\alpha-1}}{([\alpha]_q-\mu)}\Big(2^{-\sigma}a^{-\sigma}M^{-\sigma}\psi(1,1)\frac{\Gamma_q(2)}{\Gamma_q(\alpha+2)}\\
&+\frac{b^\sigma M^{-\sigma}\phi(1,1)\Gamma_q[2+\sigma(\alpha+\beta-1)]}{\Gamma_q[\alpha+2+\sigma(\alpha+\beta-1)]}\Big)\\
=&\frac{\mu q^\alpha t^{\alpha-1}}{([\alpha]_q-\mu)M^\sigma}\Big(\frac{2^{-\sigma}a^{-\sigma}\psi(1,1)}{\Gamma_q(\alpha+2)}+\frac{b^\sigma\Gamma_q[2+\sigma(\alpha+\beta-1)]\phi(1,1)}{\Gamma_q[\alpha+2+\sigma(\alpha+\beta-1)]}\Big)\\
\geq&\frac{1}{M}t^{\alpha-1}.
\end{aligned}
\tag{3.17}
$$

Therefore, $T:Q_\omega\times Q_\omega\to Q_\omega$

Next, we prove that the operator H is mixed monotone. In fact, for any $u_1,u_2\in Q_\omega$ and $u_1\leq u_2$, we have

$$
\begin{aligned}
H(u_1,v)(t) =& \int_0^1 G(t,qs)\Big[\phi\Big(u_1(s)+\frac{1}{n},(I_q^\beta u_1)(s)+\frac{1}{n}\Big)\\
&+\psi\Big(v(s)+\frac{1}{n},(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs\\
\leq&\int_0^1 G(t,qs)\Big[\phi\Big(u_2(s)+\frac{1}{n},(I_q^\beta u_2)(s)+\frac{1}{n}\Big)\\
&+\psi\Big(v(s)+\frac{1}{n},(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs\\
=&H(u_2,v)(t),
\end{aligned}
\tag{3.18}
$$

which means

$$H(u_1,v)(t)\leq H(u_2,v)(t),\quad v\in Q_\omega \tag{3.19}$$

i.e., for any $v\in Q_\omega$, $H(u,v)$ is non-decreasing in u. Similarly, if $v_1\geq v_2, v_1,v_2,u\in Q_\omega$, it is easy to prove that $H(u,v_1)(t)\leq H(u,v_2)(t)$. Therefore, we can obtain that $H:Q_\omega\times Q_\omega\to Q_\omega$ is a mixed monotone operator.

Finally, we prove that the condition (2.5) holds for the mixed monotone operator H. In fact, combining with (H_2), for any $u,v\in Q_\omega$ and $0<c<1$, we have

$$
\begin{aligned}
H(cu,\frac{1}{c}v)(t) =& \int_0^1 G(t,qs)\Big[\phi\Big(cu(s)+\frac{1}{n},c(I_q^\beta u)(s)+\frac{1}{n}\Big)\\
&+\psi\Big(c^{-1}v(s)+\frac{1}{n},c^{-1}(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs\\
\geq&\int_0^1 G(t,qs)c^\sigma\Big[\phi\Big(u(s)+\frac{1}{n},(I_q^\beta u)(s)+\frac{1}{n}\Big)\\
&+\psi\Big(v(s)+\frac{1}{n},(I_q^\beta v)(s)+\frac{1}{n}\Big)\Big]d_qs
\end{aligned}
$$

$$=c^{\sigma}H(u,v)(t), \quad \forall t\in[0,1]. \tag{3.20}$$

Since $\alpha\in(n-1,n],\ (n\geq 3),\ and\ \beta>0$, it means $0<\sigma<\frac{1}{\alpha+\beta-1}<1$. It follows from Lemma 2.5 that there exists a unique positive solution $z_n^*\in Q_\omega$ such that $H(z_n^*,z_n^*)=z_n^*$. Thus, for each $n(n\geq 2)$, the auxiliary problem (3.6) has a unique positive solution.

Since $z_n^*\in Q_\omega$, z_n^* has uniform lower and upper bounds, it follows from the Arzela-Ascoli Theorem and the Lebesgue dominated convergence theorem that $\{z_n^*\}_{n\geq 2}$ being an equicontinuous family on $[0,1]$. Let $z_n^*\to z^*$ as $n\to\infty$, by Lemma 2.5, we know that the mixed monotone operator H has a unique fixed point $z^*\in Q_\omega$, which implies that the nonlinear fractional q-difference equation with q-integral boundary condition (1.1) has a unique solution z^* and z^* satisfies $\lambda t^{\alpha-1}=\frac{t^{\alpha-1}}{M}\leq z^*\leq Mt^{\alpha-1}=\rho t^{\alpha-1}$. Moreover, for any initial values $u_0,v_0\in Q_\omega$, we can construct successfully the following two explicit mixed iterative sequences:

$$\begin{aligned}u_m(t)&=\int_0^1 G(t,qs)[\phi(u_{m-1}(s),(I_q^{\alpha}u_{m-1})(s))+\psi(v_{m-1}(s),(I_q^{\alpha}v_{m-1})(s))]d_qs,\\ v_m(t)&=\int_0^1 G(t,qs)[\phi(v_{m-1}(s),(I_q^{\alpha}v_{m-1})(s))+\psi(u_{m-1}(s),(I_q^{\alpha}u_{m-1})(s))]d_qs,\end{aligned} \tag{3.21}$$

and $u_m(t),v_m(t)$ converge uniformly to the unique solution $z^*(t)$ on $[0,1]$ as $m\to\infty$, *i.e.*, $\|u_m-z^*\|\to 0,\|v_m-z^*\|\to 0$ as $m\to\infty$. □

4. Example

Example 4.1. Consider the following nonlinear fractional q-difference equation with q-integral boundary condition:

$$\begin{cases} D_{\frac{1}{2}}^{\frac{9}{2}}z(t)+z^{\frac{1}{7}}(t)+[(I_{\frac{1}{2}}^{\frac{1}{3}}z)(t)]^{\frac{1}{6}}+z^{-\frac{1}{3}}(t)+[(I_{\frac{1}{2}}^{\frac{1}{3}}z)(t)]^{-\frac{1}{5}}=0, & t\in(0,1),\\ D_{\frac{1}{2}}^{j}z(0)=0, \quad j=0,1,2,3, \quad z(1)=3\int_0^1 z(s)d_qs, \end{cases} \tag{4.1}$$

where $\alpha=\frac{9}{2},\ \beta=\frac{1}{3},\ q=\frac{1}{2},\ \mu=3$, and

$$\phi(u,v)=u^{\frac{1}{7}}+v^{\frac{1}{6}}, \quad \psi(u,v)=u^{-\frac{1}{3}}+v^{-\frac{1}{5}}. \tag{4.2}$$

Thus, we can choose $\sigma=\frac{1}{5}<\frac{1}{\alpha+\beta-1}$, then $\phi(cu,cv)=c^{\frac{1}{7}}u^{\frac{1}{7}}+c^{\frac{1}{6}}v^{\frac{1}{6}}\geq c^{\frac{1}{5}}\phi(u,v)$, $\psi(c^{-1}u,c^{-1}v)=(c^{-1}u)^{-\frac{1}{3}}+(c^{-1}v)^{-\frac{1}{5}}\geq c^{\frac{1}{5}}\psi(u,v)$, for any $u,v>0$ and $0<c<1$. Clearly, $\phi,\psi:(0,+\infty)^2\to[0,+\infty)$ are continuous, and $\phi(u,v)$ is non-decreasing and $\psi(u,v)$ is non-increasing in $u,v>0$ respectively. Thus by Theorem 3.1, the nonlinear fractional q-difference equation with q-integral boundary condition (4.1) has a unique positive solution z^*, and there exist two constants λ,ρ such that

$$\lambda t^{\frac{7}{2}}\leq z^*(t)\leq\rho t^{\frac{7}{2}}. \tag{4.3}$$

Moreover, for any initial $u_0,v_0\in Q_\omega$, we can construct successively two sequences

$\{u_m\}$ and $\{v_m\}$ by

$$\begin{aligned} u_m(t) &= \int_0^1 G(t, \frac{1}{2}s)[\phi(u_{m-1}(s), (I_{\frac{1}{2}}^{\frac{1}{3}} u_{m-1})(s)) + \psi(v_{m-1}(s), (I_{\frac{1}{2}}^{\frac{1}{3}} v_{m-1})(s))]d_q s, \\ v_m(t) &= \int_0^1 G(t, \frac{1}{2}s)[\phi(v_{m-1}(s), (I_{\frac{1}{2}}^{\frac{1}{3}} v_{m-1})(s)) + \psi(u_{m-1}(s), (I_{\frac{1}{2}}^{\frac{1}{3}} u_{m-1})(s))]d_q s, \end{aligned} \tag{4.4}$$

and the iterative sequences $u_m(t), v_m(t)$ converge uniformly to $z^*(t)$ on $[0, 1]$ as $m \to \infty$, *i.e.*, $\|u_m - z^*\| \to 0, \|v_m - z^*\| \to 0$ as $m \to \infty$.

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