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STRONG CONVERGENCE OF A GENERAL VISCOSITY EXPLICIT RULE FOR THE SUM OF TWO MONOTONE OPERATORS IN HILBERT SPACES

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Abstract In this paper, we study a general viscosity explicit rule for approximating the solutions of the variational inclusion problem for the sum of two monotone operators. We then prove its strong convergence under some new conditions on the parameters in the framework of Hilbert spaces. As applications, we apply our main result to the split feasibility problem and the LASSO problem. We also give some numerical examples to support our main result. The results presented in this paper extend and improve the corresponding results in the literature.

Keywords Monotone operator, Hilbert space, strong convergence, iterative method.

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1. Introduction

Let H be a real Hilbert space. In this paper, we study the *variational inclusion problem* (VIP) which is the problem of finding $z \in H$ such that

$$0_H \in (A+B)z \tag{1.1}$$

where $A : H \to H$ is an operator, $B : H \multimap H$ is a set-valued operator and 0_H is a zero vector in H. The set of solutions of VIP is denoted by $(A+B)^{-1}0_H$.

It is known that the variational inclusion problem is a generalization of variational inequalities, equilibrium problem, split feasibility problem, convex minimization problem and linear inverse problem (see [23, 33, 37]). Moreover, the variational inclusion problem has many applications in applied sciences, engineering, economics and medical sciences especially image and signal processing, statistical regression and machine learning (see, *e.g.* [6, 34, 39]).

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A popular method for solving the VIP is the *forward–backward algorithm* (FBA) [3,16,21,40] which is defined by the following manner: $x_1 \in H$ and

$$x_{n+1} = J_r^B(x_n - rAx_n), \quad \forall n \geq 1, \tag{1.2}$$

where $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator and $J_r^B := (I + rB)^{-1}$ is a resolvent operator of B for $r > 0$. It was shown that the sequence $\{x_n\}\}$ generated by FBA converges weakly to a solution of VIP. This method also includes, in particular, the proximal point algorithm [5,13,18,27,32] and the gradient method [4,17].

In order to obtain the strong convergence, Lopez et al. [23] introduced the following Halpern iteration for solving the VIP: $x_1 \in H$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n}^B (x_n - r_n(Ax_n + a_n) + b_n), \quad \forall n \geq 1, \tag{1.3}$$

where $u \in H$ is a given point, $\{a_n\}$ and $\{b_n\}$ are sequences in H, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. They proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a solution of VIP.

Lin and Takahashi [22] proposed the following modified FBA by using the viscosity approximation method introduced by Moudafi [29]: $x_1 \in H$ and

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n) J_{r_n}^B (x_n - r_n Ax_n), \quad \forall n \geq 1, \tag{1.4}$$

where $h : H \to H$ is a contraction, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of VIP.

In recent years, some modifications of FBA have been investigated extensively by many researchers in the several setting (see, *e.g.*, [1,8,12,14,15,19,20,30,31,35,38]).

Takahashi et al. [37] introduced the following iteration for solving the fixed point problem of a nonexpansive mapping and the variational inclusion problem:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T(\alpha_n u + (1 - \alpha_n) J_{r_n}^B (x_n - r_n Ax_n)), \quad \forall n \geq 1, \tag{1.5}$$

where $u \in H$ is a given point, T is a nonexpansive mapping, $A : H \to H$ is a monotone operator and $B : H \to H$ is a maximal monotone operator. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common solution.

On the other hand, a typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping T on a real Hilbert space H:

$$\min_{x \in F(T)} \tfrac{1}{2} \langle Gx, x \rangle - f(x), \tag{1.6}$$

where A is a linear bounded operator and f is a potential function for γh (*i.e.*, $f'(x) = \gamma h(x)$ for $x \in H$).

Using the viscosity approximation method, Marino and Xu [26] introduced the following general iterative process for a nonexpansive mapping T on H:

$$x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G) T x_n, \quad \forall n \geq 1, \tag{1.7}$$

where h is a contraction on H and $0 < \gamma < \frac{\bar{\gamma}}{\theta}$. They proved that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a unique solution of the variational inequality

$$\langle (\gamma h - G)z, x - z \rangle \leq 0, \quad \forall x \in F(T), \tag{1.8}$$

which is also the optimality condition for the minimization problem (1.6).

Very recently, Marino et al. [25] introduced the following general viscosity explicit rule in real Hilbert spaces:

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)T(t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad \forall n \geq 1, \end{cases} \tag{1.9}$$

where T is a nonexpansive mapping and h is a contraction on H. They proved that the sequence $\{x_n\}$ generated by (1.9) strongly converges to a fixed point of T, which is also the unique solution of the variational inequality (1.8).

Motivated by the works in the literature, we aim to propose a new general viscosity explicit rule for solving variational inclusion (1.1) in the framework of Hilbert spaces. We prove its strong convergence under some suitable condition on the parameters. As applications, we apply our main result to the split feasibility problem and the LASSO problem. Some numerical experiments are also given in this paper.

2. Preliminaries and lemmas

In this section, we provide some basic definitions and lemmas which will be used in our proof.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonlinear mapping. We denote $F(T)$ by the set of fixed points of T.

• A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

• A mapping $T : C \to C$ is said to be *contractive* if there exists a constant $\theta \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \theta \|x - y\|, \quad \forall x, y \in C.$$

• A mapping $G : H \to H$ is said to be *strongly positive* if there is a constant $\bar{\gamma} > 0$ such that

$$\langle Gx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Let $A : H \multimap H$ be a set-valued operator. We denote the domain of an operator A by $\mathrm{dom}(A) = \{x \in H : Ax \neq \emptyset\}$. The set of all zero points of A is denoted by $A^{-1}0_H$, *i.e.*,

$$A^{-1}0_H = \{x \in H : 0_H \in Ax\}$$

where 0_H is a zero vector of H.

• An operator A is said to be *monotone* if for each $x, y \in \mathrm{dom}(A)$,

$$\langle u - v, x - y\rangle \geq 0, \quad u \in Ax \ \text{ and } \ v \in Ay.$$

• An operator A is said to be *α-inverse strongly monotone* if for each $x, y \in \mathrm{dom}(A)$, there exists $\alpha > 0$ such that

$$\langle u - v, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad u \in Ax \ \text{ and } \ v \in Ay.$$

• A monotone operator A is said to be *maximal* if the graph of A is not property contained in the graph of any other monotone operators. It is known that a monotone operator A is maximal if and only if $\mathcal{R}(I+rA)=H$ for all $r>0$, where $\mathcal{R}(I+rA)$ is the range of $I+rA$.

In this case, we can define the *resolvent operator* of A for r by $J_r^A=(I+rA)^{-1}:H\to \mathrm{dom}(A)$. It is known that J_r^A is single-valued and nonexpansive. Moreover, $F(J_r^A)=A^{-1}0_H$ (see [36]).

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The nearest point projection of H onto C is denoted by P_C with the property

$$\|x-P_Cx\|\leq\|x-y\|$$

for all $x\in H$ and $y\in C$. Such P_C is called the *metric projection* of H onto C. It is well known that P_C satisfies

$$\langle x-P_Cx, y-P_Cx\rangle\leq 0$$

for all $x\in H$ and $y\in C$ (see [36]).

We next recall some facts which will be needed in the rest of this paper.

Lemma 2.1 ([36]). *Let H be a real Hilbert space. Then the following statements hold:*

(i) $\|x+y\|^2\leq\|x\|^2+2\langle y,x+y\rangle$ *for all* $x,y\in H$*;*

(ii) $\|tx+(1-t)y\|^2=t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$ *for* $t\in\mathbb{R}$ *and* $x,y\in H$.

Lemma 2.2 ([26]). *Assume G is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}>0$ and $0<\rho\leq\|G\|^{-1}$. Then, $\|I-\rho G\|\leq 1-\rho\bar{\gamma}$.*

Let $A:H\to H$ be an α-inverse strongly monotone and $B:H\multimap H$ be a maximal monotone operator. In what follows, we shall use the following notation:

$$T_r=J_r^B(I-rA)=(I+rB)^{-1}(I-rA),\quad r>0.$$

Lemma 2.3 ([23]). *The following statements hold:*

(i) *For* $r>0$, $F(T_r)=(A+B)^{-1}0$.

(ii) *For* $0<r\leq s$ *and* $x\in H$, $\|x-T_rx\|\leq 2\|x-T_sx\|$.

Lemma 2.4 ([23]). *Let H be a real Hilbert space. Assume that A is an α-inverse strongly monotone in H. Then, given $r>0$, we have*

$$\begin{aligned}\|T_rx-T_ry\|^2\leq{}&\|x-y\|^2-r(2\alpha-r)\|Ax-Ay\|^2\\&-\|(I-J_r^B)(I-rA)x-(I-J_r^B)(I-rA)y\|^2,\end{aligned}$$

for all $x,y\in B_r:=\{z\in H:\|z\|\leq r\}$. In particular, if $0<r<2\alpha$, then T_r is nonexpansive.

Lemma 2.5 ([26]). *Let H be a real Hilbert space. Let T be a nonexpansive mapping on H such that $F(T)\neq\emptyset$, G be a strongly positive linear bounded operator on H and h be a contraction on H with coefficient $\theta\in(0,1)$ and $0<\gamma<\bar{\gamma}/\theta$. Let $\{z_t\}$ be a net which is defined by*

$$z_t=t\gamma h(z_t)+(I-tG)Tz_t,\quad\forall t\in(0,1).$$

Then $\{z_t\}$ *converges strongly as* $t \to 0^+$ *to a point* $z \in F(T)$*, which solves the variational inequality:*

$$\langle \gamma h(z) - Gz, x - z\rangle \le 0, \ \forall x \in F(T).$$

Lemma 2.6 ([42]). *Assume that* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n)s_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$*.*

Then, $\lim_{n\to\infty} s_n = 0$*.*

Lemma 2.7 ([24]). *Let* $\{s_n\}$ *be sequences of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $s_{n_i} < s_{n_i+1}$ *for all* $i \in \mathbb{N}$*. Then there exists an increasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$s_{m_k} \le s_{m_k+1} \quad \textit{and} \quad s_k \le s_{m_k+1}.$$

In fact, $m_k := \max\{j \le k : s_j \le s_{j+1}\}$*.*

3. Main results

In this section, we introduce a new general viscosity explicit rule for solving the VIP and prove the strong convergence theorem of the proposed method in real Hilbert spaces.

Theorem 3.1. *Let* H *be a real Hilbert space. Let* $A : H \to H$ *be an* α*-inverse strongly monotone operator and* $B : H \multimap H$ *be a maximal monotone operator such that* $(A+B)^{-1}0 \neq \emptyset$*. Let* $G : H \to H$ *be a strongly positive linear bounded operator with coefficient* $\bar{\gamma} > 0$ *and* $h : H \to H$ *be a contraction with coefficient* $\theta \in (0,1)$ *such that* $0 < \gamma < \bar{\gamma}/\theta$*. Choose an initial guess* $x_1 \in H$*, let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) J_{r_n}^B (x_n - r_n A x_n), \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G) J_{r_n}^B (I - r_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}), \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where $\{r_n\} \subset \big(0, 2\alpha\big)$*, and* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$*. Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$*;*

(C3) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\alpha$*.*

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element* $z = P_{(A+B)^{-1}0}\gamma h(z)$*.*

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, without loss of generality, that $\alpha_n < \|G\|^{-1}$ for all $n \geq 1$. For each $n \geq 1$, we put $T_n := J^B_{r_n}(I - r_n A)$. Let $z \in (A+B)^{-1}0$. By the nonexpansivity of T_n, we have

$$\begin{aligned}\|\bar{x}_{n+1} - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(T_n x_n - T_n z)\|\\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|T_n x_n - T_n z\|\\ &\leq \beta_n\|x_n - z\| + (1-\beta_n)\|x_n - z\|\\ &= \|x_n - z\|.\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}&\|x_{n+1} - z\|\\ &= \|\alpha_n(\gamma h(x_n) - Gz) + (I - \alpha_n G)(T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z)\|\\ &\leq \alpha_n\|\gamma h(x_n) - Gz\| + \|I - \alpha_n G\|\|T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) - T_n z\|\\ &\leq \alpha_n\gamma\|h(x_n) - h(z)\| + \alpha_n\|\gamma h(z) - Gz\|\\ &\quad +(1-\alpha_n\bar{\gamma})\|\|t_n(z_n - z) + (1-t_n)(\bar{x}_{n+1} - z)\|\\ &\leq \alpha_n\gamma\theta\|x_n - z\| + (1-\alpha_n\bar{\gamma})(t_n\|x_n - z\| + (1-t_n)\|\bar{x}_{n+1} - z\|) + \alpha_n\|\gamma h(z) - Gz\|\\ &\leq (1 - (\bar{\gamma} - \gamma\theta)\alpha_n)\|x_n - z\| + (\bar{\gamma} - \gamma\theta)\alpha_n\frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta}\\ &\leq \max\left\{\|x_n - z\|, \frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta}\right\}.\end{aligned}$$

By induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_1 - z\|, \frac{\|\gamma h(z) - Gz\|}{\bar{\gamma} - \gamma\theta}\right\}, \quad \forall n \geq 1.$$

Hence $\{x_n\}$ is bounded.

For each $n \geq 1$, we put $z_n := t_n x_n + (1-t_n)\bar{x}_{n+1}$. By Lemma 2.4, we have

$$\begin{aligned}&\|T_n z_n - z\|^2\\ &= \|J^B_{r_n}(I - r_n A)z_n - J^B_{r_n}(I - r_n A)z\|^2\\ &\leq \|z_n - z\|^2 - r_n(2\alpha - r_n)\|Az_n - Az\|^2 - \|z_n - r_n Az_n - T_n z_n + r_n Az\|^2.\end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned}&\|z_n - z\|^2\\ &\leq t_n\|x_n - z\|^2 + (1-t_n)\|\bar{x}_{n+1} - z\|^2\\ &\leq t_n\|x_n - z\|^2 + (1-t_n)\Big[\beta_n\|x_n - z\|^2 + (1-\beta_n)\|T_n x_n - z\|^2\Big]\\ &\leq t_n\|x_n - z\|^2 + (1-t_n)\Big[\beta_n\|x_n - z\|^2 + (1-\beta_n)\Big(\|x_n - z\|^2\\ &\quad -r_n(2\alpha - r_n)\|Ax_n - Az\|^2 - \|x_n - r_n Ax_n - T_n x_n + r_n Az\|^2\Big)\Big]\\ &\leq \|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\end{aligned}$$

$$+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big). \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$\begin{aligned}
&\|T_nz_n - z\|^2\\
\le\ &\|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big)\\
&-r_n(2\alpha - r_n)\|Az_n - Az\|^2 - \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2.
\end{aligned} \tag{3.4}$$

From Lemma 2.1 (i) and (3.4), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
=\ &\|\alpha_n(\gamma h(x_n) - Gz) + (I - \alpha_nG)(T_nz_n - z)\|^2\\
\le\ &\|(I - \alpha_nG)(T_nz_n - z)\|^2 + 2\alpha_n\gamma\langle h(x_n) - h(z), x_{n+1} - z\rangle\\
&+2\alpha_n\langle\gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1 - \alpha_n\bar{\gamma})^2\|T_nz_n - z\|^2 + 2\alpha_n\gamma\theta\|x_n - z\|\|x_{n+1} - z\|\\
&+2\alpha_n\langle\gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1 - \alpha_n\bar{\gamma})^2\|T_nz_n - z\|^2 + \alpha_n\gamma\theta\big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big)\\
&+2\alpha_n\langle\gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &(1 - \alpha_n\bar{\gamma})^2\Big[\|x_n - z\|^2 - (1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - r_n(2\alpha - r_n)\|Az_n - Az\|^2\\
&-\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big] + \alpha_n\gamma\theta\big(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\big)\\
&+2\alpha_n\langle\gamma h(z) - Gz, x_{n+1} - z\rangle\\
\le\ &\big((1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma\theta\big)\|x_n - z\|^2 + \alpha_n\gamma\theta\|x_{n+1} - z\|^2\\
&-(1 - \alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - (1 - \alpha_n\bar{\gamma})^2\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2\\
&+\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) + 2\alpha_n\langle\gamma h(z) - Gz, x_{n+1} - z\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - z\|^2\\
\le\ &\frac{(1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma\theta}{1 - \alpha_n\gamma\theta}\|x_n - z\|^2\\
&-\frac{(1 - \alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)}{1 - \alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2\\
&+\|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) - \frac{(1 - \alpha_n\bar{\gamma})^2}{1 - \alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) + \frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle \\
&= \left[1 - \frac{2(\bar{\gamma} - \gamma\theta)\alpha_n}{1-\alpha_n\gamma\theta}\right]\|x_n - z\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\|x_n - z\|^2 \\
&\quad -K_n\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2 + \|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) \\
&\quad -\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\Big(r_n(2\alpha - r_n)\|Az_n - Az\|^2 + \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2\Big) \\
&\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle, \qquad (3.5)
\end{aligned}
$$

where $K_n := \frac{(1-\alpha_n\bar{\gamma})^2(1-t_n)(1-\beta_n)}{1-\alpha_n\gamma\theta}$. We note that $\liminf_{n\to\infty} K_n > 0$ and $\liminf_{n\to\infty} r_n(2\alpha - r_n) > 0$. For each $n \geq 1$, we set

$$
\begin{aligned}
s_n &:= \|x_n - z\|^2, \\
\gamma_n &:= \frac{2(\bar{\gamma} - \gamma\theta)\alpha_n}{1-\alpha_n\gamma\theta}, \\
\eta_n &:= K_n\Big(r_n(2\alpha - r_n)\|Ax_n - Az\|^2 + \|x_n - r_nAx_n - T_nx_n + r_nAz\|^2\Big) \\
&\quad + \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}(r_n(2\alpha - r_n)\|Az_n - Az\|^2 + \|z_n - r_nAz_n - T_nz_n + r_nAz\|^2), \\
\delta_n &:= \frac{2\alpha_n}{1-\alpha_n\gamma\theta}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle + \frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\theta}\|x_n - z\|^2.
\end{aligned}
$$

Then (3.5) reduces to the the following formulae:

$$
s_{n+1} \leq (1-\gamma_n)s_n - \eta_n + \delta_n, \quad \forall n \geq 1 \qquad (3.6)
$$

and

$$
s_{n+1} \leq (1-\gamma_n)s_n + \delta_n, \quad \forall n \geq 1. \qquad (3.7)
$$

We next show that $s_n \to 0$ as $n \to \infty$ by considering two possible cases:
Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{s_n\}_{n=n_0}^\infty$ is non-increasing. This implies that $\{s_n\}_{n=1}^\infty$ is convergent. From (3.6), we have

$$
0 \leq \eta_n \leq s_n - s_{n+1} + \delta_n - \gamma_n s_n.
$$

Since $\lim_{n\to\infty}\gamma_n = \lim_{n\to\infty}\delta_n = 0$, which implies that $\lim_{n\to\infty}\eta_n = 0$. Then, we obtain

$$
\lim_{n\to\infty}\|Az_n - Az\| = \lim_{n\to\infty}\|z_n - r_nAz_n - T_nz_n + r_nAz\| = 0
$$

and

$$
\lim_{n\to\infty}\|Ax_n - Az\| = \lim_{n\to\infty}\|x_n - r_nAx_n - T_nx_n + r_nAz\| = 0.
$$

Consequently,

$$
\lim_{n\to\infty}\|T_nz_n - z_n\| = 0 \text{ and } \lim_{n\to\infty}\|T_nx_n - x_n\| = 0. \qquad (3.8)
$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. Then, by Lemma 2.3 (ii), we have

$$\|T_r x_n - x_n\| \leq 2\|T_n x_n - x_n\|.$$

From (3.8), we obtain

$$\lim_{n\to\infty} \|T_r x_n - x_n\| = 0. \tag{3.9}$$

Let $z_t = t\gamma h(z_t) + (I - tG)T_r z_t,\ \forall t \in (0,1)$. Then it follows from Lemma 2.5 that $\{z_t\}$ converges strongly to a fixed point $z \in F(T_r)$. So, we obtain

$$\begin{aligned}
&\|z_t - x_n\|^2 \\
&= \|t(\gamma h(z_t) - Gx_n) + (I - tG)(T_r z_t - x_n)\|^2 \\
&\leq (1 - t\bar{\gamma})^2\|T_r z_t - x_n\|^2 + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&= (1 - t\bar{\gamma})^2\|T_r z_t - T_r x_n + T_r x_n - x_n\|^2 + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&\leq (1 - t\bar{\gamma})^2\Bigg(\|T_r z_t - T_r x_n\|^2 + 2\langle T_r x_n - x_n, T_r z_t - x_n\rangle\Bigg) \\
&\quad + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&\leq (1 - t\bar{\gamma})^2\Bigg(\|z_t - x_n\|^2 + 2\|T_r x_n - x_n\|\|T_r z_t - x_n\|\Bigg) \\
&\quad + 2t\langle \gamma h(z_t) - Gx_n, z_t - x_n\rangle \\
&= (1 - 2t\bar{\gamma} + (\bar{\gamma}t)^2)\|z_t - x_n\|^2 + 2t\langle \gamma h(z_t) - Gz_t, z_t - x_n\rangle \\
&\quad + 2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t),
\end{aligned} \tag{3.10}$$

where

$$f_n(t) = 2(1 - t\bar{\gamma})^2\|T_r z_t - x_n\|\|T_r x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Since G is strongly positive linear, we have

$$\langle Gz_t - Gx_n, z_t - x_n\rangle = \langle G(z_t - x_n), z_t - x_n\rangle \geq \bar{\gamma}\|z_t - x_n\|^2. \tag{3.12}$$

It follows (3.10) and (3.12) that

$$\begin{aligned}
&2t\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\leq (\bar{\gamma}^2 t^2 - 2t\bar{\gamma})\|z_t - x_n\|^2 + 2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t) \\
&\leq (\bar{\gamma}t^2 - 2t)\langle Gz_t - Gx_n, z_t - x_n\rangle + 2t\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t) \\
&= \bar{\gamma}t^2\langle Gz_t - Gx_n, z_t - x_n\rangle + f_n(t),
\end{aligned} \tag{3.13}$$

which implies that

$$\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \leq \frac{\bar{\gamma}t}{2}\langle Gz_t - Gx_n, z_t - x_n\rangle + \frac{1}{2t} f_n(t). \tag{3.14}$$

Taking limit $n \to \infty$ in (3.14) and noting (3.11), we have

$$\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \leq \frac{t}{2}M, \tag{3.15}$$

where $M > 0$ is large enough. Taking limit $t \to 0$ in (3.15), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \le 0. \tag{3.16}$$

Since

$$\begin{aligned}
&\langle \gamma h(z) - Gz, x_n - z\rangle \\
&= \langle \gamma h(z) - Gz, x_n - z\rangle - \langle \gamma h(z) - Gz, x_n - z_t\rangle + \langle \gamma h(z) - Gz, x_n - z_t\rangle \\
&\quad -\langle \gamma h(z) - Gz_t, x_n - z_t\rangle + \langle \gamma h(z) - Gz_t, x_n - z_t\rangle - \langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\quad +\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle \\
&\le \|\gamma h(z) - Gz\|\|z_t - z\| + (\|G\| + \gamma\theta)\|z_t - z\| \lim_{n\to\infty}\|x_n - z_t\| \\
&\quad + \limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle.
\end{aligned}$$

Then, from (3.16), we obtain that

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_n - z\rangle \\
&\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(z_t) - Gz_t, x_n - z_t\rangle \\
&\le 0.
\end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned}
\|T_n z_n - x_n\| &\le \|T_n z_n - z_n\| + \|z_n - x_n\| \\
&\le \|T_n z_n - z_n\| + (1 - t_n)(1 - \beta_n)\|T_n x_n - x_n\| \\
&\le \|T_n z_n - z_n\| + \|T_n x_n - x_n\|.
\end{aligned}$$

This together with (3.8) implies that

$$\lim_{n\to\infty}\|T_n z_n - x_n\| = 0. \tag{3.18}$$

Further, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - T_n z_n\| + \|T_n z_n - x_n\| \\
&\le \alpha_n\|h(x_n) - T_n z_n\| + \|T_n z_n - x_n\|.
\end{aligned}$$

This together with (3.18) implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.19}$$

Combining (3.17) and (3.19), we get that

$$\limsup_{n\to\infty}\langle \gamma h(z) - Gz, x_{n+1} - z\rangle \le 0. \tag{3.20}$$

Due to (3.7), we see that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0$. Utilizing Lemma 2.6, we can conclude that $\lim_{n\to\infty} s_n = 0$. Therefore $x_n \to z$ as $n \to \infty$.

Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $s_{n_i} \leq s_{n_i+1}$ for all $i \in \mathbb{N}$. By Lemma 2.7, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and

$$s_{m_k} \leq s_{m_k+1} \quad \text{and} \quad s_k \leq s_{m_k+1} \tag{3.21}$$

for all $k \in \mathbb{N}$. So, we have

$$0 \leq \eta_{m_k} \leq s_{m_k} - s_{m_k+1} + \delta_{m_k} - \gamma_{m_k} s_{m_k} \to 0 \quad \text{and} \quad k \to \infty.$$

This implies that

$$\lim_{k\to\infty} \|T_{m_k} z_{m_k} - z_{m_k}\| = 0 \quad \text{as} \quad \lim_{k\to\infty} \|T_{m_k} x_{m_k} - x_{m_k}\| = 0. \tag{3.22}$$

Following the proof line in **Case 1**, we can show that

$$\lim_{k\to\infty} \|T_r x_{m_k} - x_{m_k}\| = 0$$

and

$$\limsup_{k\to\infty} \langle \gamma h(z) - Gz, x_{m_k} - z\rangle \leq 0.$$

Since

$$\begin{aligned}\|T_{m_k} z_{m_k} - x_{m_k}\| &\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + \|z_{m_k} - x_{m_k}\| \\ &\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + (1 - s_{m_k})(1 - \beta_{m_k})\|T_{m_k} x_{m_k} - x_{m_k}\| \\ &\leq \|T_{m_k} z_{m_k} - z_{m_k}\| + \|T_{m_k} x_{m_k} - x_{m_k}\|.\end{aligned}$$

This implies by (3.18) that

$$\lim_{k\to\infty} \|T_{m_k} z_{m_k} - x_{m_k}\| = 0.$$

Note that

$$\begin{aligned}\|x_{m_k+1} - x_{m_k}\| &\leq \|x_{m_k+1} - T_{m_k} z_{m_k}\| + \|T_{m_k} z_{m_k} - x_{m_k}\| \\ &\leq \alpha_{m_k}\|h(x_{m_k}) - T_{m_k} z_{m_k}\| + \|T_{m_k} z_{m_k} - x_{m_k}\|.\end{aligned}$$

Hence, we have

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0 \tag{3.23}$$

and hence

$$\limsup_{k\to\infty} \langle \gamma h(z) - Gz, x_{m_k+1} - z\rangle \leq 0. \tag{3.24}$$

From (3.6), we have

$$s_{m_k+1} \leq (1 - \gamma_{m_k}) s_{m_k} + \delta_{m_k}. \tag{3.25}$$

This implies that

$$\gamma_{m_k} s_{m_k} \leq s_{m_k} - s_{m_k+1} + \delta_{m_k}.$$

Since $s_{m_k} \leq s_{m_k+1}$ and $\alpha_{m_k} > 0$ then $\lim_{k\to\infty} s_{m_k} = 0$. By the fact that $a^2 - b^2 \leq 2a(a-b)$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |s_{m_k+1} - s_{m_k}| &= \|x_{m_k+1} - z\|^2 - \|x_{m_k} - z\|^2 \\ &\leq 2\|x_{m_k} - z\|(\|x_{m_k+1} - z\| - \|x_{m_k} - z\|) \\ &\leq 2\|x_{m_k} - z\|\|x_{m_k+1} - x_{m_k}\|. \end{aligned}$$

This implies by (3.23) that

$$\lim_{k\to\infty}(s_{m_k+1} - s_{m_k}) = 0.$$

So, we have

$$s_k \leq s_{m_k+1} = s_{m_k} + (s_{m_k+1} - s_{m_k}) \to 0 \ \text{ as } \ k \to \infty,$$

which implies that $\lim_{k\to\infty} s_k = 0$ and so $x_k \to z$ as $k \to \infty$. This completes the proof. □

Next, we also study the following general viscosity explicit rule (3.1) with the error sequence.

Theorem 3.2. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \multimap H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $G : H \to H$ be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and $h : H \to H$ be a contraction with coefficient $\theta \in (0,1)$ such that $0 < \gamma < \bar{\gamma}/\theta$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)J_{r_n}^B(x_n - r_n A x_n), \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)J_{r_n}^B(I - r_n A)(t_n x_n + (1-t_n)\bar{x}_{n+1}) + e_n, \ \forall n \geq 1, \end{cases} \tag{3.26}$$

where $\{e_n\} \subset H$, $\{r_n\} \subset \big(0, 2\alpha\big)$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{t_n\}$ are sequences in $(0,1)$. Suppose that the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$;

(C4) $\sum_{n=1}^{\infty} \|e_n\| < \infty$ *or* $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element $z = P_{(A+B)^{-1}0}\gamma h(z)$.

Proof. For arbitrary initial guess $y_1 \in H$, we define a sequence $\{y_n\}$ as follows:

$$\begin{cases} \bar{y}_{n+1} = \beta_n y_n + (1-\beta_n)T_n y_n, \\ y_{n+1} = \alpha_n \gamma h(y_n) + (I - \alpha_n G)T_n(t_n y_n + (1-t_n)\bar{y}_{n+1}), \quad \forall n \geq 1, \end{cases}$$

where $T_n = J_{r_n}^B(I - r_n A)$. By Theorem 3.1, we know that $\{y_n\}$ converges strongly to $z = P_{(A+B)^{-1}0}\gamma h(z)$. We next show that $x_n \to z$ as $n \to \infty$. By the nonexpansiveness of T_n, we have

$$\begin{aligned} \|\bar{x}_{n+1} - \bar{y}_{n+1}\| &\leq \beta_n\|x_n - y_n\| + (1-\beta_n)\|T_n x_n - T_n y_n\| \\ &\leq \|x_n - y_n\|. \end{aligned}$$

It follows that

$$
\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&= \|\alpha_n \gamma(h(x_n) - h(y_n)) + (I - \alpha_n G)(T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) \\
&\quad - T_n(t_n y_n + (1-t_n)\bar{y}_{n+1})) + e_n\| \\
&\le \alpha_n \gamma \|h(x_n) - h(y_n)\| + (1 - \alpha_n \bar{\gamma})\|T_n(t_n x_n + (1-t_n)\bar{x}_{n+1}) \\
&\quad - T_n(t_n y_n + (1-t_n)\bar{y}_{n+1})\| + \|e_n\| \\
&\le \alpha_n \gamma \|h(x_n) - h(y_n)\| + (1-\alpha_n\bar{\gamma})\Big(t_n\|x_n - y_n\| + (1-t_n)\|\bar{x}_{n+1} - \bar{y}_{n+1}\|\Big) + \|e_n\| \\
&\le \alpha_n \gamma\theta \|x_n - y_n\| + (1 - \alpha_n\bar{\gamma})\|x_n - y_n\| + \|e_n\| \\
&= (1 - (\bar{\gamma} - \theta\gamma)\alpha_n)\|x_n - y_n\| + \|e_n\|.
\end{aligned}
$$

From $(C4)$ and Lemma 2.6, we obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Then, we conclude that $x_n \to z$. This completes the proof. □

We remark some merits of our work as follows:

(1) The method of proof in Theorem 3.1 is very different from the proof in Theorem 3.1 of Marino et al. [25] because Algorithm (3.1) deals with the problem of finding an element of $(A+B)^{-1}0$ which involves the resolvent of maximal monotone operator.

(2) The result presented in Theorem 3.1 is proved under new assumptions on $\{\beta_n\}$ and $\{t_n\}$.

(3) The result presented in Theorem 3.1 is applicable for solving the split feasibility problem and the LASSO problem (see, Section 4).

4. Some Applications

In this section, we utilize our main result to the split feasibility problem and the LASSO problem.

4.1. The split feasibility problem

Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a linear bounded operator with its adjoint T^*. The *split feasibility problem* (SFP) is to find

$$\hat{x} \in C \quad \text{such that} \quad T\hat{x} \in Q. \tag{4.1}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor-Elfving [9] in 1994 for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction (see [7]). The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (see, *e.g.*, [10, 11, 28, 41, 45, 46]).

For solving the SFP (4.7), Byrne [7] introduce the so-called *CQ-iterative algorithm* for approximating a solution of SFP, which is defined by

$$x_{n+1} = P_C(x_n - \lambda A^*(I - P_Q)Ax_n), \quad \forall n \ge 1, \tag{4.2}$$

where $0 < \lambda < 2\alpha$ with $\alpha = 1/\|A\|^2$. Here, $\|A\|^2$ is the spectral radius of A^*A. It was shown that the sequence $\{x_n\}$ converges weakly to a solution of the SFP.

It is known that $\hat{x}$ solves the SFP (4.1) if and only if $\hat{x}$ is the solution of the following minimization problem [43]:

$$\min_{x \in C} f(x),$$

where f is the proximity function defined by $f(x) := \frac{1}{2}\|(I - P_Q)Tx\|^2$ with its gradient $\nabla f = T^*(I - P_Q)T$. Further, if $\nabla f = T^*(I - P_Q)T$ is $\|T\|^2$-Lipschitz continuous, then ∇f is $1/\|T\|^2$-inverse strongly monotone, where $\|T\|^2$ is the spectral radius of T^*T (see [6]). In fact, set $A = \nabla f$ and $B = \partial i_C$ in Theorem 3.1, where i_C is the indicator function (see [37]). So we obtain the following result.

Theorem 4.1. *Suppose that the SFP* (4.1) *is consistent. For an initial guess* $x_1 \in H_1$*, let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)P_C(x_n - r_n T^*(I - P_Q)Tx_n), \\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)P_C(I - r_n T^*(I - P_Q)T)(t_n x_n + (1 - t_n)\bar{x}_{n+1}), \end{cases} \tag{4.3}$$

$\forall n \geq 1$,*where* $\{r_n\} \subset (0, \frac{2}{\|T\|^2})$, $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$ *which satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) $\liminf_{n\to\infty}(1 - t_n)(1 - \beta_n) > 0$*;*

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{\|T\|^2}$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a solution of the SFP.*

4.2. The LASSO Problem

The LASSO problem is abbreviation for the least absolute shrinkage and selection operator, which formulated as the minimization problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Tx - b\|_2^2 \quad \text{subject to} \quad \|x\|_1 \leq \lambda, \tag{4.4}$$

where $T \in \mathbb{R}^{m \times n}$ is a given matrix, $b \in \mathbb{R}^m$ is a given vector and $\lambda \geq 0$ is a tuning parameter. The lasso was introduced by Tibshirani [39] in 1996. It has been received much attention due to the involvement of the l_1 norm which promotes sparsity, phenomenon of many practical problems arising from image and signal processing, statistics model, machine learning, and so on. It is known that an equivalent formulation of (4.4) is the following regularized minimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) + g(x), \tag{4.5}$$

where $f(x) := \frac{1}{2}\|Tx - b\|_2^2$, $g(x) := \lambda\|x\|_1$ and $\lambda \geq 0$. We know that $\nabla f(x) = T^*(Tx - b)$ is $\|T^*T\|$-Lipshitz continuous. This implies that ∇f is $1/\|T^*T\|$-inverse strongly monotone. The proximal of $g(x) = \lambda\|x\|_1$ is given by

$$\text{prox}_g(x) = \text{argmin}_u \lambda\|x\|_1 + \frac{1}{2}\|u - x\|_2^2,$$

which is separable in indices. Then, for $x \in \mathbb{R}^n$,

$$\begin{aligned}\operatorname{prox}_g(x) &= \operatorname{prox}_{\lambda\|\cdot\|_1}(x)\\ &= \Big(\operatorname{prox}_{\lambda|\cdot|_1}(x_1), \operatorname{prox}_{\lambda|\cdot|_1}(x_2), ..., \operatorname{prox}_{\lambda|\cdot|_1}(x_n)\Big)\\ &= (\alpha_1, \alpha_2, ..., \alpha_n),\end{aligned}$$

where $\alpha_k = \operatorname{sgn}(x_k)\max\{|x_k| - \lambda, 0\}$ for $k = 1, 2, ..., n$.

For solving the LASSO problem, Xu [44] (see also [2]) proposed the following proximal-gradient algorithm (PGA):

$$x_{n+1} = \operatorname{prox}_{r_n g}(x_n - r_n T^*(Tx_n - b)). \tag{4.6}$$

He proved that the PGA (4.6) converges weakly to a solution of the LASSO problem (4.4).

In what follows, we present a general viscosity explicit rule for approximating solutions of the LASSO problem in infinite dimensional Hilbert spaces. Set $A = \nabla f$ and $B = \operatorname{prox}_{r_n g}$ in Theorem 3.1, we obtain the following result.

Theorem 4.2. *Suppose that the problem* (4.4) *is consistent. For an initial guess* $x_1 \in H$, *let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by*

$$\begin{cases} \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)\operatorname{prox}_{r_n g}(x_n - r_n T^*(Tx_n - b)),\\ x_{n+1} = \alpha_n \gamma h(x_n) + (I - \alpha_n G)\operatorname{prox}_{r_n g}((t_n x_n + (1-t_n)\bar{x}_{n+1})\\ -r_n T^*(T(t_n x_n + (1-t_n)\bar{x}_{n+1}) - b)), \quad \forall n \geq 1, \end{cases} \tag{4.7}$$

where $\{r_n\} \subset (0, \frac{2}{\|T^*T\|^2})$ *and* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,1)$. *Suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty}(1-t_n)(1-\beta_n) > 0$;

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < \frac{2}{\|T^*T\|^2}$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a solution of the problem* (4.4).

5. Numerical Example

We next give some numerical experiments of a general viscosity explicit rule (3.1).

Example 5.1. Let $H = \mathbb{R}^3$ with the norm $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$. Consider the mapping $G : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $G\mathbf{x} = 4\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. It is easy to see that G is a linear bounded operator on $\mathbb{R}^3$ with $\bar{\gamma} = 4$. Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $h(\mathbf{x}) = 0.1\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. It is easy to see that h is a contraction on $\mathbb{R}^3$ with $\theta = 0.1$. Then, we can choose $\gamma = 10$. For any $\mathbf{x} \in \mathbb{R}^3$, let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A\mathbf{x} = 3\mathbf{x} - (1, -2, 5)^t$ and $B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $B\mathbf{x} = 2\mathbf{x}$. We see that A is a 1/3-inverse strongly monotone and B is a maximal monotone operator. Moreover, we have for $r > 0$

$$J_r^B(\mathbf{x} - rA\mathbf{x}) = (I + rB)^{-1}(\mathbf{x} - rA\mathbf{x})$$

$$= \frac{1-3r}{1+2r}\mathbf{x} + \frac{r}{1+2r}(1,-2,5)^t,$$

for all $\mathbf{x} \in \mathbb{R}^3$. Since $\alpha = 1/3$, we can choose $r_n = 0.5$ for all $n \in \mathbb{N}$. Let $\alpha_n = \frac{1}{1000n+1}$, $\beta_n = \frac{n}{2n+3}$ and $t_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Starting $\mathbf{x}_1 = (10000, -40000, 50000)^t$ and use $\|x_{n+1} - x_n\|_2 < 10^{-6}$, for stopping criterion. Then, we obtain the following numerical results.

Table 1. Numerical results of Example 5.1 for iteration process (3.1).

Time taken	No. of iterations	$\mathbf{x}_n = (x_1^n, x_2^n, x_3^n)^t$	$\|x_{n+1} - x_n\|_2$
0.110385	2	$(-2479.77097902, 9919.58191808, -12398.85489510)^t$	8.08E+04
	3	$(203.36993275, -813.08835944, 1016.84966373)^t$	1.73E+04
	4	$(-12.88336871, 51.93363061, -64.41684356)^t$	1.41E+03
	5	$(1.02739736, -3.70989923, 5.13698682)^t$	9.01E+01
	6	$(0.14598968, -0.18417851, 0.72994841)^t$	5.71E+00
	7	$(0.20354553, -0.41436724, 1.01772764)^t$	3.72E-01
	8	$(0.19966701, -0.39882655, 0.99833507)^t$	2.51E-02
	10	$(0.19993710, -0.39987152, 0.99968548)^t$	8.69E-05
	20	$(0.19997096, -0.39994192, 0.99985479)^t$	8.88E-06
	$\vdots$	$\vdots$	$\vdots$
	50	$(0.19998878, -0.39997755, 0.99994388)^t$	1.28E-06
	$\vdots$	$\vdots$	$\vdots$
	55	$(0.19998982, -0.39997964, 0.99994909)^t$	1.05E-06
	56	$(0.19999000, -0.39998001, 0.99995002)^t$	1.02E-06
	57	$(0.19999018, -0.39998037, 0.99995091)^t$	9.8E-07

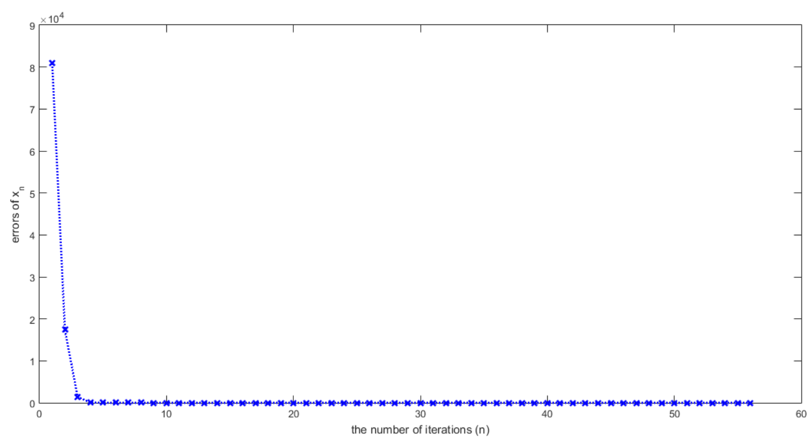

Figure 1. The error plotting of $\|x_{n+1} - x_n\|_2$ in Table 1.

6. Conclusions

In this work, we have introduced new iterative methods for solving the inclusion problem for the sum of two monotone operators in Hilbert spaces. Strong convergence was discussed under suitable conditions. Some applications to the split feasibility problem and the LASSO problem are also given. Preliminary numerical experiments are provided to support our proposed methods.

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