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PEAKON AND CUSPON SOLUTIONS OF A GENERALIZED CAMASSA-HOLM-NOVIKOV EQUATION*

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Abstract The bounded traveling wave solutions of a generalized Camassa-Holm-Novikov equation with $p = 2$ and $p = 3$ are derived via the dynamical system approach. The singular wave solutions including peakons and cuspons are obtained by the bifurcation analysis of the corresponding singular dynamical system and the orbits intersecting with or approaching the singular lines. The results show that the generalized Camassa-Holm-Novikov equation with $p = 2$ and $p = 3$ both admit smooth solitary wave, smooth periodic wave solutions, solitary peakons, periodic peakons, solitary cuspons and periodic cuspons as well. It is worth pointing out that the Novikov equation has no bounded traveling wave solutions with negative wave speed, but has a family of new periodic cuspons which are distinguished with the normal periodic cuspons for their discontinuous first-order derivatives at both maximum and minimum.

Keywords Generalized Camassa-Holm-Novikov equation, dynamical system approach, bifurcation, singular wave solutions.

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1. Introduction

A nonlinear dispersive equation arising in the modeling of the propagation of shallow water waves is given by

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which is usually known as Camassa-Holm equation, named after Camassa and Holm who proposed this equation in 1993 [3], where $u(t, x)$ represents the height of the water's free surface above a flat bottom. Actually, this equation was firstly derived in [11] as a bi-Hamiltonian system and it has been shown that this equation is

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completely integrable and thus has an infinite number of conservation laws. In 1998, Dai [7] re-derived this equation when modelling nonlinear waves in cylindrical hyperelastic rods, where $u(t,x)$ represents the radial stretch relative to a pre-stressed state.

Camassa-Holm equation is also well-known for its possession of the singular wave solution

$$u(x,t) = ce^{-|x-ct|}, \tag{1.2}$$

which was named as peakon since it has a discontinuity in the first derivative at its peak. The orbital stability of the peakons of Camassa-Holm equation in the H^1 norm has been proved by Constantin and Strauss [4], which implies that these wave patterns can be physically recognized. The finite propagation speed and compatly supported solutions of (1.1) was studied in [5,13]. Recently, other weakly dissipative wave equations describing propagation of surface waves in shallow water regimes, such as Dullin-Gottwald-Holm equation, b-family equations, or integrable modified Camassa-Holm equation, including (1.1) as a special case or relating to (1.1), have attracted a lot of attention. The blow-up of solutions, wave breaking, integrability and orbital stability of the traveling wave solutions of some nonlinear wave equations have been investigated [8, 9, 12, 14–17, 22, 24, 26–30]. The bounded traveling wave solutions of Camassa-Holm equation and its generalized forms, including smooth solitary wave solutions and periodic wave solutions, solitary peakons and periodic peakons, have been investigated by studying the bifurcation of its traveling wave system [18, 19, 31].

In this paper, we study a one-parameter family of equations known as the generalized Camassa-Holm-Novikov (gCHN) equation [1]

$$m_t + (p+1)u^{p-1}u_x m + u^p m_x = 0, \quad m = u - u_{xx}, \tag{1.3}$$

which unifies the Camassa-Holm equation ($p = 1$) and Novikov equation ($p = 2$) [23]. A conservation law

$$D_t T + D_x X = 0 \tag{1.4}$$

holds for all solutions of (1.3) [2], where $T = \frac{1}{2}(u^2 + u_x^2)$ and $X = (u - u_{xx})u^{p+1} - uu_{tx}$. This generalized Camassa-Holm-Novikov (gCHN) equation is further generalized into a very general family of peakon equations [25].

A new variable $\xi = x - ct$ is introduced where c is the wave speed. Let $u(x,t) = \phi(\xi)$, then after integrating once with respect to the new variable ξ, (1.4) reduces to the nonlinear second-order ordinary differential equation

$$2\phi(c - \phi^p)\phi^{''} - c(\phi')^2 + \phi^2(2\phi^p - c) = g, \tag{1.5}$$

where $'$ denotes the derivative with respect to ξ and g is an arbitrary constant of integration. Clearly, if $\phi(\xi)$ is a solution of (1.5) for any given constants c and g, then $u(x,t) = \phi(x - ct)$ will be a traveling wave solution of (1.3) with wave speed c. Furthermore, we know that if $\phi(\xi)$ is an analytic periodic solution of (1.5) for arbitrary constant g and c, then $\phi(x - ct)$ will be a smooth periodic traveling wave solution of (1.3); if $\phi(\xi)$ is an analytic solution of (1.5) satisfying $\lim_{x\to+\infty} \phi(\xi) = \lim_{x\to-\infty} \phi(\xi)$, then $\phi(x - ct)$ will be a smooth solitary wave solution of (1.3); if $\phi(\xi)$ is an analytic solution of (1.5) satisfying $\lim_{x\to+\infty} \phi(\xi) \neq \lim_{x\to-\infty} \phi(\xi)$, then $\phi(x - ct)$ will be a smooth kink or anti-kink wave solution of (1.3).

Let $\phi^{'} = y$, then equation (1.5) is equivalent to the dynamical system

$$\begin{cases} \phi' = y, \\ y^{'} = \frac{cy^2 - \phi^2(2\phi^p - c) + g}{2\phi(c - \phi^p)}, \end{cases} \tag{1.6}$$

provided that $\phi(c - \phi^p) \neq 0$. Note that system (1.6) has the singular line $\phi = 0$ and may have other singular lines $\phi = \phi_s$, if the equation $\phi^p - c = 0$ has the real root ϕ_s.

We observe that the nonlinear second-order differential equation (1.5) and the singular planar dynamical system (1.6) involve two arbitrary parameters c and g. Consequently, we explore the bifurcation sets and phase portraits of (1.6) with $p = 2$ and $p = 3$ in each bifurcation set to obtain the bounded traveling wave solutions of (1.3) in Section 2. Based on the analysis and results obtained in Section 2, we investigate the bounded smooth traveling wave solutions of (1.3) through the integral along the bounded orbits of (1.6) which are not disjoint with the singular lines, as well as the effects of the singular lines of system (1.6) to derive the cuspons and peakons of (1.3) in Section 3. A conclusion and further discussion is presented in the last section.

2. Bifurcation sets and phase portraits of the traveling wave system of (1.3)

By the rescaling $d\xi = 2\phi(c - \phi^p)d\eta$, system (1.6) transforms to the analytic system

$$\begin{cases} \dot{\phi} = 2y\phi(c - \phi^p), \\ \dot{y} = cy^2 - \phi^2(2\phi^p - c) + g, \end{cases} \tag{2.1}$$

where " $\cdot$ " denotes the derivative with respect to η. System (2.1) has the first integral

$$I(\phi, y) = \frac{1}{\phi}|c - \phi^p|^{\frac{1}{p}}(y^2 + \frac{g}{c} - \phi^2) \tag{2.2}$$

when p is an even integer and

$$I(\phi, y) = \frac{1}{\phi}(c - \phi^p)^{\frac{1}{p}}(y^2 + \frac{g}{c} - \phi^2) \tag{2.3}$$

when p is an odd integer.

Clearly, systems (1.6) and (2.1) have the same phase portraits except for the singular straight lines. Thus we study the equilibrium points of system (2.1) firstly to obtain their phase portraits and bifurcations. Let

$$F = 2y\phi(c - \phi^p) \tag{2.4}$$

and

$$G = cy^2 - \phi^2(2\phi^p - c) + g. \tag{2.5}$$

Suppose that (ϕ_0, y_0) is an equilibrium point of system (2.1), that is, (ϕ_0, y_0) is a solution of $F = 0$ and $G = 0$. To get the local qualitative properties of the equilibrium point (ϕ_0, y_0), we now study the eigenvalues of the coefficient matrix of the linearized system of (2.1) at this point, i.e.,

$$M(\phi_0, y_0) = \begin{vmatrix} F_\phi & F_y \\ G_\phi & G_y \end{vmatrix}_{(\phi_0, y_0)}. \tag{2.6}$$

By the theory of planar dynamical system [6], we know that if $M(\phi_0, y_0)$ has two real eigenvalues with opposite signs then the equilibrium point (ϕ_0, y_0) is a saddle point; if $M(\phi_0, y_0)$ has two real eigenvalues with same signs, then (ϕ_0, y_0) is a node; if $M(\phi_0, y_0)$ has two complex eigenvalues, then (ϕ_0, y_0) is either a center or a focus. In our case if $M(\phi_0, y_0)$ has two complex eigenvalues, then (ϕ_0, y_0) must be a center since we know that (2.1) has a first integral.

2.1. Bifurcation sets and phase portraits of (2.1) with $p = 2$

Firstly, we study the bifurcation sets and phase portraits of (2.1) with $p = 2$. After careful computations, we obtain the bifurcation curves as

$$\begin{aligned} &C_1: \ g = -\frac{c^2}{8} \quad (c > 0), \\ &C_2: \ g = c^2 \quad (c > 0), \\ &C_3: \ g = 0, \\ &C_4: \ c = 0. \end{aligned}$$

These bifurcation curves partition the parametric plane (c, g) into the following six regions:

$$\begin{aligned} &D_1 = \{(c, g)|\ c > 0,\ -\frac{c^2}{8} < g < 0\}, \\ &D_2 = \{(c, g)|\ c > 0,\ 0 < g < c^2\}, \\ &D_3 = \{(c, g)|\ c > 0,\ g > c^2\}, \\ &D_4 = \{(c, g)|\ c < 0,\ g > 0\}, \\ &D_5 = \{(c, g)|\ c < 0,\ g < 0\}, \\ &D_6 = \{(c, g)|\ c > 0,\ g < -\frac{c^2}{8}\}. \end{aligned}$$

We now study the equilibrium points of system (2.1) in every region and on every bifurcation curve and state the results in the following lemma.

Lemma 2.1. *For system (2.1) with $p = 2$, the following conclusions hold.*

(1) For $(c, g) \in D_1$, system (2.1) has ten equilibrium points $(\pm\frac{1}{2}\sqrt{c \pm \sqrt{c^2+8g}}, 0)$, $(0, \pm\sqrt{-\frac{g}{c}})$ and $(\pm\sqrt{c}, \pm\sqrt{c - \frac{g}{c}})$. Here $(0, \pm\sqrt{-\frac{g}{c}})$ are nodes; $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}}, 0)$ are centers; $(\pm\frac{1}{2}\sqrt{c-\sqrt{c^2+8g}}, 0)$ and $(\pm\sqrt{c}, \pm\sqrt{c-\frac{g}{c}})$ are saddle points.

(2) For $(c, g) \in D_2$, system (2.1) has six equilibrium points $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}}, 0)$ and $(\pm\sqrt{c}, \pm\sqrt{c-\frac{g}{c}})$. Here $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}}, 0)$ are centers; $(\pm\sqrt{c}, \pm\sqrt{c-\frac{g}{c}})$ are saddle points.

(3) For $(c,g) \in D_3$*, system (2.1) has two saddle points* $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}},\ 0)$.

(4) For $(c,g) \in D_4$*, system (2.1) has four equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$ *and* $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}},\ 0)$*. Here* $(0,\pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(\pm\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}},\ 0)$ *are saddle points.*

(5) For $(c,g) \in D_5$*, system (2.1) has no equilibrium point.*

(6) For $(c,g) \in D_6$*, system (2.1) has six equilibrium points* $(0,\ \pm\sqrt{-\frac{g}{c}})$ *and* $(\pm\sqrt{c}, \pm\sqrt{c-\frac{g}{c}})$*. Here* $(0,\pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(\pm\sqrt{c},\pm\sqrt{c-\frac{g}{c}})$ *are saddle points.*

(7) For $(c,g)\in C_1$*, system (2.1) has eight equilibrium points* $(0,\pm\sqrt{\frac{c}{8}})$, $(\pm\frac{1}{2}\sqrt{c},0)$ *and* $(\pm\sqrt{c},\pm 3\sqrt{\frac{c}{8}})$*. Here* $(0,\pm\sqrt{\frac{c}{8}})$ *are nodes;* $(\pm\frac{1}{2}\sqrt{c},0)$ *are cusps;* $(\pm\sqrt{c},\pm 3\sqrt{\frac{c}{8}})$ *are saddle points.*

(8) For $(c,g) \in C_2$*, system (2.1) has two cusps* $(\pm\sqrt{c},0)$.

(9) For $(c,g) \in C_3$ *and* $c > 0$*, system (2.1) has seven equilibrium points* $(0,0)$, $(\pm\sqrt{\frac{c}{2}},0)$ *and* $(\pm\sqrt{c},\pm\sqrt{c})$*. Here* $(0,0)$ *is a cusp;* $(\pm\sqrt{\frac{c}{2}},0)$ *are centers;* $(\pm\sqrt{c},\pm\sqrt{c})$ *are saddle points. For* $(c,g) \in C_3$ *and* $c < 0$*, system (2.1) has only a cusp* $(0,0)$.

The phase portraits of system (2.1) with $p = 2$ are obtained by using Lemma 2.1 and the first integral (2.2), $I(\phi,y) = h$ for different values of h, which are shown in Figure 1. Clearly, $I(\phi,y) = 0$ determines two straight lines $\phi = \pm\sqrt{c}$ for $c > 0$ and two hyperbolic curves $y^2 = \phi^2 - \frac{g}{c}$.

We now investigate the values of h which determine the bounded orbits in ϕ direction. Let

$$h_0 = I(\phi_0, 0) = \frac{1}{\phi_0}\sqrt{|c-\phi_0^2|}(\frac{g}{c} - \phi_0^2), \tag{2.7}$$

$e_1 = \frac{1}{2}\sqrt{c-\sqrt{c^2+8g}}$ and $e_2 = \frac{1}{2}\sqrt{c+\sqrt{c^2+8g}}$, then h_0 is an odd function of ϕ_0. Next, we study the orbits of (2.1) which are determined by $I(\phi,y) = h$ for different values of h.

Case (1) $(c,g) \in D_1$

We notice that h_0 is increasing from 0 to $I(-e_2,0)$ as ϕ_0 varies from $-\sqrt{c}$ to $-e_2$, decreasing to $I(-e_1,0)$ as ϕ_0 continuously varies to $-e_1$ and then increasing to $+\infty$ as ϕ_0 continuously increases and approaches 0. Similarly, h_0 is increasing from $-\infty$ to $I(e_1,0)$ as ϕ_0 varies from 0 to e_1, decreasing to $I(e_2,0)$ as ϕ_0 continuously increases to e_2 and then increasing to 0 as ϕ_0 continuously increases to $\sqrt{c}$. Let $\phi_1 = \sqrt{\frac{c\sqrt{c^2+8g}+c^2+4g}{2c}}$, then $I(\pm\phi_1,0) = I(\pm e_1,0)$. We consider three subcases.

Case (1.1) For $\phi_0 \in (-\phi_1,-e_2) \cup (e_2,\phi_1)$, $I(\phi,y) = h_0$ determines a smooth periodic orbit passing through $(\phi_0,0)$.

Case (1.2) For $h_0 = I(\pm e_1,0)$, $I(\phi,y) = h_0$ determines a homoclinic orbit passing through $(\pm\phi_1,0)$ and tending to $(\pm e_1,0)$, two heteroclinic orbits connecting $(\pm e_1,0)$ and $(0,\pm\sqrt{-\frac{g}{c}})$ respectively and two branches starting from $(0,\pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = \pm\sqrt{c}$ which are unbounded in y direction.

Case (1.3) For $\phi_0 \in (-\sqrt{c},-\phi_1) \cup (-e_1,0) \cup (0,e_1) \cup (\phi_1,\sqrt{c})$, $I(\phi,y) = h_0$ determines a curve passing through $(\phi_0,0)$ and $(0,\pm\sqrt{-\frac{g}{c}})$ and tending to $\phi = \pm\sqrt{c}$ which is unbounded in y direction.

Case (2) $(c,g) \in D_2$

The function h_0 is increasing from 0 to $I(-e_2,0)$ as ϕ_0 varies from $-\sqrt{c}$ to $-e_2$ and then decreasing to $-\infty$ as ϕ_0 continuously increases and approaches 0. Similarly, h_0 is decreasing from $+\infty$ to $I(e_2,0)$ as ϕ_0 varies from 0 to e_2, increasing to $I(e_2,0)$ as ϕ_0 continuously varies to $\sqrt{c}$.

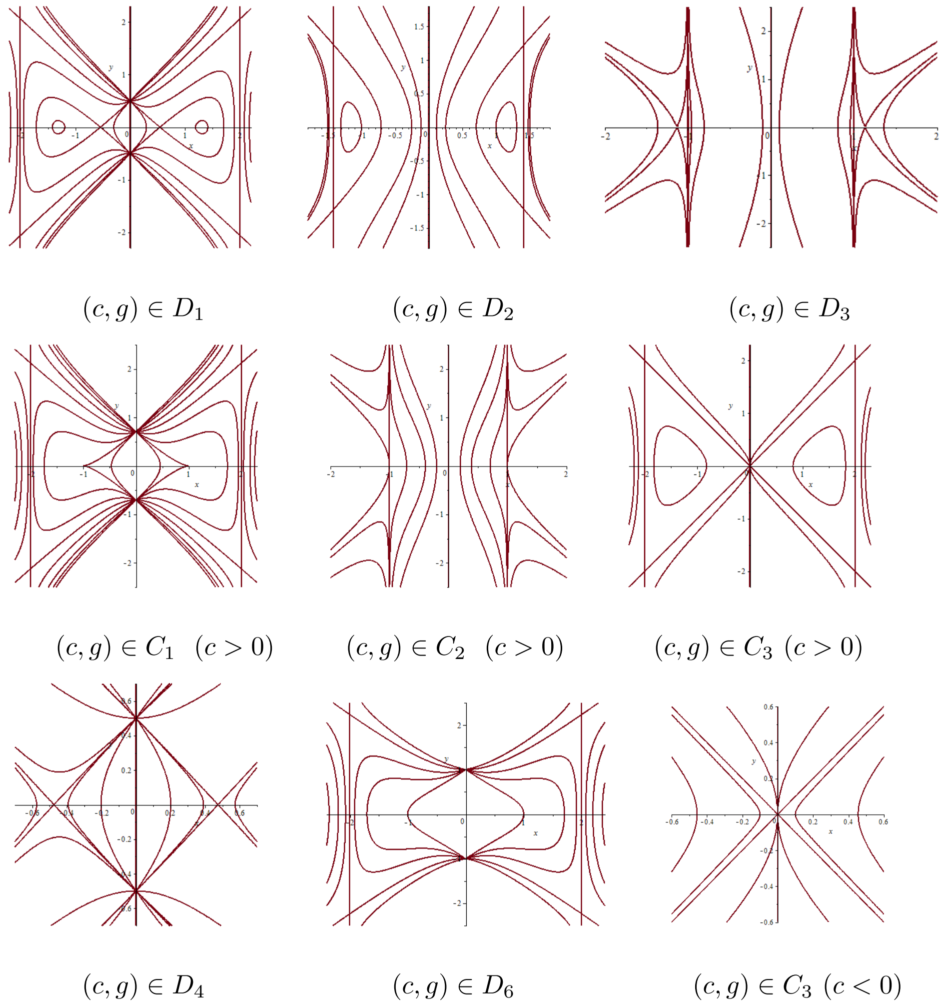


Figure 1. The phase portraits of system (2.1) with $p = 2$.

Case (2.1) For $\phi_0 \in (-e_2, -\sqrt{\frac{g}{c}}) \cup (\sqrt{\frac{g}{c}}, e_2)$, $I(\phi, y) = h_0$ determines a smooth periodic orbit passing through $(\phi_0, 0)$.

Case (2.2) For $\phi_0 = \pm\sqrt{\frac{g}{c}}$, $I(\phi, y) = 0$ determines two hyperbolic curves intersecting with the two singular lines $\phi = \pm\sqrt{c}$ at $(\pm\sqrt{c}, \pm\sqrt{c - \frac{g}{c}})$ which are separatrices of two families of periodic orbits.

Case (2.3) For $\phi_0 \in (-\sqrt{\frac{g}{c}}, 0) \cup (0, \sqrt{\frac{g}{c}})$, $I(\phi, y) = h_0$ determines a curve passing through $(\phi_0, 0)$ and approaching $\phi = \pm\sqrt{c}$, which is unbounded in y direction.

Case (3) $(c, g) \in D_3$

The function h_0 is increasing from $I(-e_2, 0)$ to 0 as ϕ_0 varies from $-e_2$ to $-\sqrt{c}$ and then decreasing to $-\infty$ as ϕ_0 continuously increases and approaches 0. Similarly, h_0 is decreasing from $+\infty$ to 0 and increasing to $I(e_2, 0)$ as ϕ_0 varies from 0 to $-\sqrt{c}$ and then increases to e_2.

Case (3.1) For $\phi_0 \in (-e_2, e_2) \backslash \{\pm\sqrt{c}, 0\}$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $\phi = \pm\sqrt{c}$, which is unbounded in y direction.

Case (3.2) For $h_0 = I(\pm e_2, 0)$, there are two orbits tending to $(\pm e_2, 0)$ and approaching $\phi = \pm\sqrt{c}$, which are unbounded in y direction.

Case (4) $(c, g) \in C_1$ with $c > 0$

The function h_0 is increasing from 0 to $+\infty$ as ϕ_0 varies from $-\sqrt{c}$ to 0; h_0 is increasing from $-\infty$ to 0 as ϕ_0 varies from 0 to $\sqrt{c}$.

Case (4.1) For $\phi_0 \in (-\sqrt{c}, \sqrt{c}) \setminus \{\pm\frac{1}{2}\sqrt{c}, 0\}$, $I(\phi, y) = h_0$ determines a curve passing through $(\phi_0, 0)$ and tending to $(0, \pm\sqrt{\frac{c}{8}})$, and two branches starting from $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = \pm\sqrt{c}$, which are unbounded in y direction.

Case (4.2) For $\phi_0 = I(\pm\frac{1}{2}\sqrt{c})$, $I(\phi, y) = h_0$ determines two heteroclinic orbits connecting the two equilibrium points $(\pm\frac{1}{2}\sqrt{c}, 0)$ and $(0, \pm\sqrt{\frac{c}{8}})$, and two branches starting from $(0, \pm\sqrt{\frac{c}{8}})$ and approaching $\phi = \pm\sqrt{c}$.

Case (5) $(c, g) \in C_2$ with $c > 0$

The function h_0 is decreasing from 0 to $-\infty$ as ϕ_0 continuously increases from $-\sqrt{c}$ to 0, and it is decreasing from $+\infty$ to 0 as ϕ_0 varies from 0 to $\sqrt{c}$. For $\phi_0 \in (-\sqrt{c}, \sqrt{c}) \setminus \{0\}$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $\phi = \pm\sqrt{c}$, which is unbounded in y direction.

Case (6) $(c, g) \in C_3$ with $c > 0$

The function h_0 is increasing from 0 to $I(-\sqrt{\frac{c}{2}}, 0)$ as ϕ_0 varies from $-\sqrt{c}$ to $-\sqrt{\frac{c}{2}}$, decreasing to $I(\sqrt{\frac{c}{2}}, 0)$ as ϕ_0 continuously varies to $\sqrt{\frac{c}{2}}$ and then increasing to 0 as ϕ_0 continuously increases to $\sqrt{c}$. For $\phi_0 \in (-\sqrt{c}, -\sqrt{\frac{c}{2}}) \cup (\sqrt{\frac{c}{2}}, \sqrt{c})$, $I(\phi, y) = h_0$ determines a smooth periodic orbit passing through $(\phi_0, 0)$.

Case (7) $(c, g) \in D_4$

The function h_0 is increasing from $I(-e_2, 0)$ to $+\infty$ as ϕ_0 varies from $-e_2$ to 0 and increasing from $-\infty$ to $I(e_2, 0)$ as ϕ_0 varies from 0 to e_2. For $\phi_0 \in (-e_2, 0) \cup (0, e_2)$, $I(\phi, y) = h_0$ determines a heteroclinic orbit connecting $(0, \pm\sqrt{\frac{-g}{c}})$, and two branches starting from $(0, \pm\sqrt{-\frac{c}{8}})$, which are unbounded in both ϕ and y directions.

Case (8) $(c, g) \in D_6$

The function h_0 is increasing from 0 to $+\infty$ as ϕ_0 varies from $-\sqrt{c}$ to 0 and increasing from $-\infty$ to 0 as ϕ_0 varies from 0 to $\sqrt{c}$. For $\phi_0 \in (-\sqrt{c}, 0) \cup (0, \sqrt{c})$, $I(\phi, y) = h_0$ determines a heteroclinic orbit connecting $(0, \pm\sqrt{-\frac{g}{c}})$, and two branches starting from $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = \pm\sqrt{c}$, which are unbounded in y direction.

Here we ignore to study the case when $c < 0$ in view of the symmetry (refer to Lemma 3.1).

2.2. Bifurcation sets and phase portraits of (2.1) with $p = 3$

By the theory of planar dynamical system and careful computations, we get the bifurcation curves of system (2.1) with $p = 3$ as follows:

$$
\begin{aligned}
C_1 &: \ g = -3\left(\frac{c}{5}\right)^{\frac{5}{3}}, \\
C_2 &: \ g = c^{\frac{5}{3}}, \\
C_3 &: \ g = 0, \\
C_4 &: \ c = 0.
\end{aligned}
$$

These bifurcation curves separate the parametric plane (c, g) into eight regions:

$$D_1 = \{(c, g) |\ 0 < g < c^{\frac{5}{3}}\},$$

$$
\begin{aligned}
D_2 &= \{(c,g)|\ g > c^{\frac{5}{3}} > 0\},\\
D_3 &= \{(c,g)|\ g < -3(\frac{c}{5})^{\frac{5}{3}} < 0\},\\
D_4 &= \{(c,g)|\ -3\left(\frac{c}{5}\right)^{\frac{5}{3}} < g < 0\},\\
D_5 &= \{(c,g)|\ c^{\frac{5}{3}} < g < 0\},\\
D_6 &= \{(c,g)|\ g < c^{\frac{5}{3}} < 0\},\\
D_7 &= \{(c,g)|\ g > -3\left(\frac{c}{5}\right)^{\frac{5}{3}} > 0\},\\
D_8 &= \{(c,g)|\ 0 < g < -3\left(\frac{c}{5}\right)^{\frac{5}{3}}\}.
\end{aligned}
$$

Lemma 2.2. *For system (2.1) with* $p = 3$*, the following conclusions hold.*

(1) For $(c,g) \in D_1$*, equation* $\phi^2(2\phi^3 - c) - g = 0$ *has only one real root, say* ϕ_{e_1}*, and thus system (2.1) has three equilibrium points* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$*.* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *are saddle points and* $(\phi_{e_1}, 0)$ *is a center.*

(2) For $(c,g) \in D_2$*, system (2.1) has only one saddle point* $(\phi_{e_1}, 0)$*.*

(3) For $(c,g) \in D_3$*, system (2.1) has five equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $(\phi_{e_1}, 0)$ *and* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$*. Here* $(0, \pm\sqrt{-\frac{g}{c}})$ *are nodes and* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$ *are saddle points.*

(4) For $(c,g) \in D_4$*, equation* $\phi^2(2\phi^3 - c) - g = 0$ *has three real roots* $(\phi_{e_i}, 0), i = 1,2,3, (\phi_{e_1} < \phi_{e_2} < \phi_{e_3})$*. Thus system (2.1) has seven equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_i}, 0), i = 1,2,3$*. Here* $(0, \pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$*,* $(\phi_{e_1}, 0)$ *and* $(\phi_{e_2}, 0)$ *are saddle points;* $(\phi_{e_3}, 0)$ *is a center.*

(5) For $(c,g) \in D_5$*, system (2.1) has three equilibrium points* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$*. Here* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *are saddle points and* $(\phi_{e_1}, 0)$ *is a center.*

(6) For $(c,g) \in D_6$*, system (2.1) has only a saddle point* $(\phi_{e_1}, 0)$*.*

(7) For $(c,g) \in D_7$*, system (2.1) has five equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $(\phi_{e_1}, 0)$ *and* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$*. Here* $(0, \pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$ *are saddle points.*

(8) For $(c,g) \in D_8$*, system (2.1) has seven equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $(\phi_{e_i}, 0)$*,* $i = 1,2,3$ *and* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$*. Here* $(0, \pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$*,* $(\phi_{e_1}, 0)$ *and* $(\phi_{e_2}, 0)$ *are saddle points;* $(\phi_{e_3}, 0)$ *is a center.*

(9) For $(c,g) \in C_1$*, system (2.1) has six equilibrium points* $(0, \pm\sqrt{-\frac{g}{c}})$*,* $((\frac{c}{5})^{\frac{1}{3}}, 0)$*,* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$*, where* ϕ_{e_1} *is the unique real root of equation* $\phi^3 + 2(\frac{c}{5})^{\frac{1}{3}}\phi^2 + 3(\frac{c}{5})^{\frac{2}{3}}\phi + \frac{3}{10}c = 0$*. Here* $(0, \pm\sqrt{-\frac{g}{c}})$ *are nodes;* $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$ *and* $(\phi_{e_1}, 0)$ *are saddle points;* $((\frac{c}{5})^{\frac{1}{3}}, 0)$ *is a cusp.*

(10) For $(c,g) \in C_2$*, system (2.1) has a cusp* $(c^{\frac{1}{3}}, 0)$*.*

(11) For $(c,g) \in C_3$*, system (2.1) has four equilibrium points* $(0,0)$*,* $(c^{\frac{1}{3}}, \pm c^{\frac{1}{3}})$ *and* $((\frac{c}{2})^{\frac{1}{3}}, 0)$*. Here* $(0,0)$ *is a cusp;* $(c^{\frac{1}{3}}, \pm c^{\frac{1}{3}})$ *are saddle;* $((\frac{c}{2})^{\frac{1}{3}}, 0)$ *is a center.*

(12) For $(c,g) \in C_4$*, system (2.1) has only a saddle* $((\frac{g}{2})^{\frac{1}{5}}, 0)$*.*

Similarly as in subsection 2.1, the phase portraits of system (2.1) with $p = 3$ in each bifurcation region are obtained by Lemma 2.2 and the first integral (2.3), $I(\phi, y) = h$ for different values of h, which are shown in Figure 2. Obviously, $I(\phi, y) = 0$ determines one straight line $\phi = c^{\frac{1}{3}}$ and two hyperbolic curves $y^2 = \phi^2 - \frac{g}{c}$. In order to find the bounded traveling wave solutions of (1.3), we next investigate the values of h which determines the orbits which are bounded in ϕ direction.

Case (1) $(c, g) \in D_1$

The function h_0 is decreasing from $+\infty$ to $I(\phi_{e_1}, 0)$ and increasing to 0 as ϕ_0 varies from 0 to ϕ_{e_1} and then to $c^{\frac{1}{3}}$ continuously.

Case (1.1) For $\phi_0 \in (\sqrt{\frac{g}{c}}, \phi_{e_1})$, $I(\phi, y) = h_0$ determines a periodic orbit passing through $(\phi_0, 0)$.

Case (1.2) For $\phi_0 = \sqrt{\frac{g}{c}}$, $I(\phi, y) = 0$ determines an orbit passing through $(\phi_0, 0)$ and intersecting with $\phi = c^{\frac{1}{3}}$ at $(c^{\frac{1}{3}}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}})$.

Case (1.3) For $\phi_0 \in (0, \sqrt{\frac{g}{c}})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (2) $(c, g) \in D_2$

The function h_0 is decreasing from $+\infty$ to $I(\phi_{e_1}, 0)$ as ϕ_0 varies from 0 to ϕ_{e_1} continuously.

Case (2.1) For $\phi_0 \in (0, c^{\frac{1}{3}}) \cup (c^{\frac{1}{3}}, \phi_{e_1})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (2.2) For $h_0 = I(\phi_{e_1}, 0)$, there are two orbits tending to $(\phi_{e_1}, 0)$ and approaching $\phi = c^{\frac{1}{3}}$, which are unbounded in y direction.

Case (3) $(c, g) \in D_3$

The function h_0 is increasing from $I(\phi_{e_1}, 0)$ to $+\infty$ as ϕ_0 varies from ϕ_{e_1} to 0 continuously. Also h_0 is increasing from $-\infty$ to 0 as ϕ_0 varies from 0 to $c^{\frac{1}{3}}$ continuously.

Case (3.1) For $\phi_0 \in (0, c^{\frac{1}{3}})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $-\infty$ in ϕ direction.

Case (3.2) For $\phi_0 \in (\phi_{e_1}, 0)$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ and then approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (3.3) For $h_0 = I(\phi_{e_1}, 0)$, there are two orbits tending to $(\phi_{e_1}, 0)$ and passing through $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = c^{\frac{1}{3}}$, which are unbounded in y direction.

Case (4) $(c, g) \in C_1$ with $c > 0$

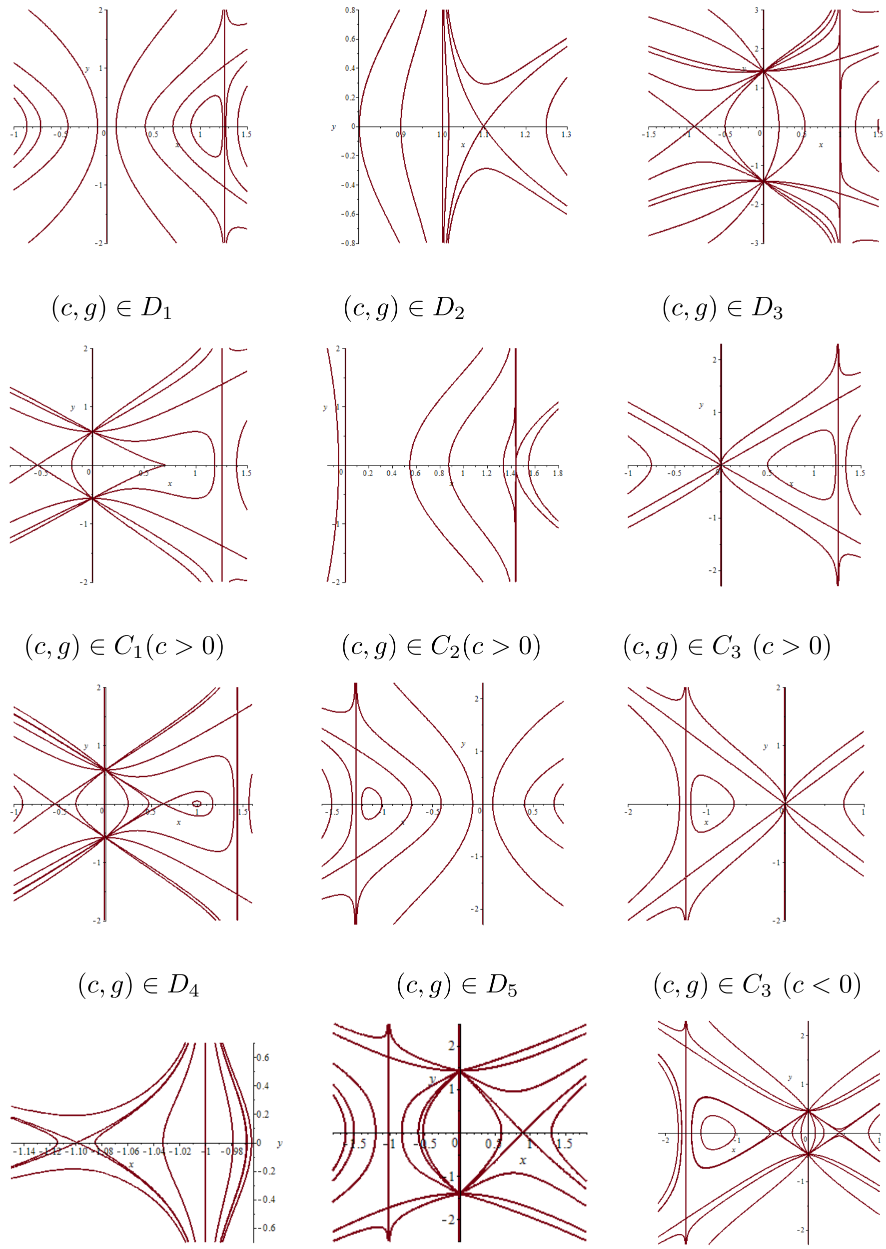
The function h_0 is increasing from $I(\phi_{e_1}, 0)$ to $+\infty$ as ϕ_0 varies from ϕ_{e_1} to 0 continuously. Also h_0 is increasing from $-\infty$ to 0 as ϕ_0 varies from 0 to $c^{\frac{1}{3}}$ continuously.

Case (4.1) For $\phi_0 \in (\phi_{e_1}, 0)$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (4.2) For $h_0 = I(\phi_{e_1}, 0)$, there are two orbits tending to $(\phi_{e_1}, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ respectively and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (4.3) For $\phi_0 \in (0, c^{\frac{1}{3}})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $-\infty$ in ϕ direction.

Case (5) $(c, g) \in C_2$ with $c > 0$



$(c, g) \in D_1$ $(c, g) \in D_2$ $(c, g) \in D_3$

$(c, g) \in C_1 (c > 0)$ $(c, g) \in C_2 (c > 0)$ $(c, g) \in C_3\ (c > 0)$

$(c, g) \in D_4$ $(c, g) \in D_5$ $(c, g) \in C_3\ (c < 0)$

$(c, g) \in D_6$ $(c, g) \in D_7$ $(c, g) \in D_8$

Figure 2. The phase portraits of system (2.1) with $p = 3$.

The function h_0 is decreasing from $+\infty$ to 0 as ϕ_0 varies from 0 to $c^{\frac{1}{3}}$ continuously. For $\phi_0 \in (0, c^{\frac{1}{3}})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (6) $(c, g) \in C_3$ with $c > 0$

The function h_0 is decreasing from 0 to $I((\frac{c}{2})^{\frac{1}{3}}, 0)$ and then increasing to 0 as ϕ_0 varies from 0 to $(\frac{c}{2})^{\frac{1}{3}}$ and to $c^{\frac{1}{3}}$ continuously. For $\phi_0 \in (0, (\frac{c}{2})^{\frac{1}{3}})$, $I(\phi, y) = h_0$ determines a periodic orbit passing through $(\phi_0, 0)$. The boundary of the family of periodic orbits consists of the straight lines $y = \pm\phi$ and $\phi = c^{\frac{1}{3}}$.

Case (7) $(c, g) \in D_4$

The function h_0 is increasing from $I(\phi_{e_1}, 0)$ to $+\infty$ as ϕ_0 varies from ϕ_{e_1} to 0 continuously. Also h_0 is increasing from $-\infty$ to $I(\phi_{e_2}, 0)$, decreasing to $I(\phi_{e_3}, 0)$ and then increasing to 0 as ϕ_0 varies from 0 to ϕ_{e_2}, to ϕ_{e_3} and then to $c^{\frac{1}{3}}$ continuously.

Case (7.1) For $\phi_0 \in (\phi_{e_1}, 0)$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $\phi = c^{\frac{1}{3}}$, which is unbounded in y direction.

Case (7.2) For $h_0 = I(\phi_{e_1}, 0)$, $I(\phi, y) = h_0$ determines two orbit tending to the saddle points $(\phi_{e_1}, 0)$ and passing through $(0, \pm\sqrt{-\frac{g}{c}})$ respectively and approaching $\phi = c^{\frac{1}{3}}$, which are unbounded in y direction.

Case (7.3) For $\phi_0 \in (0, \phi_{e_2})$, $I(\phi, y) = h_0$ determines an orbit passing through $(\phi_0, 0)$ and approaching $-\infty$ in ϕ direction.

Case (7.4) For $\phi_0 \in (\phi_{e_2}, \phi_{e_3})$, $I(\phi, y) = h_0$ determines a periodic orbit passing through $(\phi_0, 0)$.

Case (7.5) For $h_0 = I(\phi_{e_2}, 0)$, there exists unique $\phi_1 > \phi_{e_3}$ such that $h_0 = I(\phi_1, 0)$. Equation $I(\phi, y) = h_0$ determines a homoclinic orbit which is the boundary of a family of periodic orbits. It also determines two orbits passing through $(0, \pm\sqrt{-\frac{g}{c}})$ respectively and approaching $-\infty$ in ϕ direction.

Case (7.6) For $\phi_0 \in (\phi_1, c^{\frac{1}{3}})$ and $h_0 = I(\phi_{e_2}, 0)$, $I(\phi, y) = h_0$ determines an orbit passing through $(0, \pm\sqrt{-\frac{g}{c}})$ and approaching $-\infty$ in ϕ direction.

Here we ignore to study the case when $c < 0$ in consideration of the symmetry (refer to Lemma 3.1).

3. Bounded smooth traveling wave solutions of (1.3)

From the analysis in Section 2, we recall that the reduced equation (1.5) of the gCHN equation (1.3) is equivalent to system (1.6) and thus equivalent to system (2.1) by rescaling $d\xi = 2\phi(c - \phi^p)d\eta$ if $\phi(c - \phi^p) \neq 0$. Suppose that ϕ_{e_1} is a zero of $c - \phi^p = 0$, then $\phi = 0$ and $\phi = \phi_{e_1}$ are singularities of system (1.6). Clearly, it has three singular lines $\phi = 0$ and $\phi = \pm c^{1/p}$ when $c > 0$ and p is an even integer, and two singular lines $\phi = 0$ and $\phi = c^{1/p}$ when p is an odd integer. It is known that the orbits which have no intersection with the straight lines correspond to smooth traveling wave solutions.

It is easy to check that the solutions of (1.3) have the following symmetry.

Lemma 3.1. *Suppose that $\phi = \phi(\xi)$ is a solution of (1.3) for given c and g, then so is $\phi = \phi(-\xi)$. Furthermore, if $\phi = \phi(\xi)$ is a solution of (1.3) with $p = 2$ for given c and g, then $\phi = -\phi(\xi)$ is also a solution of (1.3); if $\phi = \phi(\xi)$ is a solution of (1.3) with $p = 3$ for given c and g, then $\phi = -\phi(\xi)$ is a solution of (1.3) for $-c$ and $-g$.*

Consequently, for equation (1.5) with $p=2$, we only need to consider the initial value $\phi(0)=\phi_0$ with $\phi_0 \leq 0$ or $\phi_0>0$, and when $p=3$ we only consider $c>0$. We firstly derive the smooth traveling wave solutions of gCHN equation (1.3) in terms of the analysis presented in Section 2.

3.1. Bounded smooth traveling wave solutions of (1.3) with $p=2$

For system (2.1) with $p=2$, from the first integral (2.2) we know that along each orbit when $|\phi| \neq \sqrt{c}$ and $\phi \neq 0$,

$$\frac{1}{\phi}\sqrt{|c-\phi^2|}(\phi'^2+\frac{g}{c}-\phi^2)=h, \tag{3.1}$$

which implies that these orbits are determined by the ODE

$$\phi'^2=\frac{h\phi}{\sqrt{|c-\phi^2|}}+\phi^2-\frac{g}{c}. \tag{3.2}$$

Let

$$H_1(\phi;\phi_0,c,g,h)=\int_{\phi_0}^{\phi}\frac{d\phi}{\sqrt{h\phi|c-\phi^2|^{-\frac{1}{2}}+\phi^2-\frac{g}{c}}}, \tag{3.3}$$

then $H_1(\phi;\phi_0,c,g,h)=\pm\xi$ determines implicitly the solution of (1.3) for some values of c, g and h.

From the analysis of Subsection 2.1, one knows that only when c, g and ϕ_0 satisfy Case (1.1), Case (1.2), Case (2.1) or Case (6), the orbit determined by (3.3) satisfies $|\phi|<\sqrt{c}$. Consequently, we have the following conclusion on the smooth traveling wave solutions of equation (1.3) with $p=2$.

(1) For arbitrary $(c,g)\in D_1$, $H_1(\phi;\phi_0,c,g,h_0)=\pm\xi$ with $\phi_0\in(-\phi_1,-e_2)\cup(e_2,\phi_1)$ determines implicitly a smooth periodic traveling wave solution of (1.3); $H_1(\phi;\phi_0,c,g,h_0)=\pm\xi$ with $\phi_0=\pm\phi_1$ determines implicitly a solitary wave solution of (1.3), respectively.

(2) For arbitrary $(c,g)\in D_2$, $H_1(\phi;\phi_0,c,g,h_0)=\pm\xi$ with $\phi_0\in(-e_2,-\sqrt{\frac{g}{c}})\cup(\sqrt{\frac{g}{c}},e_2)$, determines a smooth periodic traveling wave solution.

(3) For arbitrary $c>0$, $H_1(\phi;\phi_0,c,0,h_0)=\pm\xi$ with $\phi_0\in(-\sqrt{c},-\sqrt{\frac{c}{2}})\cup(\sqrt{\frac{c}{2}},\sqrt{c})$ determines a smooth periodic traveling wave solution.

3.2. Bounded smooth traveling wave solutions of (1.3) with $p=3$

For system (2.1) with $p=3$, from the first integral (2.2) we get that along each orbit

$$\frac{1}{\phi}(c-\phi^3)^{\frac{1}{3}}(\phi'^2+\frac{g}{c}-\phi^2)=h, \tag{3.4}$$

which implies that these orbits are determined by the ODE

$$\phi'^2=h\phi(c-\phi^3)^{-\frac{1}{3}}+\phi^2-\frac{g}{c}. \tag{3.5}$$

Let

$$H_2(\phi;\phi_0,c,g,h)=\int_{\phi_0}^{\phi}\frac{d\phi}{\sqrt{h\phi(c-\phi^3)^{-\frac{1}{3}}+\phi^2-\frac{g}{c}}},\tag{3.6}$$

then $H_2(\phi;\phi_0,c,g,h)=\pm\xi$ determines implicitly the solution of (1.3) for some values of c,g and h_0.

From the analysis of Subsection 2.2, we see that only when c,g and ϕ_0 satisfy Case (1.1), Case (6), Case (7.4) or Case (7.5), the orbit determined by (3.6) are bounded and does not intersect with the singular lines, that is it corresponds to the smooth bounded traveling waves of (1.3). Consequently, for the smooth traveling wave solutions of equation (1.3) with $p=3$ we have the following conclusion.

(1) For arbitrary $(c,g)\in D_1$, $H_2(\phi;\phi_0,c,g,h_0)=\pm\xi$ with $\phi_0\in(\sqrt{\frac{g}{c}},\phi_{e_1})$ gives a periodic traveling wave solution. By the symmetry (see Lemma 3.1), one can conclude that $H_2(-\phi;-\phi_0,-c,-g,h_0)=\pm\xi$ with $\phi_0\in(\sqrt{\frac{g}{c}},\phi_{e_1})$, $h=I(\phi_0,y)$ gives another periodic traveling wave solution.

(2) For arbitrary $(c,g)\in C_3$ and $c>0$, $H_2(\phi;\phi_0,c,g,h_0)=\pm\xi$ and $H_2(-\phi;-\phi_0,-c,-g,h_0)=\pm\xi$ with $\phi_0\in(0,(\frac{c}{2})^{\frac{1}{3}})$ and $h=I(\phi_0,y)$ give two periodic traveling wave solutions.

(3) For arbitrary $(c,g)\in D_4$, $H_2(\phi;\phi_0,c,g,h_0)=\pm\xi$ and $H_2(-\phi;-\phi_0,-c,-g,h_0)=\pm\xi$ with $\phi_0\in(\phi_{e_2},\phi_{e_3})$ give two smooth periodic traveling wave solutions.

(4) For arbitrary $(c,g)\in D_4$, $H_2(\phi;\phi_0,c,g,h)=\pm\xi$ and $H_2(-\phi;-\phi_0,-c,-g,h_0)=\pm\xi$ with $\phi_0=\phi_1$ give two solitary wave solutions.

4. Effects of the singular lines on wave solutions of (1.3)

Recall that the reduced equation (1.5) of the gCHN equation (1.3) is equivalent to system (1.6) when $\phi(c-\phi^p)\neq 0$. Thus we have to investigate what happens to the solutions which intersect with or tend to the straight lines, i.e. $\phi=0$ or $\phi=\phi_s$, where ϕ_s is the root of $c-\phi^p=0$. The effects of the singular lines of the singular dynamical systems and the blow-up solutions of nonlinear wave equations have been studied by some researchers [18–20, 31, 32]. In this section, the effects of the singular lines of system (1.6) are studied and then singular wave solutions are derived.

For any $\phi\in C^2(\mathbb{R})$ and $\phi(c-\phi^p)\neq 0$, it admits that

$$\begin{aligned}&\frac{d}{d\xi}\left(\frac{1}{\phi}(c-\phi^p)^{\frac{1}{p}}(\phi'^2+\frac{g}{c}-\phi^2)\right)\\=&\frac{\phi'}{\phi^2(c-\phi^p)^{1-\frac{1}{p}}}\left(2\phi(c-\phi^p)\phi''-c(\phi')^2+\phi^2(2\phi^p-c)-g\right).\end{aligned}\tag{4.1}$$

Consequently, any classical solution of (1.3) having $\phi(c-\phi^p)\neq 0$ naturally satisfies

$$\frac{1}{\phi}(c-\phi^p)^{\frac{1}{p}}(\phi'^2+\frac{g}{c}-\phi^2)=h\tag{4.2}$$

for some constant $h\in\mathbb{R}$.

Lemma 4.1. *For arbitrary real number h and an open interval $I \subset \mathbb{R}$, if $\phi(\xi) \in C^1(I)$ is defined by*

$$\phi'^2 = \frac{h\phi}{(c-\phi^p)^{\frac{1}{p}}} + \phi^2 - \frac{g}{c} \tag{4.3}$$

and $\phi(\xi)$ is not identically equal to a constant on any subinterval of I, then $\phi(\xi)$ solves (1.5).

Proof. Differentiating (4.3) once with respect to ξ gives that

$$\phi'' = \phi + sgn(c-\phi^p)\ \frac{hc}{2(c-\phi^p)^{\frac{p+1}{p}}}\ , \tag{4.4}$$

since $\phi' \not\equiv 0$. It is easy to check that substituting (4.3) and (4.4) into (1.5) makes (1.5) an identity, which implies that $\phi(\xi)$ defined by (4.3) solves (1.5). $\square$

Remark 4.1. Note that equation (4.3) is well defined for any ϕ that $c - \phi^p \neq 0$. Since here we just study the case when $c > 0$, it is well defined for $\phi = 0$, that is singularity disappears in equation (4.3) for $\phi = 0$. Consequently the singular line $\phi = 0$ of system (2.1) does not affect the smoothness of the solutions of (1.5), in another word, the solutions of (1.5) corresponding to the orbits of (2.1) which tend to $(0, \sqrt{-\frac{g}{c}})$ pass through this points smoothly for arbitrary $g \neq 0$.

Next, we will consider the problem how to define or extend these solutions of (1.3) which approach the singular lines $\phi = \pm\sqrt{c}$ for the case when $p = 2$ and the singular line $\phi = c^{\frac{1}{3}}$ for the case when $p = 3$, respectively. Firstly, we give the following definition of weak singular traveling wave solution of (1.3).

Definition 4.1. We say that $\phi(\xi) = \phi(x - ct)$ is a singular traveling wave solution of (1.3), if $\phi(\xi) \in H^1_{loc}(\mathbb{R})$ and there exist some constant h and a countable set $\{t_k | t_k \in \mathbb{R}, k \in \mathbb{Z}, -\infty \leq t_k \leq t_{k+1} \leq \infty\}$ such that

(i) $\phi(\xi)$ is a classical solution of (4.3) on each open interval (t_k, t_{k+1});

(ii) As $\xi \to t_k$, $I(\phi, \phi') \to h$, i.e.

$$\lim_{\xi \to t_k} I(\phi, \phi') = \frac{1}{\phi}|c - \phi^p|^{\frac{1}{p}}(\phi'^2 + \frac{g}{c} - \phi^2) = h. \tag{4.5}$$

Obviously, the classical solutions of (1.5) with $\phi(c - \phi^p) \neq 0$ satisfies the above definition naturally. There are two possible cases that this definition extends the solution set of (1.5).

Case (1) $\phi'(\xi) \to \pm A$ $(A \neq 0)$ and $\phi(\xi) \to \phi_s$ as $\xi \to t_{k^\pm}$;

Case (2) when $\phi'(\xi) \to \infty$ and $\phi(\xi) \to \phi_s$ as $\xi \to t_k$;

For Case (1) and Case (2), we know that $\phi'(t_k)$ fails to exist but (4.5) may hold and thus ϕ might not be a classical traveling wave solution of (1.3) but a singular traveling wave solutions of (1.3). These well-know singular traveling wave solutions, such as peakon, cuspon and peaked or cusped periodic wave solutions, are right the traveling wave solutions having some points at which the derivatives satisfy Case (1) or Case (2). Note that the singularity appears if $\phi(\xi) \to \phi_s$, that is, the corresponding orbit tends to or approaches the singular line $\phi = \phi_s$.

Suppose that $\phi = p_i(\xi; \phi_0, c, g, h)$ determined implicitly by $H_i(\phi; \phi_0, c, g, h) = \xi$, where $i = 1, 2$. In what follows, we will investigate the singular traveling wave solutions of (1.3) with $p = 2$ or $p = 3$ in terms of Definition 4.1 and the orbits tending to or approaching the singular line $\phi = \phi_s$ in the phase portraits derived in Section 2.

4.1. Singular traveling wave solutions of (1.1) with $p = 2$

At first, we study the orbits of phase portrait of (2.1) with $p = 2$ which intersect or tend to the singular lines. It is easy to see that if $\phi = \phi(\xi)$ is a solution of (1.3) with $p = 2$, then so are $\phi = -\phi(\xi)$ and $\phi = \phi(-\xi)$. Consequently, we only consider the initial value problem with $\phi(0) = \phi_0 > 0$.

(1) For the case when $c > 0$ and $g < 0$, i.e. $(c, g) \in D_1 \cup C_1 \cup D_6$, one knows that $h = 0$ corresponds to two hyperbolic curves which intersect with the singular lines $\phi = \pm\sqrt{c}$ at the points $(\pm\sqrt{c}, \pm\sqrt{c - \frac{g}{c}}\ )$. Solving $H_1(\phi) = \pm\xi$ with $\phi_0 = 0$ and $h = 0$ gives that $\phi(\xi) = \pm\sqrt{-\frac{g}{c}}\ \sinh(\xi)$, from which we construct the following singular wave solutions:

$$\phi(\xi) = \begin{cases} \sqrt{-\frac{g}{c}}\ \sinh(\xi - 4k\xi_0) & (4k-1)\xi_0 < \xi \le (4k+1)\xi_0, \\ \sqrt{-\frac{g}{c}}\ \sinh(\xi - (4k+1)\xi_0) & (4k+1)\xi_0 < \xi \le (4k+3)\xi_0, \end{cases} \tag{4.6}$$

where $k \in \mathbb{Z}$ and $\xi_0 = \sinh^{-1}(\frac{c}{\sqrt{-g}})$. It is easy to check that $\phi(\xi)$ defined by (4.6) is continuous but has discontinuous first order derivatives at $\xi = (2k+1)\xi_0$ where $\phi(\xi)$ attains its extrema $\pm\sqrt{c}$. Clearly,

$$\lim_{\xi \to (2k+1)\xi_0^+} \phi'(\xi) = -\lim_{\xi \to (2k+1)\xi_0^-} \phi'(\xi) \neq 0,$$

so it is a periodic peakon with period $4\xi_0$, amplitude $2\sqrt{c}$ and positive wave speed c.

(2) For the case when $c > 0$ and $g = 0$, one knows that $h = 0$ corresponds to two straight lines $\phi' = \pm\phi$ which intersect with the singular lines $\phi = \pm\sqrt{c}$ at the points $(\pm\sqrt{c}, \pm\sqrt{c}\ )$. Solving $H_1(\phi) = \pm\xi$ with $\phi_0 = 0$ and $h = 0$ gives that $\phi(\xi) = Ae^{\pm\xi}$, from which we construct the following singular wave solutions

$$\phi(\xi) = \pm\sqrt{c}\ e^{-|\xi|}, \tag{4.7}$$

which are two solitary peakons of (1.3).

(3) For the case when $(c, g) \in D_1 \cup C_1$, let $h = I(e_1, 0)$ and $\phi_0 = -\sqrt{c}$, then $H_1(e_1; \phi_0, c, g, h) = \infty$. Let

$$\phi(\xi) = \begin{cases} p_1(\xi; \phi_0, c, g, h) & \xi \ge 0, \\ p_1(-\xi; \phi_0, c, g, h) & \xi < 0, \end{cases} \tag{4.8}$$

then $\phi = \phi(\xi)$ is a solitary cuspon solution. Let $\phi_0 \in (0, e_1) \cup (\phi_1, \sqrt{c})$ and $h = I(\phi_0, 0)$, then $T_1 = -H_1(-\sqrt{c}; \phi_0, c, g, h) < \infty$, where $\phi_1 = \sqrt{\frac{c\sqrt{c^2+8g}+c^2+4g}{2c}}$. Let

$$\phi(\xi) = \begin{cases} p_1(-\xi; \phi_0, c, g, h) & 2kT_1 < \xi \le (2k+1)T_1, \\ p_1(\xi; \phi_0, c, g, h) & (2k-1)T_1 < \xi \le 2kT_1, \end{cases} \tag{4.9}$$

then $\phi = \phi(\xi)$ is a periodic cuspon. Its period is $2T_1$, amplitude is $\sqrt{c} + \phi_0$ and $\lim_{\xi \to (2k+1)T_1^-} \phi'(\xi) = -\lim_{\xi \to (2k+1)T_1^+} \phi'(\xi) = \infty$. The periodic cuspon loses its smoothness where it attains its minimum.

(4) For the case when $(c,g)\in D_2\cup C_2$, let $\phi_0\in(0,\sqrt{\frac{g}{c}})$ and $h=I(\phi_0,0)$, then $T_2=H_1(\sqrt{c};\phi_0,c,g,h)<\infty$. Let

$$\phi(\xi)=\begin{cases}p_1(\xi;\phi_0,c,g,h) & 2kT_2<\xi\leq(2k+1)T_2,\\ p_1(-\xi;\phi_0,c,g,h) & (2k-1)T_2<\xi\leq 2kT_2,\end{cases}\tag{4.10}$$

then $\phi=\phi(\xi)$ is a periodic cuspon. Its period is $2T_2$, amplitude is $\sqrt{c}-\phi_0$. The periodic cuspon loses its smoothness where it attains its maximum. However, $\phi=\phi(\xi)$ is a periodic peakon, where $\phi(\xi)$ is defined by (4.10) with $\phi_0=\sqrt{\frac{g}{c}}$ and $h=I(\phi_0,0)$. This solution is a peakon because

$$\lim_{\xi\to(2k+1)T_2^-}\phi'(\xi)=-\lim_{\xi\to(2k+1)T_2^+}\phi'(\xi)=\sqrt{c-\frac{g}{c}}\neq 0.$$

(5) For the case when $(c,g)\in D_3$, (4.10) with $\phi_0\in(0,\sqrt{c})$ and $h=I(\phi_0,0)$ gives a periodic cuspon. However for $\phi_0\in(\sqrt{c},e_2)$, then $-H_1(\sqrt{c};\phi_0,c,g,h)<\infty$ and thus (4.9) with $T_1=-H_1(\sqrt{c};\phi_0,c,g,h)$ gives a periodic cuspon. The amplitude of the periodic cuspon is $|\sqrt{c}-\phi_0|$. (4.8) with $h=I(e_2,0)$ and $\phi_0=\sqrt{c}$ gives a solitary cuspon of (1.3) whose amplitude is $\sqrt{c}-e_2$. The solitary cuspon loses its smoothness where it attains minimum.

(6) For the case when $(c,g)\in D_6$, (4.9) with $\phi_0\in(0,\sqrt{c})$ and $h=I(\phi_0,0)$ gives a periodic cuspon.

Theorem 4.1. *For the bounded traveling wave solutions of equation (1.3) with $p=2$, we have the following conclusions:*

(1) It has no bounded traveling wave solutions with negative wave speed.

(2) It has one family of solitary peakons and two families of periodic peakons. The solitary peakons are defined by (4.7) with $\xi=x-ct$ for arbitrary $c>0$; one family of periodic peakons, whose amplitude are $2\sqrt{c}$, are defined by (4.6) with $\xi=x-ct$ for arbitrary $c>0$ and $g<0$; another two family of periodic peakons are $\phi=\pm\phi(x-ct)$ whose amplitude are $\sqrt{c}-\sqrt{\frac{g}{c}}$, where $\phi(\xi)$ is defined by (4.10) with $\phi_0=\sqrt{\frac{g}{c}}$ and $h=I(\phi_0,0)$ for arbitrary $c>0$ and $0<g<c^2$.

(3) It has two family of solitary cuspons and three families of periodic cuspons: one family of solitary cuspons $\phi=\pm\phi(x-ct)$ whose amplitudes are $\frac{1}{2}\sqrt{c-\sqrt{c^2+8g}}+\sqrt{c}$, where $\phi(\xi)$ is defined by (4.8) with $h=I(-\frac{1}{2}\sqrt{c-\sqrt{c^2+8g}},0)$ and $\phi_0=\sqrt{c}$ for arbitrary $c>0$ and $0>g\geq-\frac{c^2}{8}$; another family of solitary cuspons $\phi=\pm\phi(x-ct)$ whose amplitudes are $\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}}-\sqrt{c}$, where $\phi(\xi)$ is defined by (4.8) with $h=I(\frac{1}{2}\sqrt{c+\sqrt{c^2+8g}},\ 0)$ and $\phi_0=\sqrt{c}$ for arbitrary $c>0$ and $g>c^2$; one family of periodic cuspons $\phi=\pm\phi(x-ct)$ with amplitude $\sqrt{c}+\phi_0$, where $\phi(\xi)$ is defined by (4.9) with $\phi_0\in(0,\ e_1)\cup(\phi_1,\ \sqrt{c})$ and $h=I(\phi_0,0)$ for arbitrary $c>0$ and $0>g\geq-\frac{c^2}{8}$, or with $\phi_0\in(0,\sqrt{c})$ and $h=I(\phi_0,\ 0)$ for $g<-\frac{c^2}{8}$ and $c>0$; one family of periodic cuspons $\phi=\pm\phi(x-ct)$ with amplitude $\phi_0-\sqrt{c}$, where $\phi(\xi)$ is defined by (4.9) with $T_1=-H_1(\sqrt{c};\phi_0,c,g,h)$, $\phi_0\in(\sqrt{c},\ e_2)$, and $h=I(\phi_0,0)$ for arbitrary $c>0$ and $g>c^2$; another family of periodic cuspons $\phi=\pm\phi(x-ct)$ with amplitude $\sqrt{c}-\phi_0$, where $\phi(\xi)$ is defined by (4.10) with $\phi_0\in(0,\ \min\{\sqrt{\frac{g}{c}},\ \sqrt{c}\})$ and $h=I(\phi_0,\ 0)$ for arbitrary $c>0$ and $0<g$.

4.2. Singular traveling wave solutions of (1.1) with $p = 3$

We now study the orbits of phase portrait of (2.1) with $p = 3$ which intersect or tend to the singular lines. Let's first consider the case when $h = 0$.

(1) For the case when $(c, g) \in \{(c, g)|\ 0 < g < c^{\frac{5}{3}}\}$, one knows that (3.4) with $h = 0$ defines two curves, one of them intersects with the singular lines $\phi = \sqrt[3]{c}$ at the points $(\sqrt[3]{c}, \pm\sqrt{c^{\frac{2}{3}} - \frac{g}{c}}\ )$. Solving $H_2(\phi) = \xi$ with $\phi_0 = \sqrt{\frac{g}{c}}$ and $h = 0$ gives that $\phi(\xi) = \pm\sqrt{\frac{g}{c}}\ \cosh(\xi)$, from which we construct the following singular wave solutions. Let

$$\phi(\xi) = \sqrt{\frac{g}{c}}\ \cosh(\xi - 2k\xi_0), \qquad (2k-1)\xi_0 < \xi \le (2k+1)\xi_0, \tag{4.11}$$

where $k \in \mathbb{Z}$ and $\xi_0 = \cosh^{-1}(\sqrt[3]{c}\sqrt{\frac{c}{g}})$. It is easy to check that $\phi(\xi)$ defined by (4.11) is continuous but has discontinuous first order derivatives at $\xi = (2k+1)\xi_0$ where $\phi(\xi)$ attains its maximum $\sqrt[3]{c}$. Clearly,

$$\lim_{\xi \to (2k+1)\xi_0^-} \phi'(\xi) = -\lim_{\xi \to (2k+1)\xi_0^+} \phi'(\xi) = \sqrt{c^{\frac{2}{3}} - \frac{g}{c}} \neq 0,$$

so it is a periodic peakon with period $2\xi_0$ and amplitude $\sqrt[3]{c} - \frac{g}{c}$.

For the case when $c > 0$ and $g = 0$, similar analysis as in Subsection 4.2, we get

$$\phi(\xi) = \sqrt[3]{c}\ e^{-|\xi|} \tag{4.12}$$

which is a solitary peakon.

(2) For the case when $(c, g) \in D_1 \cup C_2(c > 0)$, let $\phi_0 \in (0, \sqrt{\frac{g}{c}})$ and $h = I(\phi_0, 0)$, then $H_2(\sqrt[3]{c}; \phi_0, c, g, h) = T_3 < \infty$. Let

$$\phi(\xi) = \begin{cases} p_2(\xi; \phi_0, c, g, h) & 2kT_3 < \xi \le (2k+1)T_3, \\ p_2(-\xi; \phi_0, c, g, h) & (2k-1)T_3 < \xi \le 2kT_3, \end{cases} \tag{4.13}$$

then $\phi = \phi(\xi)$ is a periodic cuspon solution since

$$\lim_{\xi \to (2k+1)T_3^-} \phi'(\xi) = -\lim_{\xi \to (2k+1)T_3^+} \phi'(\xi) = \infty.$$

(3) For the case when $(c, g) \in D_2$, let $\phi_0 \in (0, \sqrt[3]{c})$ and $h = I(\phi_0, 0)$, then (4.13) gives a periodic cuspon solution; for $\phi_0 \in (\sqrt[3]{c}, \phi_{e_1})$, let $h = I(\phi_0, 0)$, then $\phi = \phi(-\xi)$, where $\phi(\xi)$ is defined by (4.13), gives a periodic cuspon solution. However, if set $h = I(\phi_{e_1}, 0)$, then $H_2(\phi_{e_1}; \sqrt[3]{c}, c, g, h) = \infty$. Let

$$\phi(\xi) = \begin{cases} p_2(\xi; \sqrt[3]{c}, c, g, h) & \xi \ge 0, \\ p_2(-\xi; \sqrt[3]{c}, c, g, h) & \xi < 0, \end{cases} \tag{4.14}$$

then $\phi = \phi(\xi)$ is a solitary cuspon solution.

(4) For the case when $(c, g) \in D_3 \cup D_4 \cup C_1(c > 0)$, (4.13) with $\phi_0 \in (\phi_{e_1}, 0)$ and $h = I(\phi_0, 0)$ gives a periodic cuspon. Its period is $2T_3$, amplitude is $|\sqrt[3]{c} - \phi_0|$. $\phi = \phi(-\xi)$ with $h = I(\phi_{e_1}, 0)$ and $\phi_0 = \sqrt[3]{c}$, where $\phi(\xi)$ is defined by (4.14), gives

a solitary cuspon of (1.3) whose amplitude is $\sqrt[3]{c} - \phi_{e_1}$. The solitary cuspon loses its smoothness where it attains its maximum $\sqrt[3]{c}$.

Taking into consideration of the symmetries of this equation (refer to Lemma 3.1) and the analysis above, we have the following conclusion.

Theorem 4.2. *For the bounded traveling wave solutions of equation (1.3) with $p = 3$, we have the following conclusions:*

(1) It has one family of solitary peakons and one family of periodic peakons. Thesolitary peakons are defined by (4.12) with $\xi = x - ct$ for arbitrary $c \neq 0$; The periodic peakons are given by

$$\phi(\xi) = sign(c)\sqrt{\frac{g}{c}}\ \cosh(x - ct - 2k\xi_0) \qquad (2k-1)\xi_0 < \xi \leq (2k+1)\xi_0, \tag{4.15}$$

where $k \in \mathbb{Z}$ and $\xi_0 = \cosh^{-1}(\sqrt[3]{c}\sqrt{\frac{c}{g}})$ for arbitrary c and g satisfying $0 < \frac{g}{c} < c^{\frac{2}{3}}$.

(2) The periodic cuspons are given by (4.13) with $\xi = x - ct$ or

$$\phi(\xi) = \begin{cases} -p_2(\xi; -\phi_0, -c, -g, h) & 2kT_3 < \xi \leq (2k+1)T_3, \\ -p_2(-\xi; -\phi_0, -c, -g, h) & (2k-1)T_3 < \xi \leq 2kT_3, \end{cases} \tag{4.16}$$

with $\xi = x + ct$, where $h = I(\phi_0, 0)$ and $\phi_0 \in\ (0,\ \min\{\sqrt{\frac{g}{c}},\ \sqrt[3]{c}\})$ for $c > 0$ & $g > 0$, or $\phi_0 \in (\phi_{e_1}, 0)$ for $c > 0$ & $g < 0$. For $g > c^{\frac{5}{3}} > 0$, set $\phi_0 \in (\sqrt[3]{c},\ \phi_{e_1})$, then $\phi = \phi(-\xi)$ is a periodic cuspon, where $\phi(\xi)$ is defined by (4.13) with $\xi = x - ct$ or (4.16) with $\xi = x + ct$.

(3) The solitary cuspons are given by (4.14) with $\xi = x - ct$ or

$$\phi(\xi) = \begin{cases} -p_2(\xi; -\sqrt[3]{c}, -c, -g, h) & \xi > 0, \\ -p_2(-\xi; -\sqrt[3]{c}, -c, -g, h) & \xi < 0, \end{cases} \tag{4.17}$$

with $\xi = x + ct$, where $h = I(e_1, 0)$ for $c > 0$ & $g < 0$. For $g > c^{\frac{5}{3}} > 0$, $\phi = \phi(-\xi)$ is a solitary cuspon, where $\phi(\xi)$ is defined by (4.14) with $\xi = x - ct$ or (4.17) with $\xi = x + ct$.

5. Conclusion

All bounded traveling wave solutions of a generalized Camassa-Holm-Novikov equation (1.3) with $p = 2$ and $p = 3$, including smooth solitary wave solutions and periodic wave solutions, solitary peakons and periodic peakons, as well as solitary cuspons and periodic cuspons, are investigated by studying the bifurcation of its traveling wave system in this paper. It is shown in this paper once again that the dynamical method can be well applied to explore not only the smooth bounded traveling wave solutions of PDEs, even higher-order equations [21, 33, 34], but also the singular weak solutions [18, 19, 31, 32] if we get to know the effects of singular lines to the global weak solutions. The result shows that these two equations admit smooth solitary wave and periodic wave solutions, as well as the singular wave solutions: solitary peakons, periodic peakons, solitary cuspons and periodic cuspons as Camassa-Holm equation and Degasperis-Procesi equation or some generalization of them. Nevertheless, the prominent difference is that the generalized

Camassa-Holm-Novikov equation with $p = 2$, i.e. Novikov equation has no any bounded traveling wave solutions with negative wave speed, but has a family of periodic cuspons which have discontinuous first-order derivative at both maximum and minimum. However, so far no compacton or kinkon, two kinds of singular waves which have been found in some nonlinear wave equations [10, 20, 32], are found in these two nonlinear wave equations.

Although we discussed two special cases, p = 2 and p = 3, we did not consider the general p yet in the current paper. However, based on the observation of previous results for special cases, p = 1, 2 or p = 3, we predict that equation (1.3) with p an odd integer should be similar to p = 2 and p an even integer should be similar to p = 3. The bounded solutions, especially the singular wave solutions of this family of equations with arbitrary integer p will be considered in our future work.

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