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ON AN EXTENDED HARDY-HILBERT'S INEQUALITY IN THE WHOLE PLANE*

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Abstract By means of weight coefficients, a complex integral formula and Hermite-Hadamard's inequality, a new extended Hardy-Hilbert's inequality in the whole plane with multi-parameters and a best possible constant factor is given. The equivalent forms, the operator expressions and a few particular cases are considered.

Keywords Hardy-Hilbert's inequality, Hermite-Hadamard's inequality, weight coefficient, equivalent form, operator expression.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p,\ b = \{b_n\}_{n=1}^{\infty} \in l^q,\ ||a||_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0, ||b||_q > 0$, then we have the following Hardy-Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}||a||_p||b||_q, \tag{1.1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible (cf. [6]). The more accurate form of (1.1) was given as follows (cf. [7], Theorem 323):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-1} < \frac{\pi}{\sin(\frac{\pi}{p})}||a||_p||b||_q\ , \tag{1.2}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

In 2011, Yang gave an extension of (1.2) as follows (cf. [22]): If $0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda,\ a_m, b_n \geq 0,\ ||a||_{p,\varphi} = \{\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\lambda_1)-1}a_m^p\}^{\frac{1}{p}} \in (0,\infty)$, $||b||_{q,\psi} = \{\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\lambda_2)-1}b_n^q\}^{\frac{1}{q}} \in (0,\infty)$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-2\alpha)^{\lambda}} < B(\lambda_1,\lambda_2)||a||_{p,\varphi}||b||_{q,\psi}\ (0 \leq \alpha \leq \frac{1}{2}), \tag{1.3}$$

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where, the constant factor $B(\lambda_1, \lambda_2)$ is the best possible and $B(u, v)$ is the beta function defined by (cf. [20])

$$B(u,v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt \ (u, v > 0). \tag{1.4}$$

For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}, \alpha = \frac{1}{2}$, (1.3) reduces to (1.2). some other results relate to (1.1)–(1.3) were provided by [1–5, 8–13, 15–17, 19, 23, 25, 26]. In 2016-2017, a few extensions of (1.1)–(1.2) in the whole plane were obtained by [21, 24, 27].

In this paper, following the way of [21, 24, 27], by means of weight coefficients, using a complex integral formula and Hermite-Hadamard's inequality, an extension of (1.1) in the whole plane similar to the type of (1.3) is given as follows: For $0 < \lambda_1, \lambda_2 < 1, \lambda_1 + \lambda_2 = \lambda \leq 1$, $\xi, \eta \in [0, \frac{1}{2}], a_m, b_n \geq 0, 0 < \sum_{|m|=1}^\infty |m - \xi|^{p(1-\lambda_1)-1} a_m^p < \infty$, $0 < \sum_{|n|=1}^\infty |n - \eta|^{q(1-\lambda_2)-1} b_n^q < \infty$, we have

$$\sum_{|n|=1}^\infty \sum_{|m|=1}^\infty \frac{a_m b_n}{|m-\xi|^\lambda + |n-\eta|^\lambda} < \frac{2\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \times \left[\sum_{|m|=1}^\infty |m-\xi|^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^\infty |n-\eta|^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}. \tag{1.5}$$

Moreover, an extended inequality of (1.5) with multi-parameters and a best possible constant factor is proved. The equivalent forms, the operator expressions and a few particular cases are considered.

2. Some lemmas and an example

Lemma 2.1. *If $\mathbf{C}$ is the set of complex numbers and $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, $z_k \in \mathbf{C} \backslash \{z | \times Rez \geq 0, \times Imz = 0\}$ $(k = 1, 2, \cdots, n)$ are different points, the function $f(z)$ is analytic in $\mathbf{C}_\infty$ except for z_i $(i = 1, 2, \cdots, n)$, and $z = \infty$ is a zero point of $f(z)$ whose order is not less than 1, then for $\alpha \in \mathbf{R}$, we have*

$$\int_0^\infty f(x) x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{k=1}^n \times Res[f(z) z^{\alpha-1}, z_k], \tag{2.1}$$

where, $0 < \times Im \ln z = \arg z < 2\pi$. In particular, if z_k $(k = 1, \cdots, n)$ are all poles of order 1, setting $\varphi_k(z) = (z - z_k) f(z)$ $(\varphi_k(z_k) \neq 0)$, then

$$\int_0^\infty f(x) x^{\alpha-1} dx = \frac{\pi}{\sin \pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha-1} \varphi_k(z_k). \tag{2.2}$$

Proof. By [18] (P.118), we have (2.1). We find

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i \sin 2\pi\alpha \\ &= -2i \sin \pi\alpha (\cos \pi\alpha + i \sin \pi\alpha) = -2i e^{i\pi\alpha} \sin \pi\alpha. \end{aligned}$$

In particular, since $f(z) z^{\alpha-1} = \frac{1}{z - z_k} (\varphi_k(z) z^{\alpha-1})$, it is obvious that

$$\times Res[f(z) z^{\alpha-1}, -a_k] = {z_k}^{\alpha-1} \varphi_k(z_k) = -e^{i\pi\alpha} (-z_k)^{\alpha-1} \varphi_k(z_k).$$

Then by (2.1), we obtain (2.2). □

Example 2.1. For $s \in \mathbf{N} = \{1, 2, \cdots\}$, $c_s \geq \cdots \geq c_1 > 0, \varepsilon > 0$, $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x, y) := \frac{1}{\prod_{k=1}^{s}(x^{\lambda/s} + c_k y^{\lambda/s})},$$

and $\widetilde{c}_k = c_k + (k-1)\varepsilon$ $(k = 1, \cdots, s)$. In view of $\widetilde{c}_s > \cdots > \widetilde{c}_1 = c_1 > 0$, by (2.2), we find

$$\begin{aligned} \widetilde{k}_s(\lambda_1) &:= \int_0^\infty \frac{1}{\prod_{k=1}^{s}(t^{\lambda/s} + \widetilde{c}_k)} t^{\lambda_1 - 1} dt \\ &\overset{u=t^{\lambda/s}}{=} \frac{s}{\lambda} \int_0^\infty \frac{1}{\prod_{k=1}^{s}(u + \widetilde{c}_k)} u^{\frac{s\lambda_1}{\lambda} - 1} du \\ &= \frac{\pi s}{\lambda \sin(\frac{\pi s \lambda_1}{\lambda})} \sum_{k=1}^{s} \widetilde{c}_k^{\frac{s\lambda_1}{\lambda} - 1} \prod_{j=1 (j \neq k)}^{s} \frac{1}{\widetilde{c}_j - \widetilde{c}_k} \in \mathbf{R}_+. \end{aligned}$$

Since we find

$$\begin{aligned} 0 &< \widetilde{k}_s(\lambda_1) \leq \frac{s}{\lambda} \int_0^\infty \frac{1}{(u + c_1)^s} u^{\frac{s\lambda_1}{\lambda} - 1} du \\ &\overset{u=c_1 v}{=} \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}} \int_0^\infty \frac{1}{(v+1)^s} v^{\frac{s\lambda_1}{\lambda} - 1} dv \\ &= \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}} B(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}) \in \mathbf{R}_+, \end{aligned}$$

it follows that

$$\begin{aligned} k_s(\lambda_1) &= \lim_{\varepsilon \to 0^+} \widetilde{k}_s(\lambda_1) \\ &= \frac{\pi s}{\lambda \sin(\frac{\pi s \lambda_1}{\lambda})} \sum_{k=1}^{s} c_k^{\frac{s\lambda_1}{\lambda} - 1} \prod_{j=1 (j \neq k)}^{s} \frac{1}{c_j - c_k} \in \mathbf{R}_+. \end{aligned} \tag{2.3}$$

In particular, for $s = 1$, we find

$$k_1(\lambda_1) = \frac{1}{\lambda} \int_0^\infty \frac{u^{(\lambda_1/\lambda) - 1}}{u + c_1} du = \frac{\pi}{\lambda c_1^{\lambda_2/\lambda} \sin(\frac{\pi \lambda_1}{\lambda})}; \tag{2.4}$$

for $c_s = \cdots = c_1$, we obtain

$$k(\lambda_1) := \int_0^\infty \frac{t^{\lambda_1 - 1}}{(t^{\lambda/s} + c_1)^s} dt = \frac{s}{\lambda c_1^{(s\lambda_2)/\lambda}} B(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}). \tag{2.5}$$

In the following, we agree that $s \in \mathbf{N}, c_s \geq \cdots \geq c_1 > 0$, $\alpha, \beta \in (0, \pi), \xi, \eta \in [0, \frac{1}{2}]$, $0 < \lambda_1, \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda \leq s (s \geq 2)$; $0 < \lambda_1, \lambda_2 < 1$, $\lambda = \lambda_1 + \lambda_2 \leq s = 1$.

For $|t| > \frac{1}{2}$, we set functions $A_{\zeta,\theta}(t) := |t - \zeta| + (t - \zeta)\cos\theta$, $(\zeta, \theta) = (\xi, \alpha)$ (or (η, β)), and

$$h(x, y) := k_\lambda(A_{\xi,\alpha}(x), A_{\eta,\beta}(y)) = \frac{1}{\prod_{k=1}^{s}(A_{\xi,\alpha}^{\lambda/s}(x) + c_k A_{\eta,\beta}^{\lambda/s}(y))}.$$

Definition 2.1. Define the following weight coefficients:

$$\omega(\lambda_2, m) := \sum_{|n|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{A_{\eta,\beta}^{1-\lambda_2}(n)}, |m| \in \mathbf{N}, \tag{2.6}$$

$$\varpi(\lambda_1, n) := \sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\eta,\beta}^{\lambda_2}(n)}{A_{\xi,\alpha}^{1-\lambda_1}(m)}, |n| \in \mathbf{N}, \tag{2.7}$$

where, $\sum_{|j|=1}^{\infty} \ldots = \sum_{j=-1}^{-\infty} \ldots + \sum_{j=1}^{\infty} \ldots (j = m, n)$.

Lemma 2.2. *With regards to the agreements, replacing* $0 < \lambda_1 \leq 1 (0 < \lambda_1 < 1)$ *by* $\lambda_1 > 0$, *setting* $h_\beta(\lambda_1) := 2k_s(\lambda_1) \csc^2 \beta$, *we still have*

$$h_\beta(\lambda_1)(1 - \theta(\lambda_2, m)) < \omega(\lambda_2, m) < h_\beta(\lambda_1), |m| \in \mathbf{N}, \tag{2.8}$$

where,

$$\begin{aligned}\theta(\lambda_2, m) &:= \frac{1}{k_s(\lambda_1)} \int_{\frac{A_{\xi,\alpha}(m)}{(1+\eta)(1+\cos\beta)}}^{\infty} \frac{u^{\lambda_1 - 1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} du \\ &= O\left(\frac{1}{A_{\xi,\alpha}^{\lambda_2}(m)}\right) \in (0,1), |m| \in \mathbf{N}. \end{aligned}\tag{2.9}$$

Proof. For $|x| > \frac{1}{2}$, we set

$$h^{(1)}(x,y) := \frac{1}{\prod_{k=1}^{s}\{A_{\xi,\alpha}^{\lambda/s}(x) + c_k[(y-\eta)(\cos\beta - 1)]^{\lambda/s}\}}, y < -\frac{1}{2},$$

$$h^{(2)}(x,y) := \frac{1}{\prod_{k=1}^{s}\{A_{\xi,\alpha}^{\lambda/s}(x) + c_k[(y-\eta)(1+\cos\beta)]^{\lambda/s}\}}, y > \frac{1}{2},$$

wherefrom, for $y > \frac{1}{2}$,

$$h^{(1)}(x,-y) = \frac{1}{\prod_{k=1}^{s}\{A_{\xi,\alpha}^{\lambda/s}(x) + c_k[(y+\eta)(1-\cos\beta)]^{\lambda/s}\}}.$$

We find

$$\begin{aligned}\omega(\lambda_2, m) &= \sum_{n=-1}^{-\infty} h^{(1)}(m,n) \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{[(n-\eta)(\cos\beta - 1)]^{1-\lambda_2}} \\ &\quad + \sum_{n=1}^{\infty} h^{(2)}(m,n) \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{[(n-\eta)(1+\cos\beta)]^{1-\lambda_2}} \\ &= \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1-\cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{h^{(1)}(m,-n)}{(n+\eta)^{1-\lambda_2}} \\ &\quad + \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1+\cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{h^{(2)}(m,n)}{(n-\eta)^{1-\lambda_2}}. \end{aligned}\tag{2.10}$$

It is evident that for fixed $m \in \mathbf{N}$ and the assumptions, both $\frac{h^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}}$ and $\frac{h^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}$ are strictly decreasing and strict convex with respect to $y \in (\frac{1}{2}, \infty)$, satisfying $\frac{h^{(i)}(m,(-1)^i y)}{[y+(-1)^i\eta]^{1-\lambda_2}} > 0$, $\frac{d}{dy}\frac{h^{(i)}(m,(-1)^i y)}{[y+(-1)^i\eta]^{1-\lambda_2}} < 0$ and

$$\frac{d^2}{dy^2}\frac{h^{(i)}(m,(-1)^i y)}{[y+(-1)^i\eta]^{1-\lambda_2}} > 0 (i = 1, 2).$$

By Hermite-Hadamard's inequality (cf. [14]), we find

$$\begin{aligned}\omega(\lambda_2, m) &< \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1-\cos\beta)^{1-\lambda_2}} \int_{\frac{1}{2}}^{\infty} \frac{h^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} dy \\ &\quad + \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1+\cos\beta)^{1-\lambda_2}} \int_{\frac{1}{2}}^{\infty} \frac{h^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}} dy.\end{aligned}$$

Setting $u = \frac{A_{\xi,\alpha}(m)}{(y+\eta)(1-\cos\beta)} (\frac{A_{\xi,\alpha}(m)}{(y-\eta)(1+\cos\beta)})$ in the above first (second) integral, by simplification and (2.3), we find

$$\begin{aligned}\omega(\lambda_2, m) &< (\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}) \int_0^{\infty} \frac{u^{\lambda_1 - 1} du}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} \\ &= 2k_s(\lambda_1)\csc^2\beta = h_\beta(\lambda_1).\end{aligned}$$

By (2.8), since both $\frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}}$ and $\frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}$ are strictly decreasing, we still have

$$\begin{aligned}\omega(\lambda_2, m) &> \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1-\cos\beta)^{1-\lambda_2}} \int_{1}^{\infty} \frac{h^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} dy \\ &\quad + \frac{A_{\xi,\alpha}^{\lambda_1}(m)}{(1+\cos\beta)^{1-\lambda_2}} \int_{1}^{\infty} \frac{h^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}} dy \\ &= \frac{1}{1-\cos\beta} \int_0^{\frac{A_{\xi,\alpha}(m)}{(1+\eta)(1-\cos\beta)}} \frac{u^{\lambda_1-1} du}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} \\ &\quad + \frac{1}{1+\cos\beta} \int_0^{\frac{A_{\xi,\alpha}(m)}{(1-\eta)(1+\cos\beta)}} \frac{u^{\lambda_1-1} du}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} \\ &\geq 2\sec^2\beta \int_0^{\frac{A_{\xi,\alpha}(m)}{(1+\eta)(1+\cos\beta)}} \frac{u^{\lambda_1-1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} du \\ &= h_\beta(\lambda_2)(1 - \theta(\lambda_2, m)) > 0,\end{aligned}$$

where, $\theta(\lambda_2, m)(< 1)$ is indicated by (2.9). We obtain

$$\begin{aligned}0 < \theta(\lambda_2, m) &< \frac{1}{k_s(\lambda_1)} \int_{\frac{A_{\xi,\alpha}(m)}{(1+\eta)(1+\cos\beta)}}^{\infty} \frac{u^{\lambda_1-1}}{u^{\lambda}} du \\ &= \frac{1}{k_s(\lambda_1)} \int_{\frac{A_{\xi,\alpha}(m)}{(1+\eta)(1+\cos\beta)}}^{\infty} u^{-\lambda_2 - 1} du \\ &= \frac{1}{\lambda_2 k_s(\lambda_1)} \left[\frac{(1+\eta)(1+\cos\beta)}{A_{\xi,\alpha}(m)}\right]^{\lambda_2},\end{aligned}$$

and then we have (2.8) and the estimation of (2.9). □

In the same way, we have

Lemma 2.3. *With regards to the agreements, replacing* $0 < \lambda_2 \leq 1 (0 < \lambda_2 < 1)$ *by* $\lambda_2 > 0$, *setting* $h_\alpha(\lambda_1) = 2k_s(\lambda_1)\csc^2\alpha$, *we still have*

$$h_\alpha(\lambda_1)(1 - \vartheta(\lambda_1, n)) < \varpi(\lambda_1, n) < h_\alpha(\lambda_1), |n| \in \mathbf{N}, \tag{2.11}$$

where,

$$\vartheta(\lambda_1, n) := \frac{1}{k_s(\lambda_1)} \int_{\frac{A_{\eta,\beta}(n)}{(1+\xi)(1+\cos\alpha)}}^{\infty} \frac{u^{\lambda_2 - 1}}{\prod_{k=1}^{s}(u^{\lambda/s} + c_k)} du$$
$$= O\left(\frac{1}{A_{\eta,\beta}^{\lambda_1}(n)}\right) \in (0, 1), |n| \in \mathbf{N}. \tag{2.12}$$

Lemma 2.4. *For* $\rho > 0, (\zeta, \theta) = (\xi, \alpha)$ *(or* (η, β)*), we have*

$$H_\rho(\zeta, \theta) := \sum_{|k|=1}^{\infty} \frac{1}{A_{\zeta,\theta}^{1+\rho}(k)} = \frac{1 + o(1)}{\rho}$$
$$\times \left[\frac{1}{(1 + \cos\theta)^{1+\rho}} + \frac{1}{(1 - \cos\theta)^{1+\rho}}\right] (\rho \to 0^+). \tag{2.13}$$

Proof. We have

$$H_\rho(\zeta, \theta) = \sum_{k=-1}^{-\infty} \frac{1}{[(k - \zeta)(\cos\theta - 1)]^{1+\rho}} + \sum_{k=1}^{\infty} \frac{1}{[(k - \zeta)(\cos\theta + 1)]^{1+\rho}}$$
$$= \frac{1}{(1 - \cos\theta)^{1+\rho}} \sum_{k=1}^{\infty} \frac{1}{(k + \zeta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}} \sum_{k=1}^{\infty} \frac{1}{(k - \zeta)^{1+\rho}}.$$

For $a = \frac{1}{(1-\zeta)^{1+\rho}} > 0$, by Hermite-Hadamard's inequality, we find

$$H_\rho(\zeta, \theta) \leq \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right] \left[a + \sum_{k=2}^{\infty} \frac{1}{(k - \zeta)^{1+\rho}}\right]$$
$$< \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right] \left[a + \int_{\frac{3}{2}}^{\infty} \frac{dy}{(y - \zeta)^{1+\rho}}\right]$$
$$= \frac{a\rho + (\frac{3}{2} - \zeta)^{-\rho}}{\rho} \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right],$$

$$H_\rho(\zeta, \theta) \geq \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right] \sum_{k=1}^{\infty} \frac{1}{(k + \zeta)^{1+\rho}}$$
$$> \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right] \int_{1}^{\infty} \frac{dy}{(y + \zeta)^{1+\rho}}$$
$$= \frac{(1 + \zeta)^{-\rho}}{\rho} \left[\frac{1}{(1 - \cos\theta)^{1+\rho}} + \frac{1}{(1 + \cos\theta)^{1+\rho}}\right].$$

Hence, we have (2.13). □

3. Main results

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0$ $(|m|, |n| \in \mathbf{N})$,

$$0 < \sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p < \infty, 0 < \sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q < \infty,$$

$$K_{\alpha,\beta}(\lambda_1) := h_{\beta}^{\frac{1}{p}}(\lambda_1) h_{\alpha}^{\frac{1}{q}}(\lambda_1) = 2k_s(\lambda_1) \csc^{\frac{2}{p}} \beta \csc^{\frac{2}{q}} \alpha, \tag{3.1}$$

then we have the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{\prod_{k=1}^{s}(A_{\xi,\alpha}^{\lambda/s}(m) + c_k A_{\eta,\beta}^{\lambda/s}(n))} a_m b_n \\ &< K_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q \right]^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} J &:= \left\{ \sum_{|n|=1}^{\infty} A_{\eta,\beta}^{p\lambda_2-1}(n) \left[\sum_{|m|=1}^{\infty} \frac{a_m}{\prod_{k=1}^{s}(A_{\xi,\alpha}^{\lambda/s}(m) + c_k A_{\eta,\beta}^{\lambda/s}(n))} \right]^p \right\}^{\frac{1}{p}} \\ &< K_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p \right]^{\frac{1}{p}}. \end{aligned} \tag{3.3}$$

In particular, for $s = c_1 = 1, \alpha = \beta = \frac{\pi}{2} (0 < \lambda_1, \lambda_2 < 1)$, (3.2) reduces to (1.5), and (3.3) reduces to the equivalent form of (1.5) as follows:

$$\begin{aligned} &\left\{ \sum_{|n|=1}^{\infty} |n-\eta|^{p\lambda_2 - 1} \left(\sum_{|m|=1}^{\infty} \frac{1}{|m-\xi|^{\lambda} + |n-\eta|^{\lambda}} a_m \right)^p \right\}^{\frac{1}{p}} \\ &< \frac{2\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})} \left[\sum_{|m|=1}^{\infty} |m-\xi|^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \end{aligned} \tag{3.4}$$

Proof. By Hölder's inequality (cf. [14]) and (2.7), we have

$$\begin{aligned} &\left(\sum_{|m|=1}^{\infty} h(m,n) a_m \right)^p \\ &= \left[\sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{(1-\lambda_1)/q}(m) a_m}{A_{\eta,\beta}^{(1-\lambda_2)/p}(n)} \frac{A_{\eta,\beta}^{(1-\lambda_2)/p}(n)}{A_{\xi,\alpha}^{(1-\lambda_1)/q}(m)} \right]^p \\ &\leq \sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{(1-\lambda_1)p/q}(m)}{A_{\eta,\beta}^{1-\lambda_2}(n)} a_m^p \left[\sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\eta,\beta}^{(1-\lambda_2)q/p}(n)}{A_{\xi,\alpha}^{1-\lambda_1}(m)} \right]^{p-1} \end{aligned}$$

$$= \frac{(\varpi(\lambda_1,n))^{p-1}}{A_{\eta,\beta}^{p\lambda_2-1}(n)} \sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{(1-\lambda_1)p/q}(m)}{A_{\eta,\beta}^{1-\lambda_2}(n)} a_m^p.$$

Then by (2.8) we have

$$\begin{aligned} J &< h_\alpha^{\frac{1}{q}}(\lambda_1) \left[\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{(1-\lambda_1)p/q}(m)}{A_{\eta,\beta}^{1-\lambda_2}(n)} a_m^p \right]^{\frac{1}{p}} \\ &= h_\alpha^{\frac{1}{q}}(\lambda_1) \left[\sum_{|m|=1}^{\infty} \sum_{|n|=1}^{\infty} h(m,n) \frac{A_{\xi,\alpha}^{(1-\lambda_1)p/q}(m)}{A_{\eta,\beta}^{1-\lambda_2}(n)} a_m^p \right]^{\frac{1}{p}} \\ &= h_\alpha^{\frac{1}{q}}(\lambda_1) \left[\sum_{|m|=1}^{\infty} \omega(\lambda_2,m) A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p \right]^{\frac{1}{p}}. \end{aligned} \tag{3.5}$$

In view of (2.8) we have (3.3).

By Hölder's inequality (cf. [14]), we have

$$\begin{aligned} I &= \sum_{|n|=1}^{\infty} \left[A_{\eta,\beta}^{\lambda_2-\frac{1}{p}}(n) \sum_{|m|=1}^{\infty} h(m,n) a_m \right] A_{\eta,\beta}^{\frac{1}{p}-\lambda_2}(n) b_n \\ &\le J \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

Then by (3.3) we have (3.2).

On the other hand, assuming that (3.2) is valid, we set

$$b_n := A_{\eta,\beta}^{p\lambda_2-1}(n) \left(\sum_{|m|=1}^{\infty} h(m,n) a_m \right)^{p-1}, \quad |n| \in \mathbf{N}.$$

and then

$$J = \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q \right]^{\frac{1}{p}}.$$

By (3.5) we find $J < \infty$. If $J = 0$, then (3.3) is evidently valid; if $J > 0$, then by (3.2), we have

$$\begin{aligned} 0 &< \sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q = J^p = I \\ &< K_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q \right]^{\frac{1}{q}}, \end{aligned}$$

$$J = \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n) b_n^q \right]^{\frac{1}{p}} < K_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m) a_m^p \right]^{\frac{1}{p}},$$

namely, (3.3) follows, which is equivalent to (3.2). □

Theorem 3.2. *With regards to the assumptions of Theorem 3.1, the constant factor $K_{\alpha,\beta}(\lambda_1)$ in (3.2) and (3.3) is the best possible.*

Proof. For $\varepsilon \in (0, q\lambda_2)$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}(> 0), \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (0,1))$, and

$$\widetilde{a}_m := A_{\xi,\alpha}^{(\lambda_1-\frac{\varepsilon}{p})-1}(m) = A_{\xi,\alpha}^{\widetilde{\lambda}_1-\varepsilon-1}(m)\,(|m| \in \mathbf{N}),$$
$$\widetilde{b}_n := A_{\eta,\beta}^{(\lambda_2-\frac{\varepsilon}{q})-1}(n) = A_{\eta,\beta}^{\widetilde{\lambda}_2-1}(n)\ (|n| \in \mathbf{N}).$$

Then by (2.13) and (2.8), we find

$$\begin{aligned}
\widetilde{I}_1 &:= \left[\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m)\widetilde{a}_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{q(1-\lambda_2)-1}(n)\widetilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left(\sum_{|m|=1}^{\infty} A_{\xi,\alpha}^{-1-\varepsilon}(m)\right)^{\frac{1}{p}} \left(\sum_{|n|=1}^{\infty} A_{\eta,\beta}^{-1-\varepsilon}(n)\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}} (1+o_1(1))^{\frac{1}{p}} \\
&\times \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}} (1+o_2(1))^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} h(m,n)\widetilde{a}_m\widetilde{b}_n = \sum_{|m|=1}^{\infty}\sum_{|m|=1}^{\infty} h(m,n)\frac{A_{\xi,\alpha}^{\widetilde{\lambda}_1-\varepsilon-1}(m)}{A_{\eta,\beta}^{1-\widetilde{\lambda}_2}(n)} \\
&= \sum_{|m|=1}^{\infty} \frac{\omega(\widetilde{\lambda}_2, m)}{A_{\xi,\alpha}^{1+\varepsilon}(m)} > h_\beta(\widetilde{\lambda}_1)\sum_{|m|=1}^{\infty} \frac{1-\theta(\widetilde{\lambda}_2, m)}{A_{\xi,\alpha}^{1+\varepsilon}(m)} \\
&= h_\beta(\widetilde{\lambda}_1)\left[\sum_{|m|=1}^{\infty}\frac{1}{A_{\xi,\alpha}^{1+\varepsilon}(m)} - \sum_{|m|=1}^{\infty}\frac{1}{O(A_{\xi,\alpha}^{(\frac{\varepsilon}{p}+\lambda_2)+1}(m))}\right] \\
&= \frac{h_\beta(\widetilde{\lambda}_1)}{\varepsilon}\left\{[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}](1+o_1(1)) - \varepsilon O(1)\right\}.
\end{aligned}$$

If there exists a constant $k \le K_{\alpha,\beta}(\lambda_1)$, such that (3.2) is valid when replacing $K_{\alpha,\beta}(\lambda_1)$ by k, then in particular, we have $\varepsilon\widetilde{I} < k\widetilde{I}_1$, namely,

$$\begin{aligned}
&h_\beta(\widetilde{\lambda}_1)\left\{[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}](1+o_1(1)) - \varepsilon O(1)\right\} \\
&< k\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}} (1+o_1(1))^{\frac{1}{p}} \\
&\times \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}} (1+o_2(1))^{\frac{1}{q}}.
\end{aligned}$$

It follows that

$$4k_s(\lambda_1)\csc^2\beta\csc^2\alpha \le 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta\ (\varepsilon \to 0^+),$$

namely, $K_{\alpha,\beta}(\lambda_1) = 2k_s(\lambda_1)\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha \le k$. Hence, $k = K_{\alpha,\beta}(\lambda_1)$ is the best possible constant factor in (3.2). The constant factor $K_{\alpha,\beta}(\lambda_1)$ in (3.3) is still the best possible. Otherwise, we would reach a contradiction by (3.6) that the constant factor in (3.2) is not the best possible. □

4. Operator expressions and a few particular cases

For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we set functions $\Phi(m)$ and $\Psi(n)$ as follows:

$$\Phi(m) := A_{\xi,\alpha}^{p(1-\lambda_1)-1}(m)\ (|m| \in \mathbf{N}), \Psi(n) := A_{\eta,\beta}^{q(1-\lambda_2)-1}(n)\ (|n| \in \mathbf{N}),$$

wherefrom, $\Psi^{1-p}(n) = A_{\eta,\beta}^{p\lambda_2-1}(n)\ (|n| \in \mathbf{N})$. We also set the following weight normed spaces:

$$l_{p,\Phi} := \left\{ a = \{a_m\}_{|m|=1}^{\infty}; ||a||_{p,\Phi} = \left(\sum_{|m|=1}^{\infty} \Phi(m)|a_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty}; ||b||_{q,\Psi} = \left(\sum_{|n|=1}^{\infty} \Psi(n)|b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty}; ||c||_{p,\Psi^{1-p}} = \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)|c_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Then for $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}, c = \{c_n\}_{|n|=1}^{\infty}, c_n = \sum_{|m|=1}^{\infty} h(m,n)a_m$, in view of (3.3), we have $||c||_{p,\Psi^{1-p}} < K_{\alpha,\beta}(\lambda_1)||a||_{p,\Phi}$, namely, $c \in l_{p,\Psi^{1-p}}$.

Definition 4.1. Define a Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: for any $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, there exists a unique representation $c = Ta \in l_{p,\Psi^{1-p}}$. We also define the formal inner product of Ta and $b = \{b_n\}_{|n|=1}^{\infty} \in l_{q,\Psi}$ $(b_n \ge 0)$ as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} h(m,n)a_m b_n. \tag{4.1}$$

Then for $a_m \ge 0$ $(|m| \in \mathbf{N})$, we may rewrite (3.2) and (3.3) as the following equivalent forms:

$$(Ta, b) < K_{\alpha,\beta}(\lambda_1)||a||_{p,\Phi}||b||_{q,\Psi}, \tag{4.2}$$

$$||Ta||_{p,\Psi^{1-p}} < K_{\alpha,\beta}(\lambda_1)||a||_{p,\Phi}. \tag{4.3}$$

We define the norm of operator T as follows:

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\Phi}} \frac{||Ta||_{p,\Psi^{1-p}}}{||a||_{p,\Phi}}. \tag{4.4}$$

Since by Theorem 3.2, the constant factor $K_{\alpha,\beta}(\lambda_1)$ in (4.3) is the best possible, we have

$$||T|| = K_{\alpha,\beta}(\lambda_1) = 2k_s(\lambda_1)\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha. \tag{4.5}$$

Remark 4.1. (i) For $\xi = \eta = 0$, (3.2) reduces to

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}[(|m| + m\cos\alpha)^{\lambda/s} + c_k(|n| + n\cos\beta)^{\lambda/s}]}$$

$$< K_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \tag{4.6}$$

Hence, (3.2) is an extension of (4.6). In particular, for $\alpha = \beta = \frac{\pi}{2}$ in (4.6), we have the following new inequality:

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}(|m|^{\lambda/s} + c_k|n|^{\lambda/s})}$$

$$< 2k_s(\lambda_1) \left[\sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \tag{4.7}$$

(ii) For $\alpha = \beta = \frac{\pi}{2}$, $a_{-m} = a_m, b_{-n} = b_n$ $(m, n \in \mathbf{N})$ in (3.2), we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \frac{1}{\prod_{k=1}^{s}[(m-\xi)^{\lambda/s} + c_k(n-\eta)^{\lambda/s}]} \right.$$

$$+ \frac{1}{\prod_{k=1}^{s}[(m+\xi)^{\lambda/s} + c_k(n-\eta)^{\lambda/s}]} + \frac{1}{\prod_{k=1}^{s}[(m+\xi)^{\lambda/s} + c_k(n+\eta)^{\lambda/s}]}$$

$$\left. + \frac{1}{\prod_{k=1}^{s}[(m-\xi)^{\lambda/s} + c_k(n+\eta)^{\lambda/s}]} \right\} a_m b_n$$

$$< 2k_s(\lambda_1) \left\{ \sum_{m=1}^{\infty} \left[(m-\xi)^{p(1-\lambda_1)-1} + (m+\xi)^{p(1-\lambda_1)-1}\right] a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} [(n-\eta)^{q(1-\lambda_2)-1} + (n+\eta)^{q(1-\lambda_2)-1}] b_n^q \right\}^{\frac{1}{q}}. \tag{4.8}$$

In particular, for $\xi = \eta = 0$, we have the following new Hilbert-type inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\prod_{k=1}^{s}(m^{\lambda/s} + c_k n^{\lambda/s})}$$

$$< k_s(\lambda_1) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{4.9}$$

which is an extension of (1.1). It follows that (3.2) is an extended inequality of (1.1) in the whole plane.

5. Conclusions

In this paper, by means of weight coefficients, a complex integral formula and Hermite-Hadamard's inequality, a new extended Hardy-Hilbert's inequality in the whole plane with multi-parameters and a best possible constant factor is given by Theorem 3.1 and Theorem 3.2. The equivalent forms, the operator expressions and a few particular cases are considered by Theorem 3.1, (4.2)–(4.5) and Remark 4.1. The lemmas and theorems can provide an extensive account of this type of inequalities.

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