AN APPLICATION OF JACK-FUKUI-SAKAGUCHI LEMMA

Mamoru Nunokawa\textsuperscript{1}, Janusz Sokół\textsuperscript{2,†} and Huo Tang\textsuperscript{3}

Abstract We present some applications of Jack-Fukui-Sakaguchi Lemma which become sufficient criteria for a function to be in the class of strongly starlike, strongly close-to-convex or in the other classes.

Keywords Bazilević function, close-to-convex functions, convex functions, starlike functions.


1. Introduction

Let $\mathcal{A}$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$\hspace{1cm} (1.1)

which are analytic in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions in $D$. If $f \in \mathcal{A}$ satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \; z \in D,$$

then $f(z)$ is said to be starlike with respect to the origin in $D$ and it is denoted by $f(z) \in \mathcal{S}^*$. It is known that $\mathcal{S}^* \subset \mathcal{S}$. For further properties of starlike functions and other functions having a geometric property we refer to [3, 11]. To prove the main results we apply techniques of differential subordinations widely described in the book [10]. We say that an analytic function $f(z)$ is subordinate to an analytic function $g(z)$, univalent or not, and write $f(z) \prec \prec g(z)$, if and only if there exists a function $\omega(z)$, analytic in $D$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$ and $f(z) = g(\omega(z))$. If we additionally assume that $g(z)$ is univalent in $D$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(|z| < 1) \subset g(|z| < 1). \hspace{1cm} (1.2)$$

The differential subordinations were deeply developed in the monograph [10] as well as in many of recent papers, see for example in [2, 8, 9, 16, 17]. The following lemma is a generalization of well known Jack lemma, [5].

\textsuperscript{†}the corresponding author. Email address:jsokol@ur.edu.pl (J. Sokół)
\textsuperscript{1}Department of Mathematics, University of Gunma, Hoshikuki-cho 798-8, Chau-ward, Chiba, 260-0808, Japan
\textsuperscript{2}College of Natural Sciences, University of Rzeszów, Prof. Pigonia’s Street 1, 35-310 Rzeszów, Poland
\textsuperscript{3}School of Mathematics and Statistics, Chifeng University, Chifeng 024000, Inner Mongolia, China
Lemma 1.1 (4). Let \( w(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \), \( a_p \neq 0 \), \( 1 \leq p \) be analytic in \( \mathbb{D} \). If the maximum of \( |w(z)| \) on the circle \( |z| = r < 1 \) is attained at \( z = z_0 \), then \( z_0 w'(z_0)/w(z_0) \) is a real number and
\[
\frac{z_0 w'(z_0)}{w(z_0)} \geq p.
\]

A related boundary behavior of analytic functions is considered also in [14]. In this paper we present some applications of the above Jack-Fukui-Sakaguchi Lemma to obtain several sufficient criteria for a function to be in the class of strongly starlike, strongly close-to-convex or in the other classes. A related Sakaguchi’s result was recently considered in [12,13].

2. Main results

Theorem 2.1. Let \( q(z) = 1 + c_n z^n + \cdots \) be analytic in \( \mathbb{D} \) with \( c_n \neq 0 \). Assume that for all \( z \in \mathbb{D} \) we have \( q(z) \neq -1 \), \( q(z) \neq 0 \) and for all \( z \in \mathbb{D} \setminus \{0\} \) we have \( q(z) \neq 1 \). Furthermore, suppose that
\[
\left| \frac{zq'(z)}{q(z)} \right| < n, \quad (z \in \mathbb{D}),
\]
for some positive integer \( n \). Then we have
\[
q(z) < \frac{1 + z^n}{1 - z^n}.
\]

Proof. Let us consider the function \( w(z) \) such that
\[
w^n(z) = \begin{cases} \frac{q(z)-1}{q(z)+1}, & z \neq 0, \\ 0, & z = 0. \end{cases}
\]
Then we have
\[
w(z) = \sqrt[n]{c_n} z + \cdots,
\]
which gives
\[
q(z) = \frac{1 + w^n(z)}{1 - w^n(z)},
\]
then it follows that \( w(z) \) is analytic in \( \mathbb{D} \) and to prove (2.2) we need to show \( |w(z)| < 1 \).

From Fukui and Sakaguchi’s Lemma 1.1, we have that if there exists a point \( z_0 \in \mathbb{D} \) such that
\[
|w(z)| < |z_0| \quad \text{for} \quad |z| < |z_0|
\]
and
\[
|w(z_0)| = |z_0| \quad w(z_0) = e^{i\theta},
\]
where \( \theta \) is a real number and \( 0 \leq \theta < 2\pi \), then
\[
\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1.
\]
From (2.3) and (2.4) it follows that $\theta \neq 0$ and $\theta \neq \pi$ and so $w(z_0) \neq \pm 1$. On the other hand, from (2.4), we have

$$\frac{zq'(z)}{q(z)} = \frac{2nzw'(z)w^{n-1}(z)}{1 - w^{2n}(z)}.$$  \hfill (2.7)

Therefore, by (2.5) and (2.6) the equality (2.7) becomes

$$\left| \frac{z_0q'(z_0)}{q(z_0)} \right| = \left| \frac{2nz_0w'(z_0)w^{n-1}(z_0)}{1 - w^{2n}(z_0)} \right| \geq \left| \frac{2nk w^n(z_0)}{1 - w^{2n}(z_0)} \right| = 2nk \left| \frac{1}{1 - e^{2in\theta}} \right| = n.$$  \hfill (2.8)

This contradicts (2.1) and so, it completes the proof. \hfill \Box

Notice that in the subordination (2.2) the function $\left(1 + z^n\right)/\left(1 - z^n\right)$ is not univalent and it makes the calculations more difficult. Usually in $p(z) \prec q(z)$ it is considered univalent function $q(z)$. Several classes of functions connected with subordination under not-univalent function of the type

$$q(z) = \frac{1 + Ae^n}{1 + Bz^n},$$

where $A, B$ are some complex numbers, were considered in [6] and [7].

If $q^\alpha(z) = zf'(z)/f(z)$, the Theorem 2.1 becomes the following corollary.

**Corollary 2.1.** Let $(zf'(z)/f(z))^{1/\alpha} = 1 + c_n z^n + \cdots$, $(zf'(z)/f(z))^{1/\alpha} \neq -1$, $zf'(z)/f(z) \neq 0$ be analytic in $D$ for some positive real $\alpha$. Suppose also that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \alpha n, \quad (z \in D),$$  \hfill (2.8)

for some positive integer $n$. Then we have

$$zf'(z) < \left( \frac{1 + z^n}{1 - z^n} \right)^\alpha.$$  \hfill (2.9)

It is easy to see, that (2.9) implies

$$zf'(z) < \left( \frac{1 + z}{1 - z} \right)^\alpha,$$  \hfill (2.10)

which means that $f(z)$ is strongly starlike functions of order $\alpha$. We say that a function $f \in S^*$ is strongly starlike of order $\beta$ if and only if

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi}{2} \beta, \quad (z \in D),$$

for some $\beta$ ($0 < \beta \leq 1$), where the function arg is chosen with values in an interval between $-\pi$ and $\pi$. Let $SS^*(\beta)$ denote the class of strongly starlike functions of order $\beta$. The class $SS^*(\beta)$ was introduced independently in [18, 19] and in [1]. Recall also, that if there exists a function $g(z) \in S^*$ for which the function $f(z) \in A$ satisfies the condition

$$\left| \arg \left( \frac{zf'(z)}{g(z)} \right) \right| < \frac{\pi}{2} \alpha, \quad (z \in D),$$
then we say that $f(z)$ is strongly close-to-convex of order $\alpha$, $0 < \alpha \leq 1$. For some recent results on strongly starlike functions we refer to [15]. Putting $q^\alpha(z) = zf'(z)/g(z)$ in Theorem 2.1 we get the following sufficient condition for $f(z)$ to be strongly close-to-convex of order $\alpha$.

**Corollary 2.2.** Assume that $f(z) \in \mathcal{A}$, $g(z) \in S^*$ and that for some positive real $\alpha$, $0 < \alpha \leq 1$ the function $(zf'(z)/g(z))^{1/\alpha} = 1 + c_nz^n + \cdots$ is analytic in $D$ with $(zf'(z)/g(z))^{1/\alpha} \neq -1$, $zf'(z)/g(z) \neq 0$. Furthermore, suppose that

$$
\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right| < \alpha n, \quad (z \in D),
$$

for some positive integer $n$. Then we have

$$
\frac{zf'(z)}{g(z)} < \left( \frac{1 + z^n}{1 - z^n} \right)^\alpha,
$$

which follows that $f(z)$ is strongly close-to-convex of order $\alpha$.

Moreover, if there exists a function $g(z) \in S^*$ such that $f(z) \in \mathcal{A}$ satisfies the condition

$$
\left| \arg \left( \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} \right) \right| < \frac{\pi}{2\alpha}, \quad (z \in D),
$$

then we call that $f(z)$ is strongly Bazilevič function of type $\beta$, $0 < \beta$ and of order $\alpha$, $0 < \alpha \leq 1$.

**Corollary 2.3.** Assume that $f(z) \in \mathcal{A}$, $g(z) \in S^*$ and that for some positive real $\alpha$, $0 < \alpha \leq 1$ the function $(zf'(z)/f^{1-\beta}(z)g^\beta(z))^{1/\alpha} = 1 + c_nz^n + \cdots$ is analytic in $D$ with $(zf'(z)/f^{1-\beta}(z)g^\beta(z))^{1/\alpha} \neq -1$, $zf'(z)/f^{1-\beta}(z)g^\beta(z) \neq 0$. Furthermore, suppose that

$$
\left| 1 + \frac{zf''(z)}{f'(z)} - (1 - \beta)\frac{zf'(z)}{f(z)} - \beta\frac{zg'(z)}{g(z)} \right| < \alpha n, \quad (z \in D),
$$

for some positive integer $n$. Then we have

$$
\frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} < \left( \frac{1 + z^n}{1 - z^n} \right)^\alpha,
$$

which follows that $f(z)$ is strongly Bazilevič function of type $\beta$, $0 < \beta$ and of order $\alpha$.

If $g(z)$ is of the form $g(z) = (p(z))^{1/\alpha}$, then Theorem 2.1 becomes the following corollary.

**Corollary 2.4.** Let $(p(z))^{1/\alpha} = 1 + c_nz^n + \cdots$, $(p(z))^{1/\alpha} \neq -1$, $p(z) \neq 0$ be analytic in $D$, and suppose that

$$
\left| \frac{zp'(z)}{p(z)} \right| < \alpha n, \quad (z \in D),
$$

for some positive real $\alpha$ and for some positive integer $n$. Then we have

$$
p(z) < \left( \frac{1 + z^n}{1 - z^n} \right)^\alpha.
$$
Theorem 2.2. Let \((\log \{p(z)\})^{1/\alpha} = 1 + c_1 z + \cdots, \log \{p(z)\} \neq 0, (\log \{p(z)\})^{1/\alpha} \neq -1\), be analytic in \(\mathbb{D}\) and suppose that
\[
\frac{|zp'(z)|}{p(z)} < \frac{\alpha}{2 \left( \frac{1+\alpha}{2} \right)^{(1+\alpha)/2} + \left( \frac{1-\alpha}{2} \right)^{(1-\alpha)/2}}, \quad (z \in \mathbb{D}), \quad (2.17)
\]
for some positive real \(\alpha < 1\). Then we have \(p(z) = e + d_1 z + \cdots\) and
\[
p(z) < e^{\left( \frac{i+\omega(z)}{2} \right)^\alpha}, \quad (z \in \mathbb{D}). \quad (2.18)
\]

Proof. Let us put
\[
w(z) = \frac{(\log \{p(z)\})^{1/\alpha} - 1}{(\log \{p(z)\})^{1/\alpha} + 1}, \quad w(0) = 0 \quad (2.19)
\]
or
\[
p(z) = e^{\left( \frac{i+\omega(z)}{2} \right)^\alpha}, \quad (2.20)
\]
then it follows that \(w(z)\) is analytic in \(\mathbb{D}\), \(w(0) = 0\) and
\[
zp'(z) = \alpha z \left( \frac{1+w(z)}{1-w(z)} \right) \left( \frac{1+w(z)}{1-w(z)} \right)^{\alpha-1} = \frac{2\alpha zw'(z)}{(1-w(z))^2} \left( \frac{1+w(z)}{1-w(z)} \right)^{\alpha-1} = \frac{2\alpha zw'(z)}{1-w^2(z)} \left( \frac{1+w(z)}{1-w(z)} \right)^{\alpha}.
\]

To prove (2.18) we need \(|w(z)| < 1\) in \(\mathbb{D}\). If there exists a point \(z_0 \in \mathbb{D}\) such that
\[
|w(z)| < 1 \quad \text{for} \quad |z| < |z_0|
\]
and
\[
|w(z_0)| = 1, \quad w(z_0) = e^{i\theta}, \quad (2.21)
\]
for some real \(\theta, \theta \in [0, 2\pi) \setminus \{0, \pi\}\) because from the hypothesis and from (2.19) it follows that \(w(z_0) \neq \pm 1\). Then from Jack [5] and Fukui and Sakaguchi’s [4] Lemma 1.1, we have that
\[
\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1.
\]

Then it follows that
\[
\frac{z_0p'(z_0)}{p(z_0)} = \frac{2\alpha z_0 w'(z_0)}{1-w^2(z_0)} \left( \frac{1+w(z_0)}{1-w(z_0)} \right) = 2\alpha k \frac{w(z_0)}{1-w^2(z_0)} \left( \frac{1+w(z_0)}{1-w(z_0)} \right)\]
and for \(\theta \in (0, 2\pi) \setminus \{\pi\}\)
\[
\frac{w(z_0)}{1-w^2(z_0)} = \frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{i}{2\sin \theta}, \quad \frac{1+w(z_0)}{1-w(z_0)} = \frac{2i \sin \theta}{2(1-\cos \theta)} = i \frac{\cos(\theta/2)}{\sin(\theta/2)}.
\]
Therefore, we have
\[
\left| \frac{2z_0w'(z_0)}{1 - w^2(z_0)} \left( \frac{1 + w(z_0)}{1 - w(z_0)} \right)^\alpha \right| = \left| \frac{i}{2 \sin \theta} \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|^{\alpha} \right| = \frac{1}{2} \left( \frac{1}{\left| \sin(\theta/2) \right|} \right) \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|^{\alpha} = \frac{1}{2} \left( \frac{1}{\left| \sin(\theta/2) \right|^{1+\alpha} | \cos(\theta/2)|^{1-\alpha} } \right).
\]

Putting
\[
g(x) = (\sin x)^{1+\alpha} (\cos x)^{1-\alpha}, \quad 0 < x < \pi/2
\]
\[
h(x) = (\sin x)^{1+\alpha} (-\cos x)^{1-\alpha}, \quad \pi/2 < x < \pi
\]
shows that
\[
g'(x) = (1 + \alpha) \left( \frac{\sin x}{\cos x} \right)^\alpha \left\{ \cos^2 x - \frac{1 - \alpha}{1 + \alpha} \sin^2 x \right\} = (1 + \alpha) \left( \frac{\sin x}{\cos x} \right)^\alpha \left\{ 1 - \frac{2}{1 + \alpha} \sin^2 x \right\}.
\]
Therefore, for \(0 \leq x < \pi/2\) we have
\[
g'(x) = 0 \iff \left( \sin x = 0 \lor \sin x = \sqrt{\frac{1 + \alpha}{2}} \right)
\]
which gives
\[
g'(x) = 0 \iff \left( x = 0 \lor x = \sin^{-1} \sqrt{\frac{1 + \alpha}{2}} \right).
\]
It follows that
\[
\max_{0 < x < \pi/2} |g(x)| = \left( \frac{1 + \alpha}{2} \right)^{(1+\alpha)/2} + \left( \frac{1 - \alpha}{2} \right)^{(1-\alpha)/2}.
\]
Also
\[
\max_{\pi/2 < x < \pi} |h(x)| = \left( \frac{1 + \alpha}{2} \right)^{(1+\alpha)/2} + \left( \frac{1 - \alpha}{2} \right)^{(1-\alpha)/2}.
\]
Therefore, we have
\[
\left| \frac{z_0p'(z_0)}{p(z_0)} \right| \geq \frac{\alpha k}{2 \left( \frac{1+\alpha}{2} \right)^{(1+\alpha)/2} \left( \frac{1-\alpha}{2} \right)^{(1-\alpha)/2}} \geq \frac{\alpha}{2 \left( \frac{1+\alpha}{2} \right)^{(1+\alpha)/2} \left( \frac{1-\alpha}{2} \right)^{(1-\alpha)/2}}.
\]
This contradicts (2.17) and so, we have (2.18).

\section*{References}

An application of Jack-Fukui-Sakaguchi


