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ASYMPTOTIC AUTONOMY OF RANDOM ATTRACTORS FOR BBM EQUATIONS WITH LAPLACE-MULTIPLIER NOISE*

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Abstract We study asymptotic autonomy of random attractors for possibly non-autonomous Benjamin-Bona-Mahony equations perturbed by Laplace-multiplier noise. We assume that the time-indexed force converges to the time-independent force as the time-parameter tends to negative infinity, and then show that the time-indexed force is backward tempered and backward tail-small. These properties allow us to show that the asymptotic compactness of the non-autonomous system is uniform in the past, and then obtain a backward compact random attractor when the attracted universe consists of all backward tempered sets. More importantly, we prove backward convergence from time-fibers of the non-autonomous attractor to the autonomous attractor. Measurability of solution mapping, absorbing set and attractor is rigorously proved by using Egoroff, Lusin and Riesz theorems.

Keywords Random attractor, asymptotic autonomy, backward compactness, Benjamin-Bona-Mahony equation, Laplace-multiplier noise, backward tempered set, measurability.

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1. Introduction

We develop a new subject on **asymptotic autonomy** of random attractors for the following non-autonomous stochastic Benjamin-Bona-Mahony (BBM) equation:

$$\begin{cases} du - d(\Delta u) - \nu \Delta u dt + \nabla \cdot \overrightarrow{F}(u) dt = g(t,x)dt + \mathcal{S}u \circ dW, \\ u(t,\tau)|_{\partial Q} = 0, \quad u(\tau,\tau,x) = u_\tau(x), \ x \in Q, \ t \geq \tau, \ \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\nu > 0$, $\mathcal{S} = I - \Delta$ and Q is an unbounded 3D-channel: $Q = D \times \mathbb{R}$, D is bounded in $\mathbb{R}^2$.

When the equation is deterministic ($\mathcal{S} = 0$) and autonomous ($g(t) \equiv g_\infty \in L^2(Q)$), it was first proposed in [3] as a nonlinear dispersive model to describe the physical phenomenon of long waves in shallow water. Both well-posedness and global attractor had been extensively investigated (cf. [1,10,12,14,29,33]). Wang [31] obtained a random attractor for the BBM equation with additive noise ($\mathcal{S}u = h$).

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We take the *Laplace-multiplier noise* ($\mathcal{S} = I - \Delta$) instead of the usual multiplicative noise ($\mathcal{S} = I$, see [5, 9, 13, 18, 21, 22, 44]). From the viewpoint of physics, this operator-type noise vibrates in resonance with the dispersive wave ($d(\Delta u)$). From the viewpoint of mathematics, it is possible to translate the stochastic equation with Laplace-multiplier noise such that the differential of the Wiener process W disappears. Therefore, we can obtain a *non-autonomous random dynamical system* (NRDS) Φ in the sense of Wang [32], where, measurability of the system is rigorously proved by showing Lusin continuity in the sample, see Proposition 2.1.

The main purpose of this paper is to consider not only existence of a non-autonomous random attractor $\mathcal{A} = \{\mathcal{A}(\tau, \omega)\}$, but also upper semi-continuity from $\mathcal{A}(\tau, \omega)$ to $\mathcal{A}_\infty(\omega)$ as $\tau \to -\infty$, that is,

$$\lim_{\tau \to -\infty} \operatorname{dist}_{H_0^1(Q)}(\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \ P\text{-a.s. } \omega \in \Omega, \tag{1.2}$$

where, Ω is a probability space, and $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega)\}$ is the random attractor (obtained by [25]) for the RDS Φ_∞ generated from the autonomous BBM equation with the time-independent force $g_\infty(x)$ instead of $g(t, x)$ in (1.1).

Such an asymptotically autonomous problem in the non-random case (omitting the sample in (1.2)) had been investigated by Kloeden et al. [15–17] or [6, 11]. They established some abstract results by using the uniform convergence of the system and the uniform compactness of the pullback attractor. Two uniformness conditions had been reduced by Li et al. [23], in which, it was shown that the asymptotic autonomy only relates to backward or forward compactness of a pullback attractor.

The above abstract results can be partly generalized to the random case, where we must consider variety of the sample. In fact, in order to establish the asymptotic autonomy as given in (1.2), on one hand, we need to show the convergence from the NRDS Φ to the RDS Φ_∞, on the other hand, we need to show that the NRDS Φ is *backward asymptotically compact*, which means that the asymptotic compactness is uniform in the past, see Theorem 5.1.

Interestingly, the above two properties can be available by using only one assumption on two forces.

Hypothesis G. (*Convergence condition*). $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(Q))$ and $g_\infty \in L^2(Q)$ such that

$$\lim_{\tau \to -\infty} \int_{-\infty}^{\tau} \|g(s) - g_\infty\|^2 ds = 0, \quad \text{where } \|\cdot\| \text{ is the } L^2\text{-norm.} \tag{1.3}$$

In fact, under the hypothesis **G**, we can prove that Φ backward converges to Φ_∞, see Lemma 2.2. Moreover, we can show that the hypothesis **G** can imply that the time-dependent force g is backward tempered and backward tail-small (see Lemma 2.1). These properties are enough to ensure that the random attractor $\mathcal{A}(\tau, \omega)$ is *backward compact*, which means that $\cup_{s \leq \tau} \mathcal{A}(s, \omega)$ is pre-compact.

Another novelty is the option of attracted universes. A bi-parametric set $\mathcal{D} = \{\mathcal{D}(\tau, \omega)\}$ in $H_0^1(Q)$ is called *backward tempered* if

$$\lim_{t \to +\infty} e^{-\frac{\delta}{3} t} \sup_{s \leq \tau} \|\mathcal{D}(s - t, \theta_{-t}\omega)\|^2_{H^1} = 0, \ \forall \ (\tau, \omega) \in \mathbb{R} \times \Omega, \tag{1.4}$$

where $\delta = \min(\frac{\nu}{2}, \frac{\nu\lambda_0}{4})$ and λ_0 is the Poincaré constant. We take the universe $\mathfrak{D}$ by the collection of all backward tempered sets, instead of the usual tempered sets (i.e. the supremum in (1.4) is omitted, see [2, 19, 30, 37, 39, 41–43]), also, instead of the bounded sets [15, 16, 22]. The usual universe cannot work when proving the backward asymptotic compactness of the NRDS Φ.

A difficulty arises from proving measurability of the absorbing set, which is a union of some random sets over an uncountable index set. Fortunately, both Egoroff and Lusin theorems can solve the problem, see Proposition 3.1.

The tail-estimates can be realized by using square of the usual cut-off function and by treating carefully the biquadrate of solutions. Those tail-estimates are further proved to be uniform in the past.

Final application results are summarized in Theorem 5.2, where we show backward compactness and asymptotic autonomy of the random attractor for Eq.(1.1). Furthermore, Riesz theorem and measure-preserving property imply the convergence of $\mathcal{A}(s_n, \theta_{s_n}\omega)$ as $s_n \to -\infty$, where the sample is varying.

2. NRDS from BBM equations

2.1. Two backward properties of the time-indexed force

We show that the hypothesis **G** can imply the following conditions.

(I) g is **tempered**: $\int_{-\infty}^{0} e^{ar}\|g(r)\|^2 dr < +\infty$ for all $a > 0$. This is a common condition to ensure the existence of a pullback attractor, see [20, 28] and the references therein.

(II) g is **backward tempered**: for all $a > 0$ and $\tau \in \mathbb{R}$,

$$G(a,\tau) = \sup_{s\leq\tau} \int_{-\infty}^{s} e^{a(r-s)}\|g(r)\|^2 dr < +\infty.$$

This is a basic condition to guarantee existence of a backward compact attractor, see [7, 38] for deterministic PDEs.

(III) g is **backward tail-small**: for all $a > 0$ and $\tau \in \mathbb{R}$,

$$\lim_{k\to+\infty} \sup_{s\leq\tau} \int_{-\infty}^{s} e^{a(r-s)} \int_{Q(|x_3|\geq k)} |g(r,x)|^2 dx dr = 0.$$

This is a condition to ensure the existence of a backward compact attractor when a PDE is defined on an unbounded domain, see [24, 26, 34, 40] for some deterministic PDEs.

Lemma 2.1. *Let the time-indexed force g satisfy the hypothesis* **G**. *Then,*

(i) g is backward tempered, which obviously implies that g is tempered.

(ii) g is backward tail-small.

Proof. (i) Let $a > 0$ and $\tau \in \mathbb{R}$. By (1.3), we can find a $\tau_0 < \tau$ such that $\int_{-\infty}^{\tau_0} \|g(r) - g_\infty\|^2 dr < 1$. By $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(Q))$ and $g_\infty \in L^2(Q)$, we have

$$\begin{aligned}\int_{-\infty}^{\tau} \|g(r) - g_\infty\|^2 dr &= \int_{-\infty}^{\tau_0} \|g(r) - g_\infty\|^2 dr + \int_{\tau_0}^{\tau} \|g(r) - g_\infty\|^2 dr \\ &\leq 1 + 2\int_{\tau_0}^{\tau} \|g(r)\|^2 dr + 2(\tau - \tau_0)\|g_\infty\|^2 < +\infty. \qquad (2.1)\end{aligned}$$

Therefore, by $e^{a(r-s)} \leq 1$ for all $r \leq s$, we have

$$G(a,\tau) = \sup_{s\leq\tau} \int_{-\infty}^{s} e^{a(r-s)}\|g(r)\|^2 dr$$

$$\leq 2\sup_{s\leq\tau}\int_{-\infty}^{s} e^{a(r-s)}\|g(r)-g_\infty\|^2 dr + 2\sup_{s\leq\tau}\int_{-\infty}^{s} e^{a(r-s)}\|g_\infty\|^2 dr$$
$$\leq 2\sup_{s\leq\tau}\int_{-\infty}^{s}\|g(r)-g_\infty\|^2 dr + 2\|g_\infty\|^2\int_{-\infty}^{0} e^{ar}dr$$
$$\leq 2\int_{-\infty}^{\tau}\|g(r)-g_\infty\|^2 dr + \frac{2}{a}\|g_\infty\|^2 < +\infty.$$

(ii) It is similar to the above proof that for each $k \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{s\leq\tau}\int_{-\infty}^{s} e^{a(r-s)}\int_{Q(|x_3|\geq k)}|g(r,x)|^2 dxdr\\
\leq& 2\sup_{s\leq\tau}\int_{-\infty}^{s} e^{a(r-s)}\int_{Q(|x_3|\geq k)}|g(r,x)-g_\infty(x)|^2 dxdr\\
&+2\sup_{s\leq\tau}\int_{-\infty}^{s} e^{a(r-s)}\int_{Q(|x_3|\geq k)}|g_\infty(x)|^2 dxdr\\
\leq& 2\int_{-\infty}^{\tau}\int_{Q(|x_3|\geq k)}|g(r,x)-g_\infty(x)|^2 dxdr + \frac{2}{a}\int_{Q(|x_3|\geq k)}|g_\infty(x)|^2 dx.
\end{aligned} \tag{2.2}$$

Let $h(r) = \|g(r)-g_\infty\|^2$. By (2.1), we know $\int_{-\infty}^{\tau} h(r)dr < +\infty$. Note that

$$h_k(r) := \int_{Q(|x_3|\geq k)}|g(r,x)-g_\infty(x)|^2 dx \leq h(r), \text{ and } h_k(r)\to 0,$$

as $k\to\infty$ for all $r\in(-\infty,\tau]$. The Lebesgue controlled convergence theorem gives

$$\lim_{k\to\infty}\int_{-\infty}^{\tau} h_k(r)dr = 0.$$

By the absolute continuity of the integral, we have $\int_{Q(|x_3|\geq k)}|g_\infty(x)|^2dx \to 0$ as $k\to\infty$. Thereby, (2.2) implies that g is backward tail-small as required. □

We give the nonlinearity assumption as follows.

Hypothesis F. (*Nonlinearity condition*). $\overrightarrow{F}(s) := (F_1(s), F_2(s), F_3(s))$ such that all components F_k are smooth, and for two constants $\gamma_1,\gamma_2 > 0$,

$$F_k(0)=0,\quad |F_k'(s)|\leq \gamma_1+\gamma_2|s|,\ \ s\in\mathbb{R},\ \ k=1,2,3. \tag{2.3}$$

By the hypothesis **F**, the nonlinearity $\overrightarrow{F}$ has the following properties (cf. [25,31]):

Lemma 2.2. *(a) Let $f_k(s) := \int_0^s F_k(t)dt$ for $k=1,2,3$. Then,*

$$|F_k(s)|\leq\gamma_1|s|+\gamma_2|s|^2,\quad |f_k(s)|\leq\gamma_1|s|^2+\gamma_2|s|^3,\ \ k=1,2,3. \tag{2.4}$$

(b) For all $u_1,u_2,u_3\in H_0^1(Q)$, we have

$$\begin{aligned}
&|(\nabla\cdot\overrightarrow{F}(u_1)-\nabla\cdot\overrightarrow{F}(u_2),u_3)|\\
\leq& c(1+\|u_1\|_{H^1}+\|u_2\|_{H^1})\|\|u_1-u_2\|_{H^1}\|u_3\|_{H^1}.
\end{aligned} \tag{2.5}$$

2.2. Lusin continuity and measurability of systems

We identify the Wiener process $W(\cdot,\omega)$ with $\omega(\cdot)$ on the metric dynamical system $(\Omega,\mathcal{F},P,\theta_t)$, where,

$$\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\ \omega(0)=0,\lim_{t\to\pm\infty}\frac{w(t)}{t}=0\},$$

equipped with the Frechét metric: given $\omega_1,\omega_2\in\Omega$,

$$\rho(\omega_1,\omega_2):=\sum_{i=1}^{\infty}\frac{1}{2^n}\frac{\|\omega_1-\omega_2\|_n}{1+\|\omega_1-\omega_2\|_n},\ \|\omega_1-\omega_2\|_n:=\sup_{-n\le t\le n}|\omega_1(t)-\omega_2(t)|.$$

$\mathcal{F}$ is the Borel sigma-algebra on (Ω,ρ), P is the two-sided Wiener measure on $(\Omega,\mathcal{F})$ and θ_t is a group defined by $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$ for all $(\omega,t)\in\Omega\times\mathbb{R}$.

It is well known that $z(\theta_t\omega)=-\int_{-\infty}^{0}e^{\tau}(\theta_t\omega)(\tau)d\tau$ is the pathwise-continuous solution of the stochastic equation $dz+zdt=dW(t)$. Also,

$$\lim_{t\to\pm\infty}\frac{z(\theta_t\omega)}{t}=\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds=0,\ \text{for each }\omega\in\Omega. \tag{2.6}$$

In order to deal with *Laplace-multiplier noise*, we make an exponential change of variables:

$$v(t,\tau,\omega,v_\tau):=e^{-z(\theta_t\omega)}u(t,\tau,\omega,u_\tau). \tag{2.7}$$

In this case, we have

$$\begin{aligned}\mathcal{S}u\circ dW=&e^{z(\theta_t\omega)}v(t,\omega)\circ dz(\theta_t\omega)+z(\theta_t\omega)e^{z(\theta_t\omega)}v(t,\omega)dt\\&-e^{z(\theta_t\omega)}\Delta v(t,\omega)\circ dz(\theta_t\omega)-z(\theta_t\omega)e^{z(\theta_t\omega)}\Delta v(t,\omega)dt.\end{aligned}$$

We substitute the above equality into Eq.(1.1) to find that

$$\begin{aligned}&v_t-\Delta v_t-\nu\Delta v\\&=-e^{-z(\theta_t\omega)}\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}v)+z(\theta_t\omega)(v-\Delta v)+e^{-z(\theta_t\omega)}g(t,x),\end{aligned} \tag{2.8}$$

with the initial conditions: $v(\tau,\tau,\omega,v_\tau)=v_\tau=e^{-z(\theta_\tau\omega)}u_\tau$.

We establish some energy inequalities, which will be useful frequently.

Lemma 2.3. *The solution of Eq.*(2.8) *satisfies: given* $\delta:=\min(\frac{\nu}{2},\frac{\nu\lambda_0}{4})$,

$$\frac{d}{dt}\|v\|_{H^1}^2+(\delta-2z(\theta_t\omega))\|v\|_{H^1}^2\le\frac{2}{\nu\lambda_0}e^{2|z(\theta_t\omega)|}\|g(t)\|^2. \tag{2.9}$$

Proof. Taking the inner product of Eq.(2.8) with v in $L^2(Q)$, we have

$$\begin{aligned}\frac{d}{dt}\|v\|_{H^1}^2+2\nu\|\nabla v\|^2=&-2e^{-z(\theta_t\omega)}\int_Q v\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}v)dx\\&+2z(\theta_t\omega)\|v\|_{H^1}^2+2e^{-z(\theta_t\omega)}\int_Q g(t,x)vdx.\end{aligned}$$

By the boundary condition in (1.1) and $f_i(0) = 0$ for $i = 1, 2, 3$,

$$\int_Q u\nabla \cdot \overrightarrow{F}(u)dx = -\int_Q \overrightarrow{F}(u)\cdot \nabla u = -\int_Q \nabla \cdot \overrightarrow{f}(u) = -\int_{\partial Q} \overrightarrow{f}(u)\cdot \overrightarrow{n} = 0,$$

where $\overrightarrow{n}$ is the outer unit normal vector. Then, by the Poincaré inequality $\|\nabla v\|^2 \geq \lambda_0\|v\|^2$, the energy inequality (2.9) follows immediately. □

Based on the above estimate, the similar argument as given in [29] shows the well-posedness.

Lemma 2.4. *For each $(\tau, \omega, v_\tau) \in \mathbb{R}\times\Omega\times H_0^1(Q)$, the problem (2.8) has a unique solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), H_0^1(Q))$ with $v(\tau, \tau, \omega, v_\tau) = v_0$ such that v is continuous in $H_0^1(Q)$ with respect to v_τ.*

By the *sub-exponential growth* of $\omega(\cdot)$ (see [4, Lemma 11]), we can write $\Omega = \cup_{N\in\mathbb{N}}\Omega_N$, where,

$$\Omega_N := \{\omega \in \Omega : |\omega(t)| \leq Ne^{|t|},\ \forall t \in \mathbb{R}\},\ \forall N \in \mathbb{N}. \tag{2.10}$$

Slightly generalizing [8, Corollary 22], we have the following continuity on each closed subspace Ω_N of Ω. The proof is similar and so omitted.

Lemma 2.5. *For each $N \in \mathbb{N}$, suppose $\omega_k, \omega_0 \in \Omega_N$ such that $\rho(\omega_k, \omega_0) \to 0$ as $k \to \infty$. Then, for each $(\tau, T) \in \mathbb{R}\times\mathbb{R}^+$,*

$$\sup_{t\in[\tau,\tau+T]}\Big(|z(\theta_t\omega_k) - z(\theta_t\omega_0)| + |e^{z(\theta_t\omega_k)} - e^{z(\theta_t\omega_0)}|\Big) \to 0 \quad \textit{as } k \to \infty, \tag{2.11}$$

$$\sup_{k\in\mathbb{N}}\sup_{t\in[\tau,\tau+T]}|z(\theta_t\omega_k)| \leq C(\tau, T, \omega_0). \tag{2.12}$$

Now, we can show the Lusin continuity of the solution mapping in samples.

Proposition 2.1. *For each $N \in \mathbb{N}$, the mapping $\omega \mapsto v(t, \tau, \omega, v_\tau)$ is continuous from (Ω_N, ρ) to $H_0^1(Q)$, uniformly in $t \in [\tau, \tau+T]$, $T > 0$.*

Proof. Suppose $\omega_k, \omega_0 \in \Omega_N$ such that $\rho(\omega_k, \omega_0) \to 0$ as $k \to \infty$. Let $V_k := v_k - v_0$ with $v_k = v(t, \tau, \omega_k, v_\tau)$ and $v_0 = v(t, \tau, \omega_0, v_\tau)$ for $t \in [\tau, \tau+T]$. It deduces from (2.8) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V_k\|_{H^1}^2 + \nu\|\nabla V_k\|^2 =& I_k + (z(\theta_t\omega_k)(v_k - \Delta v_k) - z(\theta_t\omega_0)(v_0 - \Delta v_0), V_k)\\ &+ (e^{-z(\theta_t\omega_k)} - e^{-z(\theta_t\omega_0)})(g, V_k),\end{aligned} \tag{2.13}$$

where the nonlinear term I_k are defined and split into three parts:

$$I_k := (e^{-z(\theta_t\omega_0)}\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega_0)}v_0) - e^{-z(\theta_t\omega_k)}\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega_k)}v_k), V_k) = I_{k1} + I_{k2} + I_{k3}.$$

By (2.4), $H^1(Q) \hookrightarrow L^4(Q)$ and Lemmas 2.5, 2.4,

$$\begin{aligned}I_{k1} :=& (e^{-z(\theta_t\omega_0)} - e^{-z(\theta_t\omega_k)})(\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega_0)}v_0), V_k)\\ \leq& CJ_k(\|v_0\|\|\nabla V_k\| + \|v_0\|_4^2\|\nabla V_k\|)\\ \leq& \|V_k\|_{H^1}^2 + CJ_k^2(1 + \|v_0\|_{H^1}^4) \leq \|V_k\|_{H^1}^2 + CJ_k^2,\end{aligned}$$

where $J_k := \sup\{|e^{-z(\theta_t\omega_k)} - e^{-z(\theta_t\omega_0)}| : t \in [\tau, \tau+T]\}$. By Lemmas 2.2 (b), 2.4, 2.5,

$$\begin{aligned} I_{k2} :&= e^{-z(\theta_t\omega_k)}(\nabla\cdot \overrightarrow{F}(e^{z(\theta_t\omega_0)}v_0) - \nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega_0)}v_k), V_k) \\ &\le ce^{c|z(\theta_t\omega_k)|+c|z(\theta_t\omega_0)|}(1+\|v_k\|_{H^1}+\|v_0\|_{H^1})\|V_k\|_{H^1}^2 \le C(1+\|v_k\|_{H^1})\|V_k\|_{H^1}^2. \end{aligned}$$

By (2.5) in Lemma 2.2 and the Young inequality,

$$\begin{aligned} I_{k3} :&= e^{-z(\theta_t\omega_k)}(\nabla\cdot \overrightarrow{F}(e^{z(\theta_t\omega_0)}v_k) - \nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega_k)}v_k), V_k) \\ &\le cJ_k e^{c|z(\theta_t\omega_k)|+c|z(\theta_t\omega_0)|}(1+\|v_k\|_{H^1})\|v_k\|_{H^1}\|V_k\|_{H^1} \\ &\le \|V_k\|_{H^1}^2 + CJ_k^2(1+\|v_k\|_{H^1}^4). \end{aligned}$$

The force term in (2.13) is bounded by

$$(e^{-z(\theta_t\omega_k)} - e^{-z(\theta_t\omega_0)})(g, V_k) \le \|V_k\|_{H^1}^2 + CJ_k^2\|g(t)\|^2.$$

Let $\tilde{J}_k := \sup\{|z(\theta_t\omega_k) - z(\theta_t\omega_0)| : \ t \in [\tau,\tau+T]\}$. Then, the rest term on the right-hand side of (2.13) is bounded by

$$\begin{aligned} &(z(\theta_t\omega_k)(v_k - \Delta v_k) - z(\theta_t\omega_0)(v_0-\Delta v_0), V_k) \\ &= (z(\theta_t\omega_k) - z(\theta_t\omega_0))(v_k - \Delta v_k, V_k) + z(\theta_t\omega_0)(V_k - \Delta V_k, V_k) \\ &\le \tilde{J}_k(\|v_k\|\|V_k\| + \|\nabla v_k\|\|\nabla V_k\|) + |z(\theta_t\omega_0)|\|V_k\|_{H^1}^2 \\ &\le C\|V_k\|_{H^1}^2 + C\tilde{J}_k^2(1+\|v_k\|_{H^1}^4). \end{aligned}$$

We substitute all estimates into (2.13) to see that

$$\frac{d}{dt}\|V_k\|_{H^1}^2 \le C(1+\|v_k\|_{H^1})\|V_k\|_{H^1}^2 + C(J_k^2+\tilde{J}_k^2)(1+\|v_k\|_{H^1}^4 + \|g(t)\|^2). \quad (2.14)$$

Noting that $V_k(\tau) = 0$, then applying the Gronwall inequality to (2.14) over (τ, t), we have, for all $t \in [\tau, \tau+T]$,

$$\begin{aligned} \|V_k(t)\|_{H^1}^2 &\le C(J_k^2+\tilde{J}_k^2)e^{C\int_\tau^{\tau+T}(1+\|v_k(s)\|_{H^1})ds}\int_\tau^{\tau+T}(1+\|v_k(s)\|_{H^1}^4+\|g(s)\|^2)ds \\ &\le C(J_k^2+\tilde{J}_k^2)e^{C\int_\tau^{\tau+T}\|v_k(s)\|_{H^1}ds}(1+\int_\tau^{\tau+T}\|v_k(s)\|_{H^1}^4 ds). \end{aligned} \quad (2.15)$$

By the energy inequality (2.9) with v_k instead of v, it follows from Lemma 2.5 that

$$\frac{d}{dt}\|v_k\|_{H^1}^2 \le C\|v_k\|_{H^1}^2 + C\|g(t)\|^2, \text{ for all } t \in [\tau, \tau+T].$$

Then, the Gronwall lemma gives

$$\sup_{k\in\mathbb{N}}\sup_{t\in[\tau,\tau+T]}\|v_k(t)\|_{H^1}^2 \le e^{CT}\|v_\tau\|_{H^1}^2 + Ce^{CT}\int_\tau^{\tau+T}\|g(s)\|^2 ds \le C.$$

We substitute it into (2.15) to find that $\|V_k(t)\|_{H^1}^2 \le C(J_k^2+\tilde{J}_k^2)$. By Lemma 2.5, $J_k, \tilde{J}_k \to 0$ and thus $\|V_k(t)\|_{H^1} \to 0$ as $k \to \infty$, uniformly in $t \in [\tau, \tau+T]$. □

By Lemma 2.4, we can define a mapping Φ: $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ by

$$\Phi(t,\tau,\omega)v_\tau = v(t+\tau, \tau, \theta_{-\tau}\omega, v_\tau), \quad \forall (t,\tau,\omega) \in \mathbb{R}^+\times\mathbb{R}\times\Omega, \ v_\tau \in X, \quad (2.16)$$

where $X = H_0^1(Q)$. The Lusin continuity in Proposition 2.1 gives the $\mathcal{F}$-measurability of Φ. Therefore, we obtain

Theorem 2.1. *The mapping Φ as given by* (2.16) *is a non-autonomous random dynamical system (NRDS) on X in the following sense.*

(i) Φ is $(\mathfrak{B}(\mathbb{R}^+)\times\mathfrak{B}(\mathbb{R})\times\mathcal{F}\times\mathfrak{B}(X),\mathfrak{B}(X))$ measurable,

(ii) Φ satisfies the cocycle property: $\Phi(0,\tau,\omega)=I$, and

$$\Phi(t+s,\tau,\omega)=\Phi(t,\tau+s,\theta_s\omega)\Phi(s,\tau,\omega),\quad t,s\geq 0. \tag{2.17}$$

2.3. Backward convergence of NRDS

We consider the autonomous BBM equation with Laplace-multiplier noise:

$$\begin{cases} d\hat{u}-d(\Delta\hat{u})-\nu\Delta\hat{u}dt+\nabla\cdot\overrightarrow{F}(\hat{u})dt=g_\infty(x)dt+\mathcal{S}\hat{u}\circ dW,\\ \hat{u}(t,x)|_{\partial Q}=0,\quad \hat{u}(0,x)=\hat{u}_0(x),\quad x\in Q,\ t\geq 0. \end{cases} \tag{2.18}$$

Let $\hat{v}(t,\omega)=e^{-z(\theta_t\omega)}\hat{u}(t,\omega)$. Eq.(2.18) can be rewritten as

$$\begin{aligned} &\hat{v}_t-\Delta\hat{v}_t-\nu\Delta\hat{v}\\ =&-e^{-z(\theta_t\omega)}\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}\hat{v})+z(\theta_t\omega)(\hat{v}-\Delta\hat{v})+e^{-z(\theta_t\omega)}g_\infty \end{aligned} \tag{2.19}$$

with the initial conditions: $\hat{v}(0,\omega)=\hat{v}_0=e^{-z(\omega)}\hat{u}_0\in H_0^1(Q)$.

Proposition 2.2. *The solution v of* (2.8) *backward converges to the solution $\hat{v}$ of* (2.19)*, that is,*

$$\lim_{\tau\to-\infty}\|v(T+\tau,\tau,\theta_{-\tau}\omega,v_\tau)-\hat{v}(T,\omega,\hat{v}_0)\|_{H^1}=0,\ \forall T>0,\ \omega\in\Omega, \tag{2.20}$$

whenever $\|v_\tau-\hat{v}_0\|_{H^1}\to 0$ as $\tau\to-\infty$.

Proof. Let $V^\tau(t):=v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau)-\hat{v}(t,\omega,\hat{v}_0)$ for $t\geq 0$. By (2.19) and (2.8), we have

$$\begin{aligned} V_t^\tau-\Delta V_t^\tau-\nu\Delta V^\tau=&e^{-z(\theta_t\omega)}(\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}\hat{v})-\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}v))\\ &+z(\theta_t\omega)(V^\tau-\Delta V^\tau)+e^{-z(\theta_t\omega)}(g(t+\tau)-g_\infty). \end{aligned} \tag{2.21}$$

Taking the inner product of (2.21) with V^τ in $L^2(Q)$, we have,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|V_\tau\|_{H^1}^2+\nu\|\nabla V^\tau\|^2=&e^{-z(\theta_t\omega)}(\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}\hat{v})-\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}v),V^\tau)\\ &+z(\theta_t\omega)\|V^\tau\|_{H^1}^2+e^{-z(\theta_t\omega)}(g(t+\tau)-g_\infty,V^\tau). \end{aligned} \tag{2.22}$$

By Lemma 2.2 (b),

$$\begin{aligned} &e^{-z(\theta_t\omega)}(\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}\hat{v})-\nabla\cdot\overrightarrow{F}(e^{z(\theta_t\omega)}v),V^\tau)\\ \leq&c(e^{-z(\theta_t\omega)}+\|\hat{v}\|_{H^1}+\|v\|_{H^1})\|V^\tau\|_{H^1}^2. \end{aligned} \tag{2.23}$$

The Young inequality implies that

$$\begin{aligned} &z(\theta_t\omega)\|V^\tau\|_{H^1}^2+e^{-z(\theta_t\omega)}(g(t+\tau)-g_\infty,V^\tau)\\ \leq&\|g(t+\tau)-g_\infty\|^2+ce^{c|z(\theta_t\omega)|}\|V^\tau\|_{H^1}^2. \end{aligned} \tag{2.24}$$

Substituting (2.23)-(2.24) into (2.22), we have for each $t \in \mathbb{R}^+$,

$$\frac{d}{dt}\|V^\tau\|^2_{H^1} \leq c(e^{c|z(\theta_t\omega)|} + \|\hat{v}\|_{H^1} + \|v(t+\tau)\|_{H^1})\|V^\tau\|^2_{H^1} + c\|g(t+\tau) - g_\infty\|^2. \tag{2.25}$$

Applying the Gronwall inequality to (2.25) over $(0, T)$, we have

$$\|V^\tau(T)\|^2_{H^1} \leq ce^{J(T,\tau)}\Big(\|V^\tau(0)\|^2_{H^1} + \int_0^T \|g(t+\tau) - g_\infty\|^2 dt\Big),$$

where, there exists a constant $C = C(T, \hat{v}_0)$ such that

$$\begin{aligned} J(T,\tau) : &= c\int_0^T (e^{c|z(\theta_t\omega)|} + \|\hat{v}(t)\|_{H^1} + \|v(t+\tau)\|_{H^1})dt \\ &\leq C + c\int_0^T \|v(t+\tau)\|_{H^1} dt. \end{aligned}$$

By the hypothesis **G**, we have

$$\int_0^T \|g(t+\tau) - g_\infty\|^2 dt \leq \int_{-\infty}^{\tau+T} \|g(s) - g_\infty\|^2 ds \to 0 \ \text{ as } \tau \to -\infty. \tag{2.26}$$

By using the energy inequality (2.9) on $v(t+\tau, \tau, \theta_{-\tau}, v_\tau)$ for $t \in [0, T]$, we obtain

$$\frac{d}{dt}\|v(t+\tau)\|^2_{H^1} \leq C_1\|v(t+\tau)\|^2_{H^1} + C_2\|g(t+\tau)\|^2,$$

where C_1, C_2 are independent of τ. The Gronwall inequality implies that for all $t \in [0, T]$,

$$\begin{aligned} \|v(t+\tau)\|^2_{H^1} &\leq C_2 e^{C_1 T}\Big(\|v_\tau\|^2_{H^1} + \int_0^T \|g(s+\tau)\|^2 ds\Big) \\ &\leq 2C_2 e^{C_1 T}\Big(\|v_\tau\|^2_{H^1} + T\|g_\infty\|^2 + \int_0^T \|g(s+\tau) - g_\infty\|^2 ds\Big), \end{aligned}$$

which is bounded (as $\tau \to -\infty$) in view of (2.26). So, $J(T, \tau)$ is bounded as $\tau \to -\infty$. Note that $\|V^\tau(0)\|^2_{H^1} = \|v_\tau - \hat{v}_0\|^2_{H^1} \to 0$ as $\tau \to -\infty$. We obtain $\|V^\tau(T)\|_{H^1} \to 0$ as $\tau \to -\infty$. □

3. Increasing random absorbing sets

In this section, we show existence of a $\mathfrak{D}$-random absorbing set, where $\mathfrak{D}$ is the backward tempered universe as given in (1.4). The main difficult is to verify measurability of the absorbing set because the absorbing radius is a supremum of some random functions over an uncountable index set.

Lemma 3.1. *For each $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$, there is a $T := T(\tau, \omega, \mathcal{D}) > 0$ such that*

$$\sup_{s\leq\tau}\sup_{t\geq T}\sup_{v_0\in\mathcal{D}(s-t,\theta_{-t}\omega)} \|v(s, s-t, \theta_{-s}\omega, v_0)\|^2_{H^1} \leq 1 + \frac{2}{\nu\lambda_0}R(\tau, \omega), \tag{3.1}$$

where $R(\tau,\omega)$ *is given by*

$$R(\tau,\omega):=\sup_{s\le\tau}\int_{-\infty}^{0}e^{\delta r+2|z(\theta_r\omega)|+2\int_r^0 z(\theta_\sigma\omega)d\sigma}\|g(r+s)\|^2dr. \tag{3.2}$$

Moreover, for all $\hat{s}\ge s-t$, $t\ge 0$ *and* $v_0\in H_0^1(Q)$,

$$\begin{aligned}&\|v(\hat{s},s-t,\theta_{-s}\omega,v_0)\|_{H^1}^2\\ \le& e^{-\delta(\hat{s}-s+t)+2\int_{-t}^{\hat{s}-s}z(\theta_\sigma\omega)d\sigma}\|v_0\|_{H^1}^2\\ &+\frac{2}{\nu\lambda_0}\int_{-t}^{\hat{s}-s}e^{\delta(r+s-\hat{s})+2|z(\theta_r\omega)|+2\int_r^{\hat{s}-s}z(\theta_\sigma\omega)d\sigma}\|g(r+s)\|^2dr.\end{aligned} \tag{3.3}$$

Proof. We rewritten the energy inequality (2.9) for $v(r)=v(r,s-t,\theta_{-s}\omega,v_0)$. The result is

$$\frac{d}{dr}\|v\|_{H^1}^2+(\delta-2z(\theta_{r-s}\omega))\|v\|_{H^1}^2\le\frac{2}{\nu\lambda_0}e^{2|z(\theta_{r-s}\omega)|}\|g(r)\|^2. \tag{3.4}$$

Applying the Gronwall inequality to (3.4) with respect to $r\in(s-t,\hat{s})$, we obtain (3.3) immediately. Letting $\hat{s}=s$ in (3.3) yields

$$\|v(s,s-t,\theta_{-s}\omega,v_0)\|_{H^1}^2\le e^{-\delta t+2\int_{-t}^0 z(\theta_\sigma\omega)d\sigma}\|v_0\|_{H^1}^2+\frac{2}{\nu\lambda_0}R(\tau,\omega), \tag{3.5}$$

for all $s\le\tau$. Since $v_0\in\mathcal{D}(s-t,\theta_{-t}\omega)$ and $\mathcal{D}$ is backward tempered, it follows from (2.6) and (1.4) that there exists a $T=T(\tau,\omega,\mathcal{D})$ such that for all $t\ge T$,

$$e^{-\delta t+2\int_{-t}^0 z(\theta_\sigma\omega)d\sigma}\sup_{s\le\tau}\|v_0\|_{H^1}^2\le e^{-\frac{\delta}{3}t}\sup_{s\le\tau}\|\mathcal{D}(s-t,\theta_{-t}\omega)\|_{H^1}^2\le 1.$$

Therefore, by taking the maximum on $s\in(-\infty,\tau]$ in (3.5), we show (3.1) as required. □

Recall that a bi-parametric set $\mathcal{K}$ is said to be a **$\mathfrak{D}$-pullback absorbing** set (briefly, an absorbing set) if for each $(\mathcal{D},\tau,\omega)\in\mathfrak{D}\times\mathbb{R}\times\Omega$ there is a $T:=T(\mathcal{D},\tau,\omega)$ such that

$$\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{D}(\tau-t,\theta_{-t}\omega)\subset\mathcal{K}(\tau,\omega),\ \forall t\ge T.$$

Proposition 3.1. *There is an increasing random absorbing set* $\mathcal{K}$ *given by*

$$\mathcal{K}(\tau,\omega):=\Big\{w\in H_0^1(Q):\|w\|_{H^1}^2\le 1+\frac{2}{\nu\lambda_0}R(\tau,\omega)\Big\},\ \forall\tau\in\mathbb{R}, \tag{3.6}$$

where $R(\tau,\omega)$ *is defined by* (3.2). *Moreover,* $\mathcal{K}$ *is backward tempered, i.e.* $\mathcal{K}\in\mathfrak{D}$.

Proof. By Lemma 2.1, g is backward tempered. So, it follows from the convergence (2.6) that

$$R(\tau,\omega)\le c\sup_{s\le\tau}\int_{-\infty}^0 e^{\frac{\delta}{2}r}\|g(r+s)\|^2dr<+\infty.$$

It is easy to show that $\mathcal{K}$ is tempered. Since $\tau\to R(\tau,\omega)$ is obviously an increasing function, $\mathcal{K}(\tau,\omega)$ is increasing. Then, $\mathcal{K}$ is an increasing tempered set and thus backward tempered, that is, $\mathcal{K} \in \mathfrak{D}$. The absorption follows from Lemma 3.1 immediately.

Next, we prove the measurability of the absorbing set $\mathcal{K}$. It suffices to prove that $\omega \to R(\tau,\omega)$ is a measurable function for each $\tau \in \mathbb{R}$, where we need to carefully treat the supremum when $s \in (-\infty,\tau]$, this interval is an uncountable set. For this end, we actually prove that $\omega \to R(\tau,\omega)$ is Lusin continuous.

By the Egoroff theorem, the convergence given in (2.6) is basically uniform on Ω, that is, for each $N \in \mathbb{N}$, there is a measurable set $\tilde{\Omega}_N \subset \Omega$ such that $P(\Omega \setminus \tilde{\Omega}_N) < 1/N$ and

$$\lim_{t\to\pm\infty} \sup_{\omega\in\tilde{\Omega}_N} \Big|\frac{z(\theta_t\omega)}{t}\Big| = \lim_{t\to\pm\infty} \frac{1}{t} \sup_{\omega\in\tilde{\Omega}_N} \Big|\int_0^t z(\theta_s\omega)ds\Big| = 0. \tag{3.7}$$

Let $E_N = \tilde{\Omega}_N \cap \Omega_N$, where $\Omega_N = \{\omega \in \Omega : |\omega(s)| \le Ne^{|s|}, \forall s \in \mathbb{R}\}$ as given in (2.10). Because $\Omega = \bigcup_{N=1}^{\infty} \Omega_N$ and $\Omega_N \subset \Omega_{N+1}$, we have $P(\Omega_N) \to 1$ as $N \to \infty$, it follows that

$$\lim_{N\to\infty} P(\Omega \setminus E_N) \le \lim_{N\to\infty} P(\Omega \setminus \tilde{\Omega}_N) + \lim_{N\to\infty} P(\Omega \setminus \Omega_N) = 0.$$

Now, suppose $\rho(\omega_k,\omega_0) \to 0$ as $k \to \infty$, where $\omega_k, \omega_0 \in E_N$. Let

$$h(r,\omega) := 2|z(\theta_r\omega)| + 2\int_r^0 z(\theta_\sigma\omega)d\sigma, \quad \text{for } r \le 0, \omega \in E_N.$$

By the uniform convergence (3.7) on $\tilde{\Omega}_N \supset E_N$, there is an $r_0 < 0$ such that

$$\sup_k |h(r,\omega_k)| \le -\frac{\delta}{8}r,\ |h(r,\omega_0)| \le -\frac{\delta}{8}r, \text{ for all } r \le r_0.$$

Given $\varepsilon > 0$, we take $r_1 \le r_0 < 0$ such that $e^{\frac{\delta}{4}r_1} < \varepsilon$. Then, the above inequality implies that

$$\sup_{r\le r_1} e^{\frac{\delta}{2}r}|e^{h(r,\omega_k)} - e^{h(r,\omega_0)}| \le \sup_{r\le r_1} e^{\frac{\delta}{4}r} < \varepsilon,\ \forall k \in \mathbb{N}.$$

By the same method as given in [8, Corollary 22] (see (2.11) in Lemma 2.5), we have

$$\sup_{r_1\le r\le 0} |e^{h(r,\omega_k)} - e^{h(r,\omega_0)}| \to 0 \text{ as } k \to \infty.$$

Then, there is a $k_0 \in \mathbb{N}$ such that

$$\sup_{r_1\le r\le 0} e^{\frac{\delta}{2}r}|e^{h(r,\omega_k)} - e^{h(r,\omega_0)}| \le \sup_{r_1\le r\le 0} |e^{h(r,\omega_k)} - e^{h(r,\omega_0)}| < \epsilon,\ \forall k \ge k_0.$$

Hence, the above two estimates yield

$$\sup_{r\le 0} e^{\frac{\delta}{2}r}|e^{h(r,\omega_k)} - e^{h(r,\omega_0)}| \to 0 \text{ as } k \to \infty. \tag{3.8}$$

By using the inequality $|\sup_{s\le\tau} A(s) - \sup_{s\le\tau} B(s)| \le \sup_{s\le\tau} |A(s) - B(s)|$, it follows from the definition (3.2) of $R(\tau,\omega)$ that

$$|R(\tau,\omega_k) - R(\tau,\omega_0)|$$

$$
\begin{aligned}
=&\Big|\sup_{s\le\tau}\int_{-\infty}^{0}e^{\delta r+h(r,\omega_k)}\|g(r+s)\|^2dr-\sup_{s\le\tau}\int_{-\infty}^{0}e^{\delta r+h(r,\omega_0)}\|g(r+s)\|^2dr\Big|\\
\le&\sup_{s\le\tau}\int_{-\infty}^{0}e^{\delta r}|e^{h(r,\omega_k)}-e^{h(r,\omega_0)}|\|g(r+s)\|^2dr\\
\le&\sup_{r\le0}e^{\frac{\delta}{2}r}|e^{h(r,\omega_k)}-e^{h(r,\omega_0)}|\sup_{s\le\tau}\int_{-\infty}^{0}e^{\frac{\delta}{2}r}\|g(r+s)\|^2dr,
\end{aligned}
$$

which tends to zero as $k\to\infty$, in view of (3.8) and that g is backward tempered. Therefore, $\omega\to R(\tau,\omega)$ is continuous in E_N and thus Lusin continuous in Ω, which further implies the measurablity. □

For the later purpose, we need an auxiliary estimate, which is similar to the autonomous case given by [25, Lemma 5.1], and so we omit the proof.

Lemma 3.2. *For each $(s,t,\omega)\in\mathbb{R}\times\mathbb{R}^+\times\Omega$ and $v_0\in H_0^1(Q)$, we have,*

$$
\|v(s,s-t,\omega,v_0)\|_{H^1}^2\le ce^{c|z(\theta_s\omega)|}(1+\|v(s,s-t,\omega,v_0)\|_{H^1}^4+\|g(s)\|^2). \qquad (3.9)
$$

4. Backward tail-estimates and backward flattening

Now, we intend to give the backward tail-estimate when the third component of space-variable is large enough. We will use the square of the usual cut-off function:

$$
\rho_k(x):=\rho(\frac{x_3^2}{k^2}),\ x=(x_1,x_2,x_3)\in Q,\ k\ge1. \qquad (4.1)
$$

where $\rho:\mathbb{R}^+\mapsto[0,1]$ is smooth such that $\rho(s)\equiv0$ on $[0,1]$ and $\rho(s)\equiv1$ on $[4,+\infty)$.

Lemma 4.1. *For each $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$,*

$$
\lim_{k,t\to+\infty}\sup_{s\le\tau}\sup_{v_0\in\mathcal{D}(s-t,\theta_{-t}\omega)}\|v(s,s-t,\theta_{-s}\omega,v_0)\|_{H^1(Q_k^c)}^2=0,
$$

where, $Q_k^c=Q\setminus Q_k$ with $Q_k=\{x=(x_1,x_2,x_3)\in Q:|x_3|<k\}$ for each $k\ge1$.

Proof. Taking the inner product of Eq.(2.8) with ρ_k^2v in $L^2(Q)$, we see that

$$
\begin{aligned}
&\frac{d}{ds}\int_Q\rho_k^2(|v|^2+|\nabla v|^2)dx+2\nu\int_Q\rho_k^2|\nabla v|^2dx\\
=&2z(\theta_s\omega)\int_Q\rho_k^2(|v|^2+|\nabla v|^2)dx+I_1+I_2+I_3,
\end{aligned}\qquad (4.2)
$$

where I_1,I_2,I_3 are defined later. By (2.4), $\|\nabla\rho_k^2\|_\infty\le\frac{c}{k}$ and $H^1(Q)\hookrightarrow L^3(Q)$, we have

$$
\begin{aligned}
I_1:&=-2e^{-z(\theta_s\omega)}\int_Q\rho_k^2v\nabla\cdot\overrightarrow{F}(u)dx\\
&=2e^{-2z(\theta_s\omega)}\int_Q\rho_k^2(\nabla u\cdot\overrightarrow{F}(u))dx+2e^{-2z(\theta_s\omega)}\int_Q u(\nabla\rho_k^2\cdot\overrightarrow{F}(u))dx\\
&=2e^{-2z(\theta_s\omega)}\int_Q\rho_k^2(\nabla\cdot\overrightarrow{f}(u))dx+2e^{-2z(\theta_s\omega)}\int_Q u(\nabla\rho_k^2\cdot\overrightarrow{F}(u))dx
\end{aligned}
$$

$$
\begin{aligned}
&= -2e^{-2z(\theta_s\omega)}\int_Q \nabla\rho_k^2\cdot\overrightarrow{f}(u)dx + 2e^{-2z(\theta_s\omega)}\int_Q u(\nabla\rho_k^2\cdot\overrightarrow{F}(u))dx\\
&\leq \frac{c}{k}e^{c|z(\theta_s\omega)|}\int_Q(|u|^2+|u|^3)dx \leq \frac{c}{k}e^{c|z(\theta_s\omega)|}(\|u\|_{H^1}^2+\|u\|_{H^1}^3)\\
&\leq \frac{c}{k}e^{c|z(\theta_s\omega)|}(1+\|v\|_{H^1}^4),
\end{aligned}
$$

Similarly, by $\|\nabla\rho_k^2\|_\infty \leq \frac{c}{k}$ and Lemma 3.2, we have

$$
\begin{aligned}
I_2 :&= -2\int_Q v(\nabla v_s\cdot\nabla\rho_k^2)dx + 2(z(\theta_s\omega)-\nu)\int_Q v(\nabla v\cdot\nabla\rho_k^2)dx\\
&\leq \frac{c}{k}(1+|z(\theta_s\omega)|)(\|v\|_{H^1}^2+\|v_s\|_{H^1}^2) \leq \frac{c}{k}e^{c|z(\theta_s\omega)|}(1+\|g(s)\|^2+\|v\|_{H^1}^4).\\
I_3 :&= 2e^{-z(\theta_s\omega)}\int_Q \rho_k^2 v g(s,x)dx \leq \frac{\nu\lambda_0}{4}\int_Q\rho_k^2|v|^2dx + ce^{c|z(\theta_s\omega)|}\int_Q\rho_k^2|g(s,x)|^2dx.
\end{aligned}
$$

Applying the Poincaré inequality on $\rho_k v$, we have

$$
2\nu\int_Q\rho_k^2|\nabla v|^2dx \geq \nu\int_Q\rho_k^2|\nabla v|^2dx + \frac{\nu\lambda_0}{2}\int_Q\rho_k^2|v|^2dx - \frac{c}{k}\|v\|^2.
$$

Substituting all above estimates into (4.2) and recalling $\delta := \min(\frac{\nu}{2},\frac{\nu\lambda_0}{4})$, we have,

$$
\begin{aligned}
&\frac{d}{ds}\int_Q\rho_k^2(v^2+|\nabla v|^2)dx + (\delta-2z(\theta_s\omega))\int_Q\rho_k^2(v^2+|\nabla v|^2)dx\\
\leq&\frac{c}{k}e^{c|z(\theta_s\omega)|}(1+\|g(s)\|^2+\|v\|_{H^1}^4) + ce^{c|z(\theta_s\omega)|}\int_{Q_k^c}|g(s,x)|^2dx. \qquad (4.3)
\end{aligned}
$$

Applying the Gronwall inequality to (4.3) over $(s-t,s)$ and replacing ω by $\theta_{-s}\omega$, we have,

$$
\sup_{s\leq\tau}\int_Q\rho_k^2(|v(s,s-t,\theta_{-s}\omega,v_0)|^2+|\nabla v|^2)dx \leq J_1+J_2+\frac{c}{k}(J_3+J_4), \qquad (4.4)
$$

where J_1,J_2,J_3,J_4 are given and estimated as follows. Since $v_0\in\mathcal{D}(s-t,\theta_{-t}\omega)$ for all $s\leq\tau$, by (2.6) and (1.4), we have

$$
\begin{aligned}
J_1 :&= \sup_{s\leq\tau}e^{-\delta t+2\int_{-t}^0 z(\theta_\sigma\omega)d\sigma}\int_Q\rho_k^2(|v_0|^2+|\nabla v_0|^2)dx\\
&\leq \sup_{s\leq\tau}ce^{-\frac{\delta}{3}t}\|v_0\|_{H^1}^2 \leq ce^{-\frac{\delta}{3}t}\sup_{s\leq\tau}\|\mathcal{D}(s-t,\theta_{-t}\omega)\|_{H^1}^2 \to 0,
\end{aligned}
$$

as $t\to+\infty$. By Lemma 2.1, the hypothesis **G** implies that g is backward tail-small. So, by (2.6), we have

$$
\begin{aligned}
J_2 :&= c\sup_{s\leq\tau}\int_{s-t}^s e^{\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|+2\int_{\hat{s}}^s z(\theta_{\sigma-s}\omega)d\sigma}\int_{Q_k^c}|g(\hat{s},x)|^2dxd\hat{s}\\
&\leq \sup_{s\leq\tau}\int_{-\infty}^0 e^{\delta\hat{s}+c|z(\theta_{\hat{s}}\omega)|+2\int_{\hat{s}}^0 z(\theta_\sigma\omega)d\sigma}\int_{Q_k^c}|g(\hat{s}+s,x)|^2dxd\hat{s}\\
&\leq c\sup_{s\leq\tau}\int_{-\infty}^s e^{\frac{\delta}{4}(r-s)}\int_{Q_k^c}|g(r,x)|^2dxdr \to 0,
\end{aligned}
$$

as $k \to +\infty$. Since g is backward tempered as given in Lemma 2.1, it follows that the following term

$$J_3 := \sup_{s\le\tau}\int_{s-t}^{s} e^{\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|+2\int_{\hat{s}}^{s} z(\theta_{\sigma-s}\omega)d\sigma}(1+\|g(\hat{s})\|^2)d\hat{s} \tag{4.5}$$

is finite, and so $cJ_3/k \to 0$ as $k \to \infty$. It suffices to prove finiteness of the following term:

$$J_4 := \sup_{s\le\tau}\int_{s-t}^{s} e^{\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|+2\int_{\hat{s}}^{s} z(\theta_{\sigma-s}\omega)d\sigma}\|v(\hat{s},s-t,\theta_{-s}\omega,v_0)\|^4 d\hat{s}, \tag{4.6}$$

where we need to deal with the biquadrate. By using (3.3) in Lemma 3.1, we can split $J_4 \le \hat{J}_4 + \tilde{J}_4$ with

$$\begin{aligned}
\hat{J}_4 :&= \sup_{s\le\tau}\int_{s-t}^{s} e^{\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|+c\int_{\hat{s}}^{s} z(\theta_{\sigma-s}\omega)d\sigma} e^{-2\delta(\hat{s}-s+t)+c\int_{-t}^{\hat{s}-s} z(\theta_\sigma\omega)d\sigma} d\hat{s}\|v_0\|_{H^1}^4\\
&\le \sup_{s\le\tau}\int_{s-t}^{s} e^{\frac{1}{4}\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|}d\hat{s}\cdot e^{-\frac{3}{4}\delta t+c\int_{-t}^{0} z(\theta_\sigma\omega)d\sigma}\|v_0\|_{H^1}^4\\
&= \int_{-t}^{0} e^{\frac{1}{4}\delta\hat{s}+c|z(\theta_{\hat{s}}\omega)|}d\hat{s}\cdot e^{-\frac{3}{4}\delta t+c\int_{-t}^{0} z(\theta_\sigma\omega)d\sigma}\sup_{s\le\tau}\|v_0\|_{H^1}^4.
\end{aligned}$$

The first integral in the last line is obviously finite. Also, by $v_0 \in \mathcal{D}(s-t,\theta_{-t}\omega)$,

$$\begin{aligned}
&e^{-\frac{3}{4}\delta t+c\int_{-t}^{0} z(\theta_\sigma\omega)d\sigma}\sup_{s\le\tau}\|v_0\|_{H^1}^4\\
\le& e^{-\frac{2}{3}\delta t}\sup_{s\le\tau}\|v_0\|_{H^1}^4 \le (e^{-\frac{1}{3}\delta t}\sup_{s\le\tau}\|\mathcal{D}(s-t,\theta_{-t}\omega)\|_{H^1}^2)^2 \to 0,
\end{aligned}$$

as $t \to +\infty$. So, $\hat{J}_4 < +\infty$. Another term $\tilde{J}_4$ is given by

$$\begin{aligned}
\tilde{J}_4 :=& \sup_{s\le\tau}\int_{s-t}^{s} e^{\delta(\hat{s}-s)+c|z(\theta_{\hat{s}-s}\omega)|+c\int_{\hat{s}}^{s} z(\theta_{\sigma-s}\omega)d\sigma}\\
&\cdot\Big(\int_{-t}^{\hat{s}-s} e^{\delta(r+s-\hat{s})+c|z(\theta_r\omega)|+c\int_r^{\hat{s}-s} z(\theta_\sigma\omega)d\sigma}\|g(r+s)\|^2dr\Big)^2 d\hat{s}.
\end{aligned}$$

Let $C_1 = \int_{-\infty}^{0} e^{\frac{1}{3}\delta\hat{s}+c|z(\theta_{\hat{s}}\omega)|+c\int_{\hat{s}}^{0} z(\theta_\sigma\omega)d\sigma}d\hat{s} < +\infty$. Then, since g is backward tempered, it easily follows that

$$\tilde{J}_4 \le cC_1\Big(\sup_{s\le\tau}\int_{-t}^{0} e^{\frac{1}{3}\delta r+c|z(\theta_r\omega)|+c\int_r^{0} z(\theta_\sigma\omega)d\sigma}\|g(r+s)\|^2dr\Big)^2 < +\infty.$$

Therefore, $J_4 \le \hat{J}_4 + \tilde{J}_4 < +\infty$. By (4.4),

$$\sup_{s\le\tau}\|v(s,s-t,\theta_{-s}\omega,v_0)\|_{H^1(Q_{2k}^c)}^2 \le \sup_{s\le\tau}\int_Q \rho_k^2(|v|^2+|\nabla v|^2)dx \to 0,$$

as $k,t \to +\infty$, uniformly in $v_0 \in \mathcal{D}(s-t,\theta_{-t}\omega)$. The proof is completed. □

Next, we give backward flattening estimates in the bounded domain. For each $k \ge 1$, we let

$$\xi_k(x) := 1-\rho_k(x) = 1-\rho(\frac{x_3^2}{k^2}),\ x=(x_1,x_2,x_3)\in Q.$$

Let $\bar{v} := \xi_k v$ for $v := v(s, s-t, \omega, v_\tau) \in H_0^1(Q)$. Then, $\bar{v} \in H_0^1(Q_{2k})$, which has the orthogonal decomposition:

$$\bar{v} = P_i \bar{v} \oplus (I - P_i)\bar{v} =: \bar{v}_{i,1} + \bar{v}_{i,2}, \quad \text{for each } i \in \mathbb{N}, \tag{4.7}$$

where, $P_i : L^2(Q_{2k}) \mapsto H_i := \text{span}\{e_1, e_2 \cdots, e_i\} \subset H_0^1(Q_{2k})$ is a canonical projection and $\{e_j\}_{j=1}^{\infty}$ is the family of eigenfunctions for $-\Delta$ in $L^2(Q_{2k})$ with corresponding positive eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty$ as $j \to \infty$.

It is easy to calculate that $\xi_k \Delta v = \Delta \bar{v} - v\Delta\xi_k - 2\nabla\xi_k \cdot \nabla v$ and $\xi_k \Delta v_s = \Delta \bar{v}_s - v_s \Delta \xi_k - 2\nabla \xi_k \cdot \nabla v_s$. Hence, we multiply (2.8) by ξ_k, the equation can be rewritten as

$$\begin{aligned} &\bar{v}_s - \Delta \bar{v}_s - \nu \Delta \bar{v} = z(\theta_s\omega)(\bar{v} - \Delta \bar{v}) - e^{-z(\theta_s\omega)} \xi_k \nabla \cdot \overrightarrow{F}(e^{z(\theta_s\omega)} v) + e^{-z(\theta_s\omega)} \xi_k g \\ &\quad - v_s \Delta \xi_k - 2\nabla \xi_k \cdot \nabla v_s + (z(\theta_s\omega) - \nu) v \Delta \xi_k + 2(z(\theta_s\omega) - \nu)\nabla \xi_k \cdot \nabla v. \end{aligned} \tag{4.8}$$

Lemma 4.2. *Let $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ and $k \ge 1$ be fixed. Then*

$$\lim_{i,t \to +\infty} \sup_{s \le \tau} \sup_{v_0 \in \mathcal{D}(s-t, \theta_{-t}\omega)} \|(I - P_i)\bar{v}(s, s-t, \theta_{-s}\omega, \bar{v}_{0,2})\|^2_{H^1(Q_{2k})} = 0,$$

where $\bar{v}_{0,2} = (I - P_i)(\xi_k v_0)$.

Proof. Applying $I - P_i$ to Eq.(4.8) and taking the inner product of the resulting equation with $\bar{v}_{i,2}$ in $L^2(Q_{2k})$, it yields from the orthogonal decomposition (4.7) that

$$\frac{d}{ds}\|\bar{v}_{i,2}\|^2_{H^1} + 2\nu \|\nabla \bar{v}_{i,2}\|^2 - 2z(\theta_s\omega)\|\bar{v}_{i,2}\|^2_{H^1} = I_1 + I_2 + I_3, \tag{4.9}$$

where I_1, I_2, I_3 are defined and estimated as follows. By $\|v\|_3^2 \le c\|\nabla v\|\|v\|$ and $\|\nabla \bar{v}_{i,2}\|^2 \ge \lambda_{i+1}\|\bar{v}_{i,2}\|^2$, we have

$$\begin{aligned} I_1 :&= -2e^{-z(\theta_s\omega)}(\xi_k \nabla \cdot \overrightarrow{F}(e^{z(\theta_s\omega)}v), \bar{v}_{i,2}) \quad \text{(by (2.3))} \\ &\le ce^{c|z(\theta_s\omega)|} \int_{Q_{2k}} (1 + |v|)|\nabla v||\bar{v}_{i,2}| dx \\ &\le ce^{c|z(\theta_s\omega)|}\|\nabla v\|\|\bar{v}_{i,2}\| + ce^{c|z(\theta_s\omega)|}\|v\|_6 \|\nabla v\|\|\bar{v}_{i,2}\|_3 \\ &\le ce^{c|z(\theta_s\omega)|}\|\nabla v\|\|\bar{v}_{i,2}\| + ce^{c|z(\theta_s\omega)|}\|v\|^2_{H^1}\|\nabla \bar{v}_{i,2}\|^{\frac{1}{2}}\|\bar{v}_{i,2}\|^{\frac{1}{2}} \\ &\le c\lambda_{i+1}^{-\frac{1}{2}} e^{c|z(\theta_s\omega)|}\|\nabla v\|\|\nabla \bar{v}_{i,2}\| \\ &\le \frac{\nu}{8}\|\nabla \bar{v}_{i,2}\|^2 + c(\lambda_{i+1}^{-1} + \lambda_{i+1}^{-\frac{1}{2}})e^{c|z(\theta_s\omega)|}(1 + \|v\|^4_{H^1}). \end{aligned}$$

Similarly, the Young inequality implies that

$$I_2 := 2e^{-z(\theta_s\omega)}(\xi_k g, \bar{v}_{i,2}) \le \frac{\nu}{4}\|\nabla \bar{v}_{i,2}\|^2 + c\lambda_{i+1}^{-1} e^{c|z(\theta_s\omega)|}\|g(s)\|^2.$$

By Lemma 3.2 and $\|\Delta \xi_k\|_\infty \le c$,

$$\begin{aligned} I_3 :&= 2((z - \nu)v\Delta\xi_k - v_s\Delta\xi_k - 2\nabla\xi_k \cdot \nabla v_s + 2(z - \nu)\nabla\xi_k \cdot \nabla v, \bar{v}_{i,2}) \\ &\le c\lambda_{i+1}^{-\frac{1}{2}}(1 + |z(\theta_s\omega)|)\|\nabla \bar{v}_{i,2}\|(\|v_s\|_{H^1} + \|v\|_{H^1}) \\ &\le \frac{\nu}{8}\|\nabla \bar{v}_{i,2}\|^2 + c\lambda_{i+1}^{-1}(1 + |z(\theta_s\omega)| + |z(\theta_s\omega)|^2)(\|v_s\|^2_{H^1} + \|v\|^2_{H^1}) \end{aligned}$$

$$\leq \frac{\nu}{8}\|\nabla\bar{v}_{i,2}\|^2 + c\lambda_{i+1}^{-1}e^{c|z(\theta_s\omega)|}(1 + \|g(s)\|^2 + \|v\|_{H^1}^4).$$

We assume without loss of generality that $\lambda_i \geq 1$, then $\lambda_i^{-1} \leq \lambda_i^{-1/2}$. Substituting all above estimates into (4.9) yields

$$\begin{aligned}&\frac{d}{ds}\|\bar{v}_{i,2}\|_{H^1}^2 + (\delta - 2z(\theta_s\omega))\|\bar{v}_{i,2}\|_{H^1}^2\\ \leq&\lambda_{i+1}^{-\frac{1}{2}}ce^{c|z(\theta_s\omega)|}(1 + \|g(s)\|^2 + \|v\|_{H^1}^4),\end{aligned} \tag{4.10}$$

where $\delta := \min(\frac{\nu}{2}, \frac{\nu\lambda_0}{4})$. Hence, the Gronwall lemma over (4.10) implies

$$\begin{aligned}&\sup_{s\leq\tau}\|\bar{v}_{i,2}(s, s-t, \theta_{-s}\omega, (I - P_i)(\xi_k v_0))\|_{H^1}^2\\ \leq&e^{-\delta t + 2\int_{-t}^0 z(\theta_\sigma\omega)d\sigma}\sup_{s\leq\tau}\|(I - P_i)(\xi_k v_0)\|_{H^1}^2 + c\lambda_{i+1}^{-\frac{1}{2}}(J_3 + J_4),\end{aligned} \tag{4.11}$$

where J_3 and J_4 are given by (4.5) and (4.6) respectively. By the same method as given in the proof of Lemma 4.1, both J_3 and J_4 are finite.

On the other hand, it is obvious that $\|(I - P_i)(\xi_k v_0)\|_{H^1}^2 \leq c\|v_0\|_{H^1}^2$ for all $v_0 \in \cup_{s\leq\tau}\mathcal{D}(s - t, \theta_{-t}\omega)$. Hence, by (2.6) and (1.4),

$$\begin{aligned}&e^{-\delta t + 2\int_{-t}^0 z(\theta_\sigma\omega)d\sigma}\sup_{s\leq\tau}\|(I - P_i)\xi_k v_0\|_{H^1}^2\\ \leq&ce^{-\frac{1}{3}\delta t}\sup_{s\leq\tau}\|\mathcal{D}(s - t, \theta_{-t}\omega)\|_{H^1}^2 \to 0 \text{ as } t \to +\infty.\end{aligned}$$

Therefore, (4.11) implies the needed convergence. □

5. Backward compact attractors and asymptotic autonomy

5.1. Abstract results

Let Φ be a *general* NRDS on a Banach space X over $(\Omega, \mathcal{F}, P, \theta)$, as defined in Theorem 2.1. Let $\mathfrak{D} = \{\mathcal{D}(\tau, \omega)\}$ be a universe of some bi-parametric sets. We assume that $\mathfrak{D}$ is **backward-closed**, which means $\widetilde{\mathcal{D}} \in \mathfrak{D}$ provided $\mathcal{D} \in \mathfrak{D}$ and $\widetilde{\mathcal{D}}(\tau, \omega) = \cup_{s\leq\tau}\mathcal{D}(s, \omega)$. Also, $\mathfrak{D}$ is **inclusion-closed** (see [32]).

Definition 5.1. A bi-parametric set $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is said to be a **$\mathfrak{D}$-backward compact random attractor** for a NRDS Φ if

(i) $\mathcal{A} \in \mathfrak{D}$ and $\mathcal{A}$ is a random set,
(ii) $\mathcal{A}$ is *backward compact*, that is, both $\mathcal{A}(\tau, \omega)$ and $\overline{\cup_{s\leq\tau}\mathcal{A}(s, \omega)}$ are compact,
(iii) $\mathcal{A}$ is *invariant*, that is, $\Phi(t, \tau, \omega)\mathcal{A}(\tau, \omega) = \mathcal{A}(t + \tau, \theta_t\omega)$ for $t \geq 0$,
(iv) $\mathcal{A}$ is *attracting* under the Hausdorff semi-distance, that is, for each $\mathcal{D} \in \mathfrak{D}$,

$$\lim_{t\to+\infty} \mathrm{dist}_X(\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{D}(\tau - t, \theta_{-t}\omega),\ \mathcal{A}(\tau, \omega)) = 0. \tag{5.1}$$

The backward compact random attractor has been studied in [27,35,36]. In this article, we use it as one of the criteria for asymptotic autonomy of pullback random attractors.

Definition 5.2. A non-autonomous cocycle Φ on X is said to be **$\mathfrak{D}$-backward asymptotically compact** if for each $(\tau,\omega,\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{D}$, the sequence

$$\{\Phi(t_n,s_n-t_n,\theta_{-t_n}\omega)x_n\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X,$$

whenever $s_n\le\tau$, $t_n\to+\infty$ and $x_n\in\mathcal{D}(s_n-t_n,\theta_{-t_n}\omega)$.

We then introduce a concept of a **backward limiting set** : given a bi-parametric set $\mathcal{D}$,

$$\mathcal{W}_b(\tau,\omega,\mathcal{D}):=\bigcap_{T>0}\overline{\bigcup_{t\ge T}\bigcup_{s\le\tau}\Phi(t,s-t,\theta_{-t}\omega)\mathcal{D}(s-t,\theta_{-t}\omega)},\ \forall\ (\tau,\omega)\in\mathbb{R}\times\Omega, \quad (5.2)$$

which generalizes the usual omega-limit set $\mathcal{W}(\tau,\omega,\mathcal{D})$ (see [32]).

The following results are crucial for finding a backward compact attractor.

Proposition 5.1. *Let $\mathcal{D}$ be a bi-parametric set, $\tau\in\mathbb{R}$ and $\omega\in\Omega$. Then,*

(i) $y\in\mathcal{W}_b(\tau,\omega,\mathcal{D})$ if and only if there are $\tau_n\le\tau$, $t_n\uparrow+\infty$ and $x_n\in\mathcal{D}(\tau_n-t_n,\theta_{-t_n}\omega)$ such that

$$\Phi(t_n,\tau_n-t_n,\theta_{-t_n}\omega)x_n\to y \quad \text{in } X. \quad (5.3)$$

(ii) $\mathcal{W}_b(\tau,\omega,\mathcal{D})$ is increasing in τ, that is, $\mathcal{W}_b(\tau_1,\omega,\mathcal{D})\subset\mathcal{W}_b(\tau_2,\omega,\mathcal{D})$ if $\tau_1\le\tau_2$.

(iii) The backward limit-set contains the backward union of the usual limit-set, that is,

$$\overline{\bigcup_{s\le\tau}\mathcal{W}(s,\omega,\mathcal{D})}\subset\mathcal{W}_b(\tau,\omega,\mathcal{D})=\bigcup_{s\le\tau}\mathcal{W}_b(s,\omega,\mathcal{D}). \quad (5.4)$$

(iv) If Φ is $\mathfrak{D}$-backward asymptotically compact in X, then, both limit-sets $\mathcal{W}(\tau,\omega,\mathcal{D})$ and $\mathcal{W}_b(\tau,\omega,\mathcal{D})$ are backward compact for each $\mathcal{D}\in\mathfrak{D}$.

Proof.

The assertion (i) is similar to the deterministic case (see e.g. [26]), while the assertion (ii) follows from the definition (5.2) immediately.

The assertion (iii) follows from the following inclusion:

$$\begin{aligned}\bigcup_{s\le\tau}\mathcal{W}(s,\omega,\mathcal{D})&=\bigcup_{s\le\tau}\bigcap_{T>0}\overline{\bigcup_{t\ge T}\Phi(t,s-t,\theta_{-t}\omega)\mathcal{D}(s-t,\theta_{-t}\omega)}\\&\subset\bigcap_{T>0}\overline{\bigcup_{t\ge T}\bigcup_{s\le\tau}\Phi(t,s-t,\theta_{-t}\omega)\mathcal{D}(s-t,\theta_{-t}\omega)}=\mathcal{W}_b(\tau,\omega,\mathcal{D}).\end{aligned}$$

Since $\mathcal{W}_b(\tau,\omega,\mathcal{D})$ is increasing in τ, it follows that $\bigcup_{s\le\tau}\mathcal{W}_b(s,\omega,\mathcal{D})=\mathcal{W}_b(\tau,\omega,\mathcal{D})$.

We prove (iv). Assume that Φ is backward asymptotically compact, then it is asymptotically compact. It is well-known that $\mathcal{W}(\tau,\omega,\mathcal{D})$ is nonempty. Hence, by (iii), $\mathcal{W}_b(\tau,\omega,\mathcal{D})$ is nonempty.

We take any sequence $\{y_n\}_{n=1}^{\infty}$ from $\mathcal{W}_b(\tau,\omega,\mathcal{D})$. Then, by (i), there are $\tau_n\le\tau$, $t_n\uparrow+\infty$ and $x_n\in\mathcal{D}(\tau_n-t_n,\theta_{-t_n}\omega)$ such that

$$\|\Phi(t_n,\tau_n-t_n,\theta_{-t_n}\omega)x_n-y_n\|\le\frac{1}{n}.$$

By the backward asymptotical compactness of Φ, passing to a subsequence, we have

$$\Phi(t_{n_k}, \tau_{n_k} - t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k} \to y_0, \quad \text{in } X.$$

By (i), $y_0 \in \mathcal{W}_b(\tau, \omega, \mathcal{D})$. Also, we have $y_{n_k} \to y_0$ in X, and so $\mathcal{W}_b(\tau, \omega, \mathcal{D})$ is compact. Hence, it follows from (5.4) that both limit-sets $\mathcal{W}$ and $\mathcal{W}_b$ are backward compact. □

Now, we give a unified result for asymptotic autonomy and backward compactness of a non-autonomous random attractor. Let $\mathfrak{D}_\infty = \{D(\omega)\}$ be an inclusion-closed universe of some single-parametric sets.

Theorem 5.1. *Suppose that a NRDS Φ satisfies the following two conditions:*
(a) Φ has a closed random absorbing set $\mathcal{K} \in \mathfrak{D}$;
(b) Φ is $\mathfrak{D}$-backward asymptotically compact.
Then, Φ has a unique backward compact random attractor $\mathcal{A} \in \mathfrak{D}$.

Let Φ_∞ be an RDS with a $\mathfrak{D}_\infty$-random attractor $\mathcal{A}_\infty$, and further assume that
(c) Φ backward converges to Φ_∞ in the following sense:

$$\|\Phi(t, \tau, \omega)x_\tau - \Phi_\infty(t, \omega)x_0\|_X \to 0 \quad \text{as } \tau \to -\infty, \ \forall t \geq 0, \ \omega \in \Omega_0, \tag{5.5}$$

whenever $\|x_\tau - x_0\|_X \to 0$ as $\tau \to -\infty$, where Ω_0 is a θ-invariant full-measure set;
(d) $\mathcal{K}_{\tau_0} \in \mathfrak{D}_\infty$ for some $\tau_0 < 0$, where $\mathcal{K}_{\tau_0}(\omega) := \bigcup_{\tau \leq \tau_0} \mathcal{K}(\tau, \omega)$.
Then, $\mathcal{A}$ **backward converges** *to $\mathcal{A}_\infty$:*

$$\lim_{\tau \to -\infty} dist_X(\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \ P\text{-a.s. } \omega \in \Omega. \tag{5.6}$$

Moreover, for any sequence $\tau_n \to -\infty$, there is a subsequence $\{\tau_{n_k}\}$ such that P-a.s.

$$\lim_{k \to \infty} dist(\mathcal{A}(\tau_{n_k}, \theta_{\tau_{n_k}}\omega), \mathcal{A}_\infty(\theta_{\tau_{n_k}}\omega)) = 0. \tag{5.7}$$

Proof. *Existence.* Note that backward asymptotic compactness obviously implies asymptotic compactness. By [32, Theorem 2.23], both conditions (a) and (b) imply that Φ has a unique $\mathfrak{D}$-random attractor $\mathcal{A} \in \mathfrak{D}$ given by the omega-limit set of $\mathcal{K}$, that is, $\mathcal{A}(\tau, \omega) = \mathcal{W}(\tau, \omega, \mathcal{K})$. Since Φ is $\mathfrak{D}$-backward asymptotically compact, by (iv) of Proposition 5.1, we know $\mathcal{A}$ is backward compact.

Asymptotic autonomy. In order to show the asymptotic autonomy as given in (5.6), it suffices to prove $P(\Omega_1) = 1$, where

$$\Omega_1 = \{\omega \in \Omega : \lim_{\tau \to -\infty} \text{dist}_X(\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0\}.$$

Suppose that $P(\Omega_1) < 1$, then $P(\Omega \setminus \Omega_1) > 0$. Let $\Omega_2 = (\Omega \setminus \Omega_1) \cap \Omega_0$, where Ω_0 is the θ-invariant full-measure set given in (5.5). Then, $P(\Omega_2) > 0$, and $\theta_s\Omega_2 \subset \Omega_0$ for all $s \in \mathbb{R}$.

Let $\omega \in \Omega_2$ be fixed. Since $\omega \notin \Omega_1$, there are $\eta > 0$ and $0 > \tau_n \downarrow -\infty$ such that

$$\text{dist}_X(\mathcal{A}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \geq 3\eta, \ \forall n \in \mathbb{N}.$$

By the compactness of $\mathcal{A}(\tau_n, \omega)$, for each $n \in \mathbb{N}$, we can take a $x_n \in \mathcal{A}(\tau_n, \omega)$ such that

$$d_X(x_n, \mathcal{A}_\infty(\omega)) = \text{dist}_X(\mathcal{A}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \geq 3\eta. \tag{5.8}$$

We then prove $\mathcal{A}_{\tau_0} \in \mathfrak{D}_\infty$, where $\mathcal{A}_{\tau_0}(\omega) := \bigcup_{\tau\le\tau_0} \mathcal{A}(\tau,\omega)$. Indeed, by invariance of $\mathcal{A} \in \mathfrak{D}$ and absorption of $\mathcal{K}$, we know, for large $t > 0$,

$$\mathcal{A}(\tau,\omega) = \Phi(t, \tau - t, \theta_{-t})\mathcal{A}(\tau - t, \theta_{-t}) \subset \mathcal{K}(\tau,\omega),$$

which implies that

$$\mathcal{A}_{\tau_0}(\omega) := \bigcup_{\tau\le\tau_0} \mathcal{A}(\tau,\omega) \subset \bigcup_{\tau\le\tau_0} \mathcal{K}(\tau,\omega) = \mathcal{K}_{\tau_0}(\omega).$$

By (d) and by the inclusion-closedness of $\mathfrak{D}_\infty$, we have $\mathcal{A}_{\tau_0} \in \mathfrak{D}_\infty$.

Since $\mathcal{A}_{\tau_0} \in \mathfrak{D}_\infty$ can be attracted by the attractor $\mathcal{A}_\infty$, there is an $n_0 \in \mathbb{N}$ such that $\tau_{n_0} \le \tau_0 \le 0$ and

$$\mathrm{dist}_X(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega),\ \mathcal{A}_\infty(\omega)) \le \eta.$$

Furthermore, by the continuity of $\Phi_\infty :\ X \to X$, we have

$$\mathrm{dist}_X(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\overline{\mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega)},\ \mathcal{A}_\infty(\omega)) \le \eta. \tag{5.9}$$

On the other hand, by the invariance of $\mathcal{A}$, we know

$$\mathcal{A}(\tau_n,\omega) = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\mathcal{A}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega).$$

Hence, we can rewrite $x_n \in \mathcal{A}(\tau_n,\omega)$ as

$$x_n = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_n,\ \text{for some } y_n \in \mathcal{A}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega).$$

If $n \ge n_0$, then $\tau_n - |\tau_{n_0}| \le \tau_n \le \tau_{n_0} \le \tau_0$, and thus

$$\{y_n :\ n \ge n_0\} \subset \bigcup_{\tau\le\tau_0} \mathcal{A}(\tau, \theta_{\tau_{n_0}}\omega) = \mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega).$$

Because we have proved that $\mathcal{A}$ is backward compact, we know $\mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega)$ is a pre-compact set, which further implies that $\{y_n\}$ has a convergent subsequence:

$$y_{n_k} \to y_0 \text{ as } k \to \infty \text{ for some } y_0 \in \overline{\mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega)}.$$

By the θ-invariance of Ω_0, we know $\theta_{\tau_{n_0}}\omega \in \theta_{\tau_{n_0}}\Omega_2 \subset \theta_{\tau_{n_0}}\Omega_0 \subset \Omega_0$. By (c), we can apply the backward convergence (5.5) at the sample $\theta_{\tau_{n_0}}\omega$ for $t = |\tau_{n_0}|$ and $\tau = \tau_{n_k} - |\tau_{n_0}| \to -\infty$. The result is

$$\begin{aligned}&\|x_{n_k} - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\| \\ =&\|\Phi(|\tau_{n_0}|, \tau_{n_k} - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_{n_k} - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\| \le \eta,\end{aligned}$$

if k is large enough. This, together with (5.9), implies that

$$\begin{aligned}&d_X(x_{n_k}, \mathcal{A}_\infty(\omega)) \\ \le&\|x_{n_k} - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\| + d_X(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0, \mathcal{A}_\infty(\omega)) \\ \le&\eta + \mathrm{dist}_X(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\overline{\mathcal{A}_{\tau_0}(\theta_{\tau_{n_0}}\omega)}, \mathcal{A}_\infty(\omega)) \le 2\eta,\end{aligned}$$

if k is large enough. We obtain a contradiction with (5.8), and finish the proof of (5.6).

Finally, we show the asymptotic convergence (5.7). Given $\tau_n \to -\infty$, by (5.6), we know

$$\mathrm{dist}(\mathcal{A}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \to 0 \quad \text{as } n \to \infty.$$

By the Lebesgue theorem (because $P(\Omega)$ is finite), any almost everywhere convergent sequence of measurable functions must be convergent in probability. Hence,

$$\lim_{n\to\infty} P\{\omega : \ \mathrm{dist}_X(\mathcal{A}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \geq \delta\} = 0, \quad \forall \delta > 0.$$

Note that each θ_{τ_n} is measure preserving, it follows the following convergence in probability:

$$\begin{aligned} &P\{\omega : \ \mathrm{dist}_X(\mathcal{A}(\tau_n, \theta_{\tau_n}\omega), \mathcal{A}_\infty(\theta_{\tau_n}\omega)) \geq \delta\} \\ =&P\theta_{\tau_n}\{\omega : \ \mathrm{dist}_X(\mathcal{A}(\tau_n, \theta_{\tau_n}\omega), \mathcal{A}_\infty(\theta_{\tau_n}\omega)) \geq \delta\} \\ =&P\{\omega : \ \mathrm{dist}_X(\mathcal{A}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \geq \delta\} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, by the Riesz theorem, the above convergence in probability implies that there is a subsequence satisfying (5.7) as required. □

5.2. Application results for the BBM equation

Let $\mathfrak{D}_\infty$ be the universe of all tempered sets, where a parametric set $\mathcal{P} = \{\mathcal{P}(\omega)\}$ is tempered if

$$\lim_{t\to+\infty} e^{-\frac{\delta}{3}t}\|\mathcal{P}(\theta_{-t}\omega)\|^2_{H^1} = 0, \ \text{with } \delta := \min(\frac{\nu}{2}, \frac{\nu\lambda_0}{4}), \ \forall\, \omega \in \Omega. \tag{5.10}$$

Let $\Phi_\infty : \mathbb{R}^+ \times \Omega \times H^1_0(Q) \to H^1_0(Q)$ be the RDS generated by the autonomous problem (2.19):

$$\Phi_\infty(t, \omega)\hat{v}_0 = \hat{v}(t, \omega, \hat{v}_0), \quad (t, \omega) \in \mathbb{R}^+ \times \Omega. \tag{5.11}$$

Then, by [25], Φ_∞ has a $\mathfrak{D}_\infty$-random attractor $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega)\}$.

Theorem 5.2. *Let the hypotheses* **F, G** *be satisfied. Then, the non-autonomous cocycle* Φ*, generated from the BBM equation* (2.8)*, has a backward compact random attractor* $\mathcal{A}$ *in* $H^1_0(Q)$ *such that* $\mathcal{A}$ *is a backward tempered set. This attractor backward converges to* $\mathcal{A}_\infty$*, i.e.*

$$\lim_{\tau\to-\infty} dist_{H^1(Q)}(\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \quad \omega \in \Omega. \tag{5.12}$$

For any sequence $\tau_n \to -\infty$*, there is a subsequence* $\{\tau_{n_k}\}$ *such that* P*-a.s.*

$$\lim_{k\to\infty} dist_{H^1(Q)}(\mathcal{A}(\tau_{n_k}, \theta_{\tau_{n_k}}\omega), \mathcal{A}_\infty(\theta_{\tau_{n_k}}\omega)) = 0. \tag{5.13}$$

Proof. By Propostion 3.1, Φ has an increasing random absorbing set $\mathcal{K}$ such that $\mathcal{K}$ is backward tempered, i.e. $\mathcal{K} \in \mathfrak{D}$. We then show that Φ is backward asymptotically compact in $H^1_0(Q)$.

For this end, we let $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{D}$ be fixed, and take arbitrary sequences $s_n \leq \tau$, $t_n \to +\infty$ and $v_{0,n} \in \mathcal{D}(s_n - t_n, \theta_{-t_n}\omega)$. We need to show the precompactness of the following sequence:

$$v_n = \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega)v_{0,n} = v(s_n, s_n - t_n, \theta_{-s_n}\omega, v_{0,n}).$$

Let $B_N = \{v_n :\ n \geq N\}$, $N = 1, 2, \cdots$. It suffices to show the Kuratowski measure $\kappa_{H^1(Q)}(B_N) \to 0$ as $N \to +\infty$, where $\kappa(B)$ means the minimum of diameters d such that B has a finite $d/2$-net.

For each $\eta > 0$, by Lemma 4.1, there are $N_1 \in \mathbb{N}$ and $K \geq 1$ such that

$$\|v_n\|_{H^1(Q_K^c)} \leq \eta, \quad \text{for all} \ \ n \geq N_1, \tag{5.14}$$

where we recall $Q_K^c = Q \setminus Q_K$ and $Q_K = \{x \in Q : |x_3| \leq K\}$. By Lemma 4.2, there are $i \in \mathbb{N}$ and $N_2 \geq N_1$ such that

$$\|(I - P_i)(\xi_K v_n)\|_{H^1(Q_{2K})} \leq \eta, \quad \text{for all} \ \ n \geq N_2. \tag{5.15}$$

By Lemma 3.1, the set B_{N_2} is bounded in $H_0^1(Q)$. Then, $\{\xi_K v_n :\ n \geq N_2\}$ is bounded in $H_0^1(Q_{2K})$, hence, $P_i\{\xi_K v_n :\ n \geq N_2\}$ is pre-compact in $H_0^1(Q_{2k})$ due to the finitely dimensional range of P_i. In a conclusion,

$$\kappa_{H^1(Q_{2K})}(P_i\{\xi_K v_n :\ n \geq N_2\}) = 0,$$

which along with (5.15) implies that

$$\begin{aligned}
&\kappa_{H^1(Q_{2K})}\{\xi_K v_n :\ n \geq N_2\}\\
\leq &\kappa_{H^1(Q_{2K})}(P_i\{\xi_K v_n :\ n \geq N_2\}) + \kappa_{H^1(Q_{2K})}((I - P_i)\{\xi_K v_n :\ n \geq N_2\})\\
\leq &2\eta.
\end{aligned}$$

Since $\xi_K v = v$ on Q_K, we have

$$\begin{aligned}
\kappa_{H^1(Q_K)}(B_{N_2}) &= \kappa_{H^1(Q_K)}\{\xi_K v_n :\ n \geq N_2\}\\
&\leq \kappa_{H^1(Q_{2K})}\{\xi_K v_n :\ n \geq N_2\} \leq 2\eta.
\end{aligned} \tag{5.16}$$

Since $B_{N_2} \subset B_{N_1}$, we deduce from (5.14) and (5.16) that

$$\kappa_{H^1(Q)}(B_{N_2}) \leq \kappa_{H^1(Q_K)}(B_{N_2}) + \kappa_{H^1(Q_K^c)}(B_{N_1}) \leq 4\eta.$$

So far, we have verified both conditions (a) and (b) in Theorem 5.1. Therefore, Φ possesses a backward compact random attractor $\mathcal{A} \in \mathfrak{D}$, where the measurability of $\mathcal{A}$ follows from the measurability of the NRDS and the absorbing set.

On the other hand, by Proposition 2.2, the NRDS Φ backward converges to the RDS Φ_∞, that is, the condition (c) in Theorem 5.1 holds true.

We need to verify the condition (d) in Theorem 5.1. In fact, we can prove $\mathcal{K}_{-1} \in \mathfrak{D}_\infty$, where $\mathcal{K}_{-1}(\omega) = \cup_{\tau \leq -1}\mathcal{K}(\tau, \omega)$. Indeed, since $\mathcal{K}(\tau, \omega)$ is increasing in τ, it follows $\mathcal{K}_{-1}(\omega) = \mathcal{K}(-1, \omega)$. Hence, by the definition (3.6) of $\mathcal{K}$, we have

$$e^{-\frac{\delta}{3}t}\|\mathcal{K}_{-1}(\theta_{-t}\omega)\|^2_{H^1} \leq e^{-\frac{\delta}{3}t} + ce^{-\frac{\delta}{3}t}R(-1, \theta_{-t}\omega).$$

By (2.6) and Lemma 2.1 (g is backward tempered), we have

$$\begin{aligned}
&e^{-\frac{\delta}{3}t}R(-1, \theta_{-t}\omega)\\
&= e^{-\frac{\delta}{3}t}\sup_{\tau \leq -1}\int_{-\infty}^0 e^{\delta r + 2|z(\theta_{r-t}\omega)| + 2\int_r^0 z(\theta_{\sigma-t}\omega)d\sigma}\|g(r+\tau)\|^2 dr\\
&= e^{-\frac{\delta}{3}t}\sup_{\tau \leq -1}\int_{-\infty}^{\tau} e^{\delta(r-\tau) + 2|z(\theta_{r-\tau-t}\omega)| + 2\int_{r-\tau-t}^{-t} z(\theta_\sigma\omega)d\sigma}\|g(r)\|^2 dr\\
&\leq e^{(\frac{1}{4}-\frac{1}{3})\delta t}\sup_{\tau \leq -1}\int_{-\infty}^{\tau} e^{\frac{3}{4}\delta(r-\tau)}\|g(r)\|^2 dr \to 0 \text{ as } t \to +\infty.
\end{aligned}$$

Hence, $\mathcal{K}_{-1} \in \mathfrak{D}_\infty$. So, the needed convergence (5.12)-(5.13) follows from Theorem 5.1 immediately. □

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