

# THE ALTERNATIVE LEGENDRE TAU METHOD FOR SOLVING NONLINEAR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS

Sohrab Bazm<sup>1,†</sup> and Alireza Hosseini<sup>2</sup>

**Abstract** In this paper, the alternative Legendre polynomials (ALPs) are used to approximate the solution of a class of nonlinear multi-order fractional differential equations (FDEs). First, the operational matrix of fractional integration of an arbitrary order and the product operational matrix are derived for ALPs. These matrices together with the spectral Tau method are then utilized to reduce the solution of the mentioned equations into the one of solving a system of nonlinear algebraic equations with unknown ALP coefficients of the exact solution. The fractional derivatives are considered in the Caputo sense and the fractional integration is described in the Riemann-Liouville sense. Numerical examples illustrate that the present method is very effective for linear and nonlinear multi-order FDEs and high accuracy solutions can be obtained only using a small number of ALPs.

**Keywords** Fractional-order differential equations, alternative Legendre polynomials, operational matrix, Tau method.

**MSC(2010)** 26A33, 05E35, 33C45, 44A45.

## 1. Introduction

Many problems in physics, chemistry and engineering give rise to the linear and nonlinear fractional differential equations [1, 11, 16, 18, 21, 34, 35]. Due to the lack of analytical solutions to fractional differential equations in most cases, various numerical methods developed by many authors for obtaining approximate solutions of these equations. These methods include spectral methods [2, 5, 9, 13, 22, 25, 29, 37], Laplace transform method [19], differential transform method [12, 36], Adomian decomposition method [23], variational iteration method [24, 26], wavelet method [17, 28–31, 33, 38] and etc. Among them, the spectral methods based on the application of different sets of basis functions have special attention of many researchers. The main feature of these methods lies in their accuracy for a given number of unknowns. It is proved that, for smooth problems, they offer exponential rates of convergence [5, 39].

In this paper, we intend to use the operational Tau method based on ALPs, a new

<sup>†</sup>the corresponding author. Email address: sbazm@maragheh.ac.ir (S. Bazm)

<sup>1</sup>Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553 Maragheh, Iran

<sup>2</sup>School of Mathematics, Statistics and Computer Science, University of Tehran, P.O. Box 14115-175, Tehran, Iran

family of orthogonal polynomials were recently introduced by Vladimir Chelyshkov [6], to numerically solve the following nonlinear multi-order FDE:

$$D^\nu u(x) + \sum_{i=1}^p \gamma_i D^{\beta_i} u(x) + \sum_{i=1}^q \delta_i u^i(x) = f(x), \quad (1.1)$$

with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0, 1, \dots, m-1, \quad (1.2)$$

where  $\gamma_i (i = 1, \dots, p)$  and  $\delta_i (i = 1, \dots, q)$  are real constant coefficients,  $m = \lceil \nu \rceil$ , and  $0 < \beta_1 < \beta_2 < \dots < \beta_p < \nu$ . In Eq. (1.1),  $D^\nu u(x)$  denotes the Caputo fractional derivative of order  $\nu$  for  $u(x)$  and the values of  $d_i (i = 0, \dots, m-1)$  describe the initial state of  $u(x)$  and  $f(x)$  is a given source function.

In the present method, by applying multiple fractional integration in the Riemann-Liouville sense, the FDE (1.1) is first converted to a fully integrated form. Then, the various signals involved in the integrated form equation are approximated by representing them as linear combinations of ALPs. Finally, the integrated form equation is converted to an algebraic equation by introducing the operational matrix of the product and the operational matrix of fractional integration for ALPs.

It should be noted that, up to now, the operational matrix for fractional Riemann-Liouville integration has been derived for several types of functions, such as Bernoulli wavelets [30], Legendre wavelets [17, 38], shifted Jacobi polynomials [4], Laguerre polynomials [3], modified generalized Laguerre polynomials [3] and shifted Chebyshev polynomials [2].

The remainder of this paper is organized as follows. First, in Section 2, we give necessary knowledge about the fractional calculus theory. In Section 3, we briefly describe some characteristics of the ALPs and derive the ALP operational matrix of the product and the ALP operational matrix of fractional integration in the Riemann-Liouville sense. Section 4 is devoted to the application of alternative Legendre Tau method using ALPs for the solution of problem (1.1)-(1.2). Some numerical examples are given in Section 5 to demonstrate the efficiency and accuracy of the method. Conclusions of the work are given in Section 6.

## 2. Preliminaries of the fractional calculus

In this section, we give some definitions and basic properties of fractional calculus theory that will be used in this paper [8, 27].

**Definition 2.1.** The Riemann-Liouville fractional integral operator of order  $\nu \geq 0$ , of a function  $f(t)$  is defined as

$$J^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau, \quad \nu > 0, t > 0,$$

$$J^0 f(t) = f(t),$$

where  $\Gamma(\cdot)$  denotes the Gamma function.

**Definition 2.2.** Let  $\nu > 0$  and  $m := \lceil \nu \rceil$  be the ceiling function of  $m$ . The operator  $D^\nu$ , defined by

$$D^\nu f(t) = J^{m-\nu} f^{(m)}(t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^t (t-\tau)^{m-\nu-1} f^{(m)}(\tau) d\tau, & m-1 < \nu < m, \\ f^{(m)}(t), & \nu = m, \end{cases}$$

is called the Caputo fractional differential operator of order  $\nu$ .

The Riemann-Liouville fractional integral operator  $J^\nu$  and the Caputo fractional differential operator  $D^\nu$  satisfy the following properties

$$J^\nu t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)} t^{\beta+\nu}, \quad \nu \geq 0, \beta > -1, \quad (2.1)$$

$$J^\nu D^\nu f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0) \frac{t^i}{i!}, \quad t > 0, m = \lceil \nu \rceil, \quad (2.2)$$

$$J^{\nu_1} J^{\nu_2} f(t) = J^{\nu_1+\nu_2} f(t), \quad (2.3)$$

$$J^\nu (\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 J^\nu f(t) + \lambda_2 J^\nu g(t). \quad (2.4)$$

### 3. Alternative Legendre polynomials

#### 3.1. Definition and properties

Let  $n$  be a fixed integer number. The set  $\mathcal{P}_n = \{\mathcal{P}_{nk}(x)\}_{k=0}^n$  of alternative Legendre polynomials is defined over interval  $[0, 1]$  as [6]

$$\mathcal{P}_{nk}(x) = \sum_{j=k}^n \sigma_{k,j} x^j = \sum_{j=0}^{n-k} \sigma_{k,j+k} x^{k+j}, \quad k = 0, 1, \dots, n, \quad (3.1)$$

where

$$\sigma_{k,j} = (-1)^{j-k} \binom{n-k}{j-k} \binom{n+j+1}{n-k}. \quad (3.2)$$

These polynomials are orthogonal over the interval  $[0, 1]$  with respect to the weight function  $w(x) = 1$ , and satisfy the orthogonality relationships

$$\int_0^1 \mathcal{P}_{nk}(x) \mathcal{P}_{nl}(x) dx = \frac{1}{k+l+1} \delta_{kl}, \quad k, l = 0, 1, \dots, n, \quad (3.3)$$

where  $\delta_{kl}$  denotes the Kronecker delta. In contrast to common sets of orthogonal polynomials, every member in  $\mathcal{P}_n$  has degree  $n$ .

Relation (3.1) yields Rodrigues's type representation

$$\mathcal{P}_{nk}(x) = \frac{1}{(n-k)!} \frac{1}{x^{k+1}} \frac{d^{n-k}}{dx^{n-k}} (x^{n+k+1} (1-x)^{n-k}), \quad k = 0, 1, \dots, n. \quad (3.4)$$

It follows from (3.4) that

$$\int_0^1 \mathcal{P}_{nk}(x) dx = \int_0^1 x^n dx = \frac{1}{n+1}, \quad k = 0, 1, \dots, n.$$

By using Eq. (3.1), the ALP vector

$$\Phi_n(x) = [\mathcal{P}_{n0}(x), \mathcal{P}_{n1}(x), \dots, \mathcal{P}_{nn}(x)]^T, \quad (3.5)$$

can be written in the form

$$\Phi_n(x) = QT_n(x), \quad (3.6)$$

where

$$T_n(x) = [1, x, x^2, \dots, x^n]^T,$$

and  $Q$  is the following upper triangular matrix of order  $(n+1) \times (n+1)$

$$Q = \begin{bmatrix} \sigma_{0,0} & \sigma_{0,1} & \dots & \sigma_{0,n} \\ 0 & \sigma_{1,1} & \dots & \sigma_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{n,n} \end{bmatrix}. \quad (3.7)$$

Using the orthogonal property (3.3), the dual matrix of  $\Phi(x)$  is obtained as

$$\bar{D} = \int_0^1 \Phi_n(x) \Phi_n^T(x) dx = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2n+1} \end{bmatrix}. \quad (3.8)$$

### 3.2. Function approximation

Let  $\mathcal{H} = \mathfrak{L}^2(0,1)$  be the Banach space of the measurable functions  $f : (0,1) \rightarrow \mathbb{R}$  such that  $\int_0^1 f^2(t) dt < \infty$  [5]. It is endowed with the norm

$$\|f\|_{\mathfrak{L}^2(0,1)} = \left( \int_0^1 f^2(t) dt \right)^{\frac{1}{2}}.$$

The space  $\mathfrak{L}^2(0,1)$  is a Hilbert space for the inner product [5]

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

Suppose that

$$\mathcal{H}_n = \text{span}\{\mathcal{P}_{n0}(x), \mathcal{P}_{n1}(x), \dots, \mathcal{P}_{nn}(x)\}.$$

Since  $\mathcal{H}_n$  is a finite dimensional subspace of  $\mathcal{H}$ , then it is closed [20, Theorem 2.4-3] and for every given  $f \in \mathcal{H}$  there exist a unique best approximation  $\bar{f}_n \in \mathcal{H}_n$  [20, Theorem 6.2-5] such that

$$\|f - \bar{f}_n\|_{\mathfrak{L}^2(0,1)} \leq \|f - g\|_{\mathfrak{L}^2(0,1)}, \quad \forall g \in \mathcal{H}_n.$$

Moreover, we have [7, Theorem 4.14]

$$\bar{f}_n(x) = \sum_{k=0}^n f_k \mathcal{P}_{nk}(x),$$

where

$$f_k = \frac{\langle f, \mathcal{P}_{nk} \rangle}{\langle \mathcal{P}_{nk}, \mathcal{P}_{nk} \rangle} = (2k+1)\langle f, \mathcal{P}_{nk} \rangle, \quad k = 0, 1, \dots, n. \quad (3.9)$$

Therefore, any function  $f(x) \in \mathcal{H}$  may be approximated by its projection in the subspace  $\mathcal{H}_n$  as

$$f(x) \simeq \bar{f}_n(x) = \Phi_n^T(x)F, \quad (3.10)$$

where

$$\Phi_n(x) = [\mathcal{P}_{n0}(x), \mathcal{P}_{n1}(x), \dots, \mathcal{P}_{nn}(x)]^T, \quad (3.11)$$

and

$$F = [f_0, f_1, \dots, f_n]^T.$$

Let us denote by  $\mathcal{C}^m([0, 1])$  the space of functions  $f : [0, 1] \rightarrow \mathbb{R}$  with continuous derivatives

$$f^{(i)}(x) = \frac{d^i}{dx^i} f(x), \quad x \in [0, 1],$$

for all  $i$  such that  $0 \leq i \leq m$ .

In the following theorem, we obtain an estimation of the error norm of the best approximation of smooth functions by some polynomials in  $\mathcal{H}_n$ .

**Theorem 3.1.** *Suppose that  $f(x) \in \mathcal{C}^{n+1}([0, 1])$  and  $\bar{f}(x) = \Phi^T(x)F$  be its expansion in terms of ALPs, as described by Eq. (3.10). Then*

$$\|f - \bar{f}\|_2 \leq \frac{C}{(n+1)!2^{2n+1}}, \quad (3.12)$$

where  $C$  is a constant such that

$$|f^{(n+1)}(x)| \leq C, \quad x \in [0, 1].$$

**Proof.** Assume that  $p_n(x)$  is the interpolating polynomial to  $f$  at points  $x_l$ , where  $x_l, l = 0, 1, \dots, n$ , are the roots of the  $(n+1)$ -degree shifted Chebyshev polynomial in  $\Omega$ . Then, for any  $x \in [0, 1]$ , we have [15]

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{l=0}^n (x - x_l), \quad \xi_x \in [0, 1].$$

By taking into account the estimates for Chebyshev interpolation nodes [15], we obtain

$$|f(x) - p_n(x)| \leq \frac{C}{(n+1)!2^{2n+1}}, \quad \forall x \in [0, 1].$$

Since  $\bar{f}$  is the unique best approximation of  $f$  in  $\mathcal{H}_n$ , then we have

$$\begin{aligned} \|f - \bar{f}\|_2^2 &\leq \|f - p_n\|_2^2 = \int_0^1 |f(x) - p_n(x)|^2 dx \\ &\leq \int_0^1 \left( \frac{C}{(n+1)!2^{2n+1}} \right)^2 dx \\ &= \left( \frac{C}{(n+1)!2^{2n+1}} \right)^2. \end{aligned} \quad (3.13)$$

Taking the square root of both sides of (3.13) gives the result.  $\square$

### 3.3. Operational matrices of fractional integration and product for ALPs

Many papers have employed the operational matrix for spectral methods [2, 4, 5, 14, 17, 22, 25, 28–31, 33, 38]. In this section, we will derive the Riemann-Liouville operational matrix of fractional integration and also the operational matrix of product for ALPs. These operational matrices will be applied in the next section for solving nonlinear FDEs by the alternative Legendre spectral Tau method. We need the following two lemmas—the proofs are quite easy and so we omit them.

**Lemma 3.1.** *Let  $\mathcal{P}_{nk}(x)$  and  $\mathcal{P}_{nj}(x)$  are  $k$ th and  $j$ th ALPs, respectively and  $v_1 = \min\{j, k\}$  and  $v_2 = \max\{j, k\}$ . Then the product of  $\mathcal{P}_{nk}(x)$  and  $\mathcal{P}_{nj}(x)$  is a polynomial of degree at most  $2n$  that can be written as*

$$\mathcal{Q}_{2n}^{(k,j)}(x) = \sum_{r=k+j}^{2n} \lambda_r^{(k,j)} x^r,$$

where

$$\lambda_r^{(k,j)} = \sum_{l=\max\{r-n, v_2\}}^{\min\{r-v_1, n\}} \sigma_{v_2, l} \sigma_{v_1, r-l}.$$

**Lemma 3.2.** *Let  $r$  be a nonnegative integer, then*

$$\int_0^1 x^r \mathcal{P}_{nk}(x) dx = \sum_{l=k}^n \frac{\sigma_{k,l}}{r+l+1}, \quad k = 0, 1, \dots, n.$$

By combining Lemmas 3.1 and 3.2, we can easily obtain the following result.

**Lemma 3.3.** *Let  $\mathcal{P}_{ni}(x)$ ,  $\mathcal{P}_{nj}(x)$  and  $\mathcal{P}_{nk}(x)$  are  $i$ th,  $j$ th and  $k$ th ALPs, respectively.*

*If, according to Lemma 3.1, we define  $\mathcal{P}_{nk}(x)\mathcal{P}_{nj}(x) = \sum_{r=k+j}^{2n} \lambda_r^{(k,j)} x^r$ , then*

$$\int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)\mathcal{P}_{nk}(x) dx = \sum_{r=k+j}^{2n} \lambda_r^{(k,j)} \sum_{l=k}^n \frac{\sigma_{k,l}}{r+l+1}.$$

In the next theorem, the ALP operational matrix of fractional integration in the Riemann-Liouville sense is first derived.

**Theorem 3.2.** *Let  $\Phi_n(x)$  be the ALP vector defined by (3.11) and  $\nu \geq 0$ . Then*

$$J^\nu \Phi_n(x) \simeq P^{(\nu)} \Phi_n(x), \quad (3.14)$$

where  $P^{(\nu)} = [p_{kr}^{(\nu)}]_{k,r=0}^n$  is the ALP operational matrix of integration of fractional order  $\nu$  in the Riemann-Liouville sense in which

$$p_{kr}^{(\nu)} = (2r+1) \sum_{j=k}^n \frac{j!}{\Gamma(j+1+\nu)} \sigma_{k,j} \sum_{s=r}^n \frac{1}{\nu+j+s+1} \sigma_{r,s}.$$

In (3.14), the sign  $\simeq$  means that any element in the right-hand side vector is an approximation to the corresponding one in the left-hand side vector.

**Proof.** Since, by relation (2.4), the Riemann-Liouville's fractional integration is a linear operation, then using (2.1) and (3.1) we have

$$\begin{aligned} J^\nu \mathcal{P}_{nk}(x) &= \sum_{j=k}^n \sigma_{k,j} J^\nu x^j \\ &= \sum_{j=k}^n \sigma_{k,j} \frac{j!}{\Gamma(j+1+\nu)} x^{\nu+j}. \end{aligned}$$

Using Eq. (3.10), one can approximate  $x^{\nu+j}$  in terms of ALPs as

$$x^{\nu+j} \simeq \sum_{r=0}^n \varepsilon_{jr} \mathcal{P}_{nr}(x),$$

where, by Eq. (3.9) and Lemma 3.2, we have

$$\begin{aligned} \varepsilon_{jr} &= (2r+1) \int_0^1 x^{\nu+j} \mathcal{P}_{nr}(x) dx \\ &= (2r+1) \sum_{s=r}^n \sigma_{r,s} \int_0^1 x^{\nu+j+s} dx \\ &= (2r+1) \sum_{s=r}^n \frac{\sigma_{r,s}}{\nu+j+s+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} J^\nu \mathcal{P}_{nk}(x) &\simeq \sum_{r=0}^n (2r+1) \left[ \sum_{j=k}^n \frac{j! \sigma_{k,j}}{\Gamma(j+1+\nu)} \sum_{s=r}^n \frac{\sigma_{r,s}}{\nu+j+s+1} \right] \mathcal{P}_{nr}(x) \\ &= \sum_{r=0}^n p_{kr}^{(\nu)} \mathcal{P}_{nr}(x). \end{aligned}$$

Varying  $k$  from 0 to  $n$  gives the result.  $\square$

The following theorem is of great importance, when we want to reduce the integral equation (1.1) to a set of algebraic equations in the next section.

**Theorem 3.3.** Let  $C = [c_0, c_1, \dots, c_n]^T$  be an arbitrary vector in  $\mathbb{R}^{n+1}$ . Then

$$\Phi_n(x) \Phi_n^T(x) C \simeq \hat{C} \Phi_n(x), \quad (3.15)$$

where  $\hat{C} = [\hat{c}_{ik}]_{i,k=0}^n$  is the ALP operational matrix of the product in which

$$\hat{c}_{ik} = (2k+1) \sum_{j=0}^n c_j \tau_{ijk}, \quad \tau_{ijk} = \int_0^1 \mathcal{P}_{ni}(x) \mathcal{P}_{nj}(x) \mathcal{P}_{nk}(x) dx. \quad (3.16)$$

The values  $\tau_{ijk}$  in (3.16) are computed by means of Lemma 3.3. The sign  $\simeq$  is previously defined in Theorem 3.2.

**Proof.** We have

$$\Phi_n(x) \Phi_n^T(x) C = \left[ \sum_{j=0}^n c_j \mathcal{P}_{n0}(x) \mathcal{P}_{nj}(x), \dots, \sum_{j=0}^n c_j \mathcal{P}_{nn}(x) \mathcal{P}_{nj}(x) \right]^T. \quad (3.17)$$

Using Eq. (3.10), one can approximate  $\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)$ , for  $i, j = 0, 1, \dots, n$ , in terms of ALPs as

$$\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x) \simeq \sum_{k=0}^n a_{ijk}\mathcal{P}_{nk}(x),$$

where, by Eq. (3.9), we have

$$a_{ijk} = (2k+1) \int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)\mathcal{P}_{nk}(x)dx = (2k+1)\tau_{ijk}.$$

Therefore, for every  $i = 0, 1, \dots, n$ , we have

$$\begin{aligned} \sum_{j=0}^n c_j \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x) &\simeq \sum_{j=0}^n c_j \sum_{k=0}^n (2k+1)\tau_{ijk}\mathcal{P}_{nk}(x) \\ &= \sum_{k=0}^n \left( (2k+1) \sum_{j=0}^n c_j \tau_{ijk} \right) \mathcal{P}_{nk}(x) \\ &= \sum_{k=0}^n \hat{c}_{ik} \mathcal{P}_{nk}(x). \end{aligned} \quad (3.18)$$

Substituting (3.18) into (3.17) leads to

$$\Phi_n(x)\Phi_n^T(x)C \simeq \left[ \sum_{k=0}^n \hat{c}_{0k}\mathcal{P}_{nk}(x), \dots, \sum_{k=0}^n \hat{c}_{nk}\mathcal{P}_{nk}(x) \right]^T = \hat{C}\Phi_n(x),$$

which is the desired result.  $\square$

## 4. Alternative Legendre spectral Tau method

In this section, we use the spectral Tau method based on ALPs to solve nonlinear FDE (1.1) with initial conditions (1.2). To this end, we first apply the Riemann-Liouville fractional integral of order  $\nu$  on (1.1). Then, making use of (2.1)-(2.4), we get the integrated form of equation (1.1) as

$$u(x) + \sum_{i=1}^p \gamma_i J^{\nu-\beta_i} u(x) + \sum_{i=1}^q \delta_i J^\nu u^i(x) = g(x), \quad (4.1)$$

where

$$g(x) = J^\nu f(x) + \sum_{i=0}^{m-1} \frac{d_i}{i!} x^i + \sum_{i=1}^p \gamma_i \sum_{j=0}^{m_i-1} \frac{d_j}{\Gamma(j+1+\nu-\beta_i)} x^{j+\nu-\beta_i}, \quad (4.2)$$

in which  $m_i = \lceil \beta_i \rceil$ ,  $i = 1, \dots, p$ .

In order to use the alternative Legendre Tau method for solving the fully integrated problem (4.1) with initial conditions (4.2), we approximate  $u(x)$  and  $g(x)$  by the ALPs as

$$g(x) \simeq \bar{g}_n(x) = \Phi_n^T(x)G, \quad (4.3)$$

$$u(x) \simeq \bar{u}_n(x) = \Phi_n^T(x)U, \quad (4.4)$$

where the vector  $G = [g_0, \dots, g_n]^T$  is known but  $U = [u_0, \dots, u_n]^T$  is an unknown vector. We can also use (3.15) to approximate the nonlinear term  $u^i(x)$ , for any integer  $i \geq 1$ , in terms of ALPs as

$$\begin{aligned} u^i(x) &\simeq (\Phi_n^T(x)U)^i = U^T \Phi_n(x) \Phi_n^T(x) U (\Phi_n^T(x)U)^{i-2} \\ &\simeq U^T \hat{U} \Phi_n(x) (\Phi_n^T(x)U)^{i-2} \\ &\vdots \\ &\simeq U^T \hat{U}^{i-1} \Phi_n(x), \end{aligned} \quad (4.5)$$

where  $\hat{U}$  is the  $(n+1) \times (n+1)$  ALP operational matrix of product for  $U$ .

Now, using (2.4), (4.5) and Theorems 3.2 and 3.3, the Riemann-Liouville fractional integrals of orders  $\nu - \beta_i$  and  $\nu$  can be written, respectively, as

$$\begin{aligned} J^{\nu-\beta_i} u(x) &\simeq U^T J^{\nu-\beta_i} \Phi_n(x) \\ &\simeq U^T P^{(\nu-\beta_i)} \Phi_n(x), \quad i = 1, 2, \dots, p, \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} J^\nu u^i(x) &\simeq J^\nu (U^T \hat{U}^{i-1} \Phi_n(x)) \\ &\simeq U^T \hat{U}^{i-1} J^\nu \Phi_n(x) \\ &\simeq U^T \hat{U}^{i-1} P^{(\nu)} \Phi_n(x), \quad i = 1, 2, \dots, q, \end{aligned} \quad (4.7)$$

where  $P^{(\nu)}$  is the ALP operational matrix of fractional integration of order  $\nu$  in the Riemann-Liouville sense which is derived in Theorem 3.2.

Applying Eqs. (4.3)-(4.7), the residual function  $R_n(x)$  for Eq. (4.1) can be written as

$$R_n(x) = \left( U^T + U^T \left( \sum_{i=1}^p \gamma_i P^{(\nu-\beta_i)} + \sum_{i=1}^q \delta_i \hat{U}^{i-1} P^{(\nu)} \right) - G^T \right) \Phi_n(x). \quad (4.8)$$

As in a typical Tau method, see [5], we generate  $n - m + 1$  nonlinear algebraic equations by employing

$$\langle R_n(x), \mathcal{P}_{nk}(x) \rangle = \int_0^1 R_n(x) \mathcal{P}_{nk}(x) dx = 0, \quad k = 0, 1, \dots, n - m. \quad (4.9)$$

Also from Eqs. (1.2) and (3.6), we get

$$d_i = u^{(i)}(0) \simeq i! U^T Q e_{i+1} = i! \sum_{r=0}^i u_r \sigma_{r,i}, \quad i = 0, 1, \dots, m - 1, \quad (4.10)$$

where  $Q$  is the upper triangular matrix defined in (3.7) and  $e_{i+1}$  is the  $i$ -th standard unit vector of order  $n + 1$ .

Eqs. (4.9) and (4.10) form a system of  $n + 1$  nonlinear algebraic equations. These nonlinear equations can be solved for unknown coefficients of the vector  $U$ . Consequently,  $\bar{u}_n(x)$  given in Eq. (4.4) can be calculated, which give a solution of Eq. (1.1) with the initial conditions (1.2).

## 5. Illustrative examples

In this section, the presented method is applied to five examples to show its accuracy and applicability. Experiments were performed on a personal computer using a 2.50 GHz processor and the codes were written in *Mathematica 9*. In the case of nonlinear equations, we have solved system (4.9)-(4.10) using the *Mathematica* function *FindRoot*, which uses Newton's method as the default method. In all the cases, this function has succeeded to obtain an accurate approximate solution of the system, even starting with a zero initial approximation.

**Example 5.1.** Consider the following nonlinear FDE

$$D^2 u(x) - D^{\frac{3}{2}} u(x) + u(x) - u^3(x) = x - x^3, \quad x \in [0, 1], \quad (5.1)$$

$$u(0) = 0, u'(0) = 1. \quad (5.2)$$

The exact solution of this problem is  $u(x) = x$ . If we use the technique described in Section 4 with  $n = 3$ , we get

$$U^T = \left[ 0, \frac{1}{10}, \frac{1}{2}, \frac{7}{5} \right].$$

Thus, by relation (4.4), we have

$$u(x) \simeq \bar{u}_3(x) = \sum_{k=0}^3 u_k \mathcal{P}_{3k}(x) = \left[ 0, \frac{1}{10}, \frac{1}{2}, \frac{7}{5} \right] \begin{bmatrix} 4 - 30x + 60x^2 - 35x^3 \\ 10x - 30x^2 + 21x^3 \\ 6x^2 - 7x^3 \\ x^3 \end{bmatrix} = x,$$

which is the exact solution.

**Example 5.2.** [10, 19, 32] Consider the following nonlinear initial value problem,

$$D^3 u(x) + D^{\frac{5}{2}} u(x) + u^2(x) = x^4, \quad x \in [0, 1], \quad (5.3)$$

$$u(0) = u'(0) = 0, u''(0) = 2. \quad (5.4)$$

The exact solution of this problem is  $u(x) = x^2$ . If we use the technique described in Section 4 with  $n = 4$ , we get

$$U^T = \left[ 0, 0, \frac{1}{21}, \frac{1}{3}, \frac{9}{7} \right].$$

Thus, by relation (4.4), we have

$$u(x) \simeq \bar{u}_4(x) = \sum_{k=0}^4 u_k \mathcal{P}_{4k}(x) = \left[ 0, 0, \frac{1}{21}, \frac{1}{3}, \frac{9}{7} \right] \begin{bmatrix} 5 - 60x + 210x^2 - 280x^3 + 126x^4 \\ 20x - 105x^2 + 168x^3 - 84x^4 \\ 21x^2 - 56x^3 + 36x^4 \\ 8x^3 - 9x^4 \\ x^4 \end{bmatrix} = x^2,$$

which is the exact solution.

**Example 5.3.** [3, 10, 19, 32] Consider the following linear initial value problem in the case of the inhomogeneous Bagly- Torvik equation

$$D^2u(x) + D^{\frac{3}{2}}u(x) + u(x) = 1 + x, \quad x \in [0, 1], \quad (5.5)$$

$$u(0) = u'(0) = 1. \quad (5.6)$$

The exact solution of this problem is  $u(x) = x + 1$ . If we use the technique described in Section 4 with  $n = 2$ , we get

$$U^T = \left[ \frac{1}{3}, \frac{5}{4}, \frac{35}{12} \right].$$

Thus, by relation (4.4), we have

$$u(x) \simeq \bar{u}_2(x) = \sum_{k=0}^2 u_k \mathcal{P}_{2k}(x) = \left[ \frac{1}{3}, \frac{5}{4}, \frac{35}{12} \right] \begin{bmatrix} 3 - 12x + 10x^2 \\ 4x - 5x^2 \\ x^2 \end{bmatrix} = 1 + x,$$

which is the exact solution. Also, for  $n = 3$  we obtain

$$U^T = \left[ \frac{1}{4}, \frac{17}{20}, \frac{7}{4}, \frac{63}{20} \right].$$

Therefore, by relation (4.4), we get

$$u(x) \simeq \bar{u}_3(x) = \sum_{k=0}^3 u_k \mathcal{P}_{3k}(x) = \left[ \frac{1}{4}, \frac{17}{20}, \frac{7}{4}, \frac{63}{20} \right] \begin{bmatrix} 4 - 30x + 60x^2 - 35x^3 \\ 10x - 30x^2 + 21x^3 \\ 6x^2 - 7x^3 \\ x^3 \end{bmatrix} = 1 + x,$$

which is again the exact solution.

**Example 5.4.** [2, 3] Consider the following linear initial value problem

$$D^2u(x) - 2Du(x) + D^{\frac{1}{2}}u(x) + u(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{\frac{5}{2}}, \quad x \in [0, 1], \quad (5.7)$$

$$u(0) = u'(0) = 0. \quad (5.8)$$

The exact solution of this problem is  $u(x) = x^3$ . If we use the technique described in Section 4 with  $n = 4$ , we get

$$U^T = \left[ 0, 0, \frac{1}{8}, \frac{9}{8} \right].$$

Thus, by relation (4.4), we have

$$\begin{aligned}
 u(x) &\simeq \bar{u}_4(x) = \sum_{k=0}^4 u_k \mathcal{P}_{4k}(x) \\
 &= \begin{bmatrix} 0, 0, \frac{1}{8}, \frac{9}{8} \end{bmatrix} \begin{bmatrix} 5 - 60x + 210x^2 - 280x^3 + 126x^4 \\ 20x - 105x^2 + 168x^3 - 84x^4 \\ 21x^2 - 56x^3 + 36x^4 \\ 8x^3 - 9x^4 \\ x^4 \end{bmatrix} = x^3,
 \end{aligned}$$

which is the exact solution.

**Example 5.5.** Consider the following linear initial value problem

$$D^{\frac{1}{2}}u(x) - u(x) = \frac{2}{\Gamma(2.5)}x^{1.5} - x^2 - 2, \quad x \in [0, 1], \quad (5.9)$$

$$u(0) = 2. \quad (5.10)$$

The exact solution of this problem is  $u(x) = x^2 + 2$ . If we use the technique described in Section 4 with  $n = 2, 3, 4$ , we get respectively

$$\begin{aligned}
 U^T &= \left[ \frac{2}{3}, 2, \frac{13}{3} \right], \\
 U^T &= \left[ \frac{1}{2}, \frac{3}{2}, \frac{8}{3}, \frac{14}{3} \right], \\
 U^T &= \left[ \frac{2}{5}, \frac{6}{5}, \frac{43}{21}, \frac{47}{15}, \frac{171}{35} \right],
 \end{aligned}$$

which, in all the cases, lead to the exact solution.

## 6. Conclusion

In this paper, a new Riemann-Liouville operational matrix of fractional integration based on the alternative Legendre polynomials is obtained. This matrix is used to approximate numerical solution of a class of nonlinear multi-order fractional differential equations in the Caputo sense. The spectral Tau method was employed to reduce the problem into a nonlinear system of algebraic equations. Moreover, some test problems are presented to show the efficiency and applicability of the technique. The main feature of the method is that only a small number of ALPs is needed to obtain a satisfactory result.

## Acknowledgements

The authors are grateful to the anonymous referees for their useful suggestions which improved the contents of this article.

## References

- [1] R. Bagley and P. Torvik, *Fractional calculus in the transient analysis of viscoelastically damped structures.*, AIAA Journal, 1985, 23(6), 918–925.
- [2] A. H. Bhrawy and A. S. Alofi, *The operational matrix of fractional integration for shifted Chebyshev polynomials*, Appl. Math. Lett., 2013, 26(1), 25–31.
- [3] A. H. Bhrawy, T. M. Taha and J. A. T. Machado, *A review of operational matrices and spectral techniques for fractional calculus*, Nonlinear Dynam., 2015, 81(3), 1023–1052.
- [4] A. H. Bhrawy, M. M. Tharwat and M. A. Alghamdi, *A new operational matrix of fractional integration for shifted Jacobi polynomials*, Bull. Malays. Math. Sci. Soc. (2), 2014, 37(4), 983–995.
- [5] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods in fluid dynamics*, Springer Series in Computational Physics, Springer-Verlag, New York, 1988.
- [6] V. S. Chelyshkov, *Alternative orthogonal polynomials and quadratures*, Electron. Trans. Numer. Anal., 2006, 25, 17–26 (electronic).
- [7] F. Deutsch, *Best approximation in inner product spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 7, Springer-Verlag, New York, 2001.
- [8] K. Diethelm, *The analysis of fractional differential equations*, 2004 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.
- [9] E. H. Doha and A. H. Bhrawy, *Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials*, Appl. Numer. Math., 2008, 58(8), 1224–1244.
- [10] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, *A new Jacobi operational matrix: an application for solving fractional differential equations*, Appl. Math. Model., 2012, 36(10), 4931–4943.
- [11] N. Engheta, *On fractional calculus and fractional multipoles in electromagnetism*, IEEE Trans. Antennas and Propagation, 1996, 44(4), 554–566.
- [12] V. S. Ertürk and S. Momani, *Solving systems of fractional differential equations using differential transform method*, J. Comput. Appl. Math., 2008, 215(1), 142–151.
- [13] S. Esmaili and M. Shamsi, *A pseudo-spectral scheme for the approximate solution of a family of fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(9), 3646–3654.
- [14] B. Fornberg, *A practical guide to pseudospectral methods*, 1 of *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 1996.
- [15] A. Gil, J. Segura and N. M. Temme, *Numerical methods for special functions*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

- [16] J.-H. He, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Computer Methods in Applied Mechanics and Engineering, 1998, 167(1), 57 – 68.
- [17] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini and C. Cattani, *Wavelets method for solving fractional optimal control problems*, Appl. Math. Comput., 2016, 286, 139–154.
- [18] R. Hilfer (Ed), *Applications of fractional calculus in physics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2000.
- [19] S. Kazem, S. Abbasbandy and S. Kumar, *Fractional-order Legendre functions for solving fractional-order differential equations*, Appl. Math. Model., 2013, 37(7), 5498–5510.
- [20] E. Kreyszig, *Introductory functional analysis with applications*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1989.
- [21] F. C. Meral, T. J. Royston and R. Magin, *Fractional calculus in viscoelasticity: an experimental study*, Commun. Nonlinear Sci. Numer. Simul., 2010, 15(4), 939–945.
- [22] P. Mokhtary, F. Ghoreishi and H. M. Srivastava, *The Müntz-Legendre Tau method for fractional differential equations*, Appl. Math. Model., 2016, 40(2), 671–684.
- [23] S. Momani and K. Al-Khaled, *Numerical solutions for systems of fractional differential equations by the decomposition method*, Appl. Math. Comput., 2005, 162(3), 1351–1365.
- [24] S. Momani and Z. Odibat, *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos Solitons Fractals, 2007, 31(5), 1248–1255.
- [25] S. Nemati, S. Sedaghat and I. Mohammadi, *A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math., 2016, 308, 231–242.
- [26] Z. M. Odibat and S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Int. J. Nonlinear Sci. Numer. Simul., 2006, 7(1), 27–34.
- [27] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, Elsevier Science, 1998.
- [28] P. Rahimkhani, Y. Ordokhani and E. Babolian, *An efficient approximate method for solving delay fractional optimal control problems*, Nonlinear Dynamics, 2016, 1–13.
- [29] P. Rahimkhani, Y. Ordokhani and E. Babolian, *Fractional-order Bernoulli wavelets and their applications*, Appl. Math. Model., 2016, 40(17-18), 8087–8107.
- [30] P. Rahimkhani, Y. Ordokhani and E. Babolian, *A new operational matrix based on bernoulli wavelets for solving fractional delay differential equations*, Numerical Algorithms, 2016, 1–23.

- [31] P. Rahimkhani, Y. Ordokhani and E. Babolian, *Numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet*, J. Comput. Appl. Math., 2017, 309, 493–510.
- [32] A. Saadatmandi and M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl., 2010, 59(3), 1326–1336.
- [33] H. Saeedi and M. M. Moghadam, *Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets*, Commun. Nonlinear Sci. Numer. Simul., 2011, 16(3), 1216–1226.
- [34] H. Sun, W. Chen, C. Li and Y. Chen, *Fractional differential models for anomalous diffusion*, Physica A: Statistical Mechanics and its Applications, 2010, 389(14), 2719 – 2724.
- [35] F. J. Valdes-Parada, J. A. Ochoa-Tapia and J. Alvarez-Ramirez, *Effective medium equations for fractional fick's law in porous media*, Physica A: Statistical Mechanics and its Applications, 2007, 373, 339 – 353.
- [36] X.-J. Yang, J. A. Tenreiro Machado and H. M. Srivastava, *A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach*, Appl. Math. Comput., 2016, 274, 143–151.
- [37] Y. Yang and Y. Huang, *Spectral-collocation methods for fractional pantograph delay-integrodifferential equations*, Adv. Math. Phys., 2013, Art. ID 821327, 14.
- [38] M. Yi, L. Wang and J. Huang, *Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel*, Appl. Math. Model., 2016, 40(4), 3422–3437.
- [39] M. Zayernouri and G. E. Karniadakis, *Exponentially accurate spectral and spectral element methods for fractional ODEs*, J. Comput. Phys., 2014, 257(part A), 460–480.