A LINEAR OPERATOR ASSOCIATED WITH THE MITTAG-LEFFLER FUNCTION AND RELATED CONFORMAL MAPPINGS

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Abstract In the present paper, we introduce a linear operator associated with the Mittag-Leffler function. Some convolution properties of meromorphic functions involving this operator are given.

Keywords Meromorphic functions, conformal mapping, Mittag-Leffler function, second-order differential subordination, Hadamard product (or convolution), convex functions, univalent functions.

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1. Introduction

The familiar Mittag-Leffler function $E_\alpha(z)$ introduced by Mittag-Leffler [5] and its generalization $E_{\alpha,\beta}(z)$ introduced by Wiman [12] are defined by the following series:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (z, \alpha \in \mathbb{C}; \Re(\alpha) > 0) \quad (1.1)$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0), \quad (1.2)$$

respectively. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

$$E_1(z) = E_{1,1}(z) = e^z, \quad E_2(z^2) = E_{2,1}(z^2) = \cosh z \quad \text{and}$$

$$E_2(-z^2) = E_{2,1}(-z^2) = \cos z.$$

The above-defined functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, as well as their various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems.

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and in the study of complex systems. In particular, the Mittag-Leffler function is an explicit formula for the resolvent of Riemann-Liouville fractional integrals by Hille and Tamarkin. Several properties of the Mittag-Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$, together with their generalizations, can be found in a number of recent works (see [1–3] and [7–11]).

Let $\Sigma(p)$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \quad (1.3)$$

which are analytic in the punctured open unit disk

$$\mathbb{U}_0 = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \}.$$ 

The class $\Sigma(p)$ is closed under the Hadamard product (or convolution):

$$(f_1 \ast f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p} = (f_2 \ast f_1)(z),$$

where

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in \Sigma(p) \quad (j = 1, 2).$$

For $f \in \Sigma(p)$, we consider the following operator $T_{\alpha,\beta} : \Sigma(p) \to \Sigma(p)$ associated with the Mittag-Leffler function:

$$T_{\alpha,\beta}f(z) = (\Gamma(\beta) z^{-p} E_{\alpha,\beta}(z)) \ast f(z)$$

$$= z^{-p} + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} a_{n-p} z^{n-p}, \quad (1.4)$$

where $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

Let $\mathcal{P}$ be the class of functions $h$ with $h(0) = 1$, which are analytic and convex univalent in the open unit disk $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$.

For functions $f$ and $g$ analytic in $\mathbb{U}$, we say that $f$ is subordinate to $g$, written $f \prec g$, if $g$ is univalent in $\mathbb{U}$, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Now we introduce the following new subclass of $\Sigma(p)$.

**Definition 1.1.** A function $f \in \Sigma(p)$ is said to be in the class $\mathcal{M}_{\alpha,\beta}(\lambda; h)$ if it satisfies the second order differential subordination:

$$\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta}f(z))' + \frac{\lambda}{p(p + 1)} z^{p+2} (T_{\alpha,\beta}f(z))'' \prec h(z), \quad (1.5)$$

where $\lambda, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$ and $h \in \mathcal{P}$.

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.6)$$

which are analytic in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $S^*(\gamma)$ if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \gamma \quad (z \in \mathbb{U}) \quad (1.7)$$
for some $\gamma$ ($\gamma < 1$). When $0 \leq \gamma < 1$, $S^*(\gamma)$ is the class of starlike functions of order $\gamma$ in $U$. A function $f \in A$ is said to be prestarlike of order $\gamma$ in $U$ if

$$\frac{z}{(1-z)^{2(1-\gamma)}} \ast f(z) \in S^*(\gamma) \quad (\gamma < 1). \tag{1.8}$$

We denote this class by $R(\gamma)$ (see [6]). It is obvious that a function $f \in A$ is in the class $R(0)$ if and only if $f$ is convex univalent in $U$ and $R \left(\frac{1}{2}\right) = S^* \left(\frac{1}{2}\right)$.

The study of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ is a recent interesting topic in geometric function theory. In the present paper we shall make a further contribution to the subject by showing some convolution properties for meromorphic functions involving the Mittag-Leffler functions.

The following lemmas will be used in our investigation.

Lemma 1.1 ([6]). Let $\gamma < 1$, $f \in S^*(\gamma)$ and $g \in R(\gamma)$. Then, for analytic function $F$ in $U$,

$$g \ast (fF) \frac{g \ast f}{g \ast f} (U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

Lemma 1.2 ([4]). Let $g(z) = 1 + \sum_{n=m}^{\infty} b_n z^n (m \in \mathbb{N})$ be analytic in $U$. If $\Re(g(z)) > 0 (z \in U)$, then

$$\Re(g(z)) \geq \frac{1 - |z|^m}{1 + |z|^m} \quad (z \in U).$$

2. Hadamard product properties

In this section we shall derive several Hadamard product properties for functions in the class $M_{\alpha,\beta}(\lambda; h)$.

Theorem 2.1. Let $f \in M_{\alpha,\beta}(\lambda; h)$, $g \in \Sigma(p)$ and $\Re(z^p g(z)) > \frac{1}{2} (z \in U)$. Then $f \ast g \in M_{\alpha,\beta}(\lambda; h)$.

Proof. For $f \in M_{\alpha,\beta}(\lambda; h)$ and $g \in \Sigma(p)$, we have

$$\lambda - \frac{1}{p} z^{p+1} (T_{\alpha,\beta}(f \ast g)(z))^\prime + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f \ast g)(z))^\prime$$

$$= \lambda - \frac{1}{p} (z^p g(z))^\prime (z^p f(z))^\prime + \frac{\lambda}{p(p+1)} (z^p g(z))^\prime (z^p f(z))^\prime$$

$$= (z^p g(z))^\prime (z^p f(z))^\prime \ast \psi(z), \tag{2.1}$$

where

$$\psi(z) = \lambda - \frac{1}{p} z^{p+1} (T_{\alpha,\beta}(f^p z))^\prime + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f^p z))^\prime < h(z). \tag{2.2}$$

In view of the conditions of Theorem 2.1, the function $z^p g(z)$ has the Herglotz representation:

$$z^p g(z) = \int_{|x|=1} \frac{d\mu(x)}{1 - xz} \quad (z \in U), \tag{2.3}$$
where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$. Since the function $h$ is convex univalent in $U$, it follows from (2.1) to (2.3) that

$$\lambda - \frac{1}{p} z^{p+1} (T_{\alpha,\beta}(f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f * g)(z))'' = \int_{|x|=1} \psi(xz) d\mu(x) < h(z).$$

This shows that $f * g \in M_{\alpha,\beta}(\lambda; h)$. The proof of Theorem 2.1 is completed.

**Theorem 2.2.** Let $f \in M_{\alpha,\beta}(\lambda; h)$, $g \in \Sigma(p)$ and $z^{p+1}g(z) \in R(\gamma)$ ($\gamma < 1$). Then $f * g \in M_{\alpha,\beta}(\lambda; h)$.

**Proof.** From (2.1) we can write

$$\lambda - \frac{1}{p} z^{p+1} (T_{\alpha,\beta}(f * g)(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta}(f * g)(z))'' = \left(\frac{z^{p+1}g(z)}{z^{p+1}g(z)}\right) * \left(\frac{\psi(z)}{z}\right) = \int_{|x|=1} \psi(xz) d\mu(x) < h(z).$$

where the function $\psi$ is defined as in (2.2).

Since the function $h$ is convex univalent in $U$, $\psi(z) < h(z)$, $z^{p+1}g(z) \in R(\gamma)$ and $z \in S^*(\gamma)$ ($\gamma < 1$), from (2.4) and Lemma 1.1, we obtain the desired result. The proof of Theorem 2.2 is completed.

Taking $\gamma = 0$ and $\gamma = \frac{1}{2}$ in Theorem 2.2, we have the following consequence.

**Corollary 2.1.** Let $f \in M_{\alpha,\beta}(\lambda; h)$. Also let $g \in \Sigma(p)$ satisfy either of the following conditions:

(i) $z^{p+1}g(z)$ is convex univalent in $U$

or

(ii) $z^{p+1}g(z) \in S^*(\frac{1}{2})$.

Then $f * g \in M_{\alpha,\beta}(\lambda; h)$.

**Theorem 2.3.** Let $\lambda \leq 0$ and

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p} \in M_{\alpha,\beta}(\lambda; h_j) \quad (j = 1, 2),$$

where $h_j(z) = \frac{1 + A_j z}{1 + B_j z}$ and $-1 \leq B_j < A_j \leq 1$.

If $f \in \Sigma(p)$ is defined by

$$(T_{\alpha,\beta} f(z))' = -\frac{1}{p} \left( (T_{\alpha,\beta} f_1(z))' * (T_{\alpha,\beta} f_2(z))' \right),$$

then $f \in M_{\alpha,\beta}(\lambda; h)$, where

$$h(z) = \gamma + (1 - \gamma) \frac{1 + z}{1 - z}$$

(2.8)
and $\gamma$ is given by
\[
\gamma = \begin{cases} 
1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} \left( 1 + \frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{1+u}} \, du \right) & (\lambda < 0) \\
1 - \frac{2(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)} & (\lambda = 0).
\end{cases}
\tag{2.9}
\]

The bound $\gamma$ is sharp when $B_1 = B_2 = -1$.

**Proof.** We consider the case when $\lambda < 0$. By setting
\[
H_j(z) = \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha,\beta} f_j(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha,\beta} f_j(z))'' \quad (j = 1, 2)
\]
for $f_j (j = 1, 2)$ given by (2.5), we find that
\[
H_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j} z^n < 1 + \frac{A_j}{1 + B_j} \quad (j = 1, 2)
\tag{2.10}
\]
and
\[
(T_{\alpha,\beta} f_j(z))' = \frac{p(p+1)}{\lambda} z^{(1-\lambda)(p+1)} \int_0^1 u^{-\frac{p+1}{1+u}} H_j(t) \, dt \quad (j = 1, 2). \tag{2.11}
\]

Now, if $f \in \Sigma(p)$ is defined by (2.7), we find from (2.11) that
\[
(T_{\alpha,\beta} f(z))' = -\frac{1}{p} \left( (T_{\alpha,\beta} f_1(z))' \ast (T_{\alpha,\beta} f_2(z))' \right)
\]
\[
= -\frac{1}{p} \left( \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{1+u}} H_1(uz) \, du \right) \ast \left( \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{1+u}} H_2(uz) \, du \right)
\]
\[
= \frac{p(p+1)}{\lambda} z^{-p-1} \int_0^1 u^{-\frac{p+1}{1+u}} H(uz) \, du, \tag{2.12}
\]
where
\[
H(z) = -\frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{1+u}} (H_1 \ast H_2)(uz) \, du. \tag{2.13}
\]

Also, by using (2.10) and the Herglotz theorem, we see that
\[
\Re \left\{ \left( \frac{H_1(z) - \gamma_1}{1 - \gamma_1} \right) \ast \left( \frac{1}{2} + \frac{H_2(z) - \gamma_2}{2(1 - \gamma_2)} \right) \right\} > 0 \quad (z \in \mathbb{U}),
\]
which leads to
\[
\Re \{ (H_1 \ast H_2)(z) \} > \gamma_0 = 1 - 2(1 - \gamma_1)(1 - \gamma_2) \quad (z \in \mathbb{U}),
\]
where
\[
0 \leq \gamma_j = \frac{1 - A_j}{1 - B_j} < 1 \quad (j = 1, 2).
\]
According to Lemma 1.2, we have
\[
\Re \{ (H_1 \ast H_2)(z) \} \geq \gamma_0 + (1 - \gamma_0) \frac{1 - |z|}{1 + |z|} \quad (z \in \mathbb{U}). \tag{2.14}
\]
Now it follows from (2.12) to (2.14) that
\[
\Re \left\{ \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha, \beta} f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha, \beta} f(z))'' \right\} = \Re \{H(z)\}
\]
\[
= - \frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda} - 1} \Re \{(H_1 * H_2)(u z)\} du
\]
\[
\geq - \frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda} - 1} \left( \gamma_0 + (1 - \gamma_0) \frac{1 - u|z|}{1 + u|z|} \right) du
\]
\[
> \gamma_0 - \frac{(p+1)(1 - \gamma_0)}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda} - 1} \frac{1 - u}{1 + u} du
\]
\[
= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left( 1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda} - 1}}{1 + u} du \right)
\]
\[
= \gamma,
\]
which proves that \( f \in M_{\alpha, \beta}(\lambda; h) \) for the function \( h \) given by (2.8).

When \( B_1 = B_2 = -1 \), we consider the functions \( f_j \) (\( j = 1, 2 \)) defined by
\[
(T_{\alpha, \beta} f_j(z))' = \frac{p(p+1)}{\lambda} z^{(1-\lambda)\frac{p+1}{\lambda}} \int_0^z t^{-\frac{p+1}{\lambda} - 1} \frac{1 + A_j t}{1 - t} dt \quad (j = 1, 2)
\]
for which we have
\[
H_j(z) = \frac{\lambda - 1}{p} z^{p+1} (T_{\alpha, \beta} f_j(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha, \beta} f_j(z))''
\]
\[
= \frac{1 + A_j z}{1 - z} \quad (j = 1, 2)
\]
and
\[
(H_1 * H_2)(z) = \frac{1 + A_1 z}{1 - z} * \frac{1 + A_2 z}{1 - z}
\]
\[
= 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - z}.
\]
Hence, for the function \( f \) given by (2.7), we have
\[
\frac{\lambda - 1}{p} z^{p+1} (T_{\alpha, \beta} f(z))' + \frac{\lambda}{p(p+1)} z^{p+2} (T_{\alpha, \beta} f(z))''
\]
\[
= - \frac{p+1}{\lambda} \int_0^1 u^{-\frac{p+1}{\lambda} - 1} \left( 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - u z} \right) du
\]
\[
\rightarrow 1 - (1 + A_1)(1 + A_2) \left( 1 + \frac{p+1}{\lambda} \int_0^1 \frac{u^{-\frac{p+1}{\lambda} - 1}}{1 + u} du \right)
\]
as \( z \to -1 \).

Finally, for the case when \( \lambda = 0 \), the proof of Theorem 2.3 is simple, and we choose to omit the details involved. Now the proof of Theorem 2.3 is completed. \( \square \)
References


