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THRESHOLD DYNAMICS IN A STOCHASTIC SIRS EPIDEMIC MODEL WITH NONLINEAR INCIDENCE RATE*

Yanan Zhao[1,2,†], Xiaoying Zhang[2] and Donal O'Regan[3]

Abstract We discuss the dynamic of a stochastic Susceptible-Infectious-Recovered-Susceptible (SIRS) epidemic model with nonlinear incidence rate.The crucial threshold $\tilde{R}_0$ is identified and this will determine the extinction and persistence of the epidemic when the noise is small. We also discuss the asymptotic behavior of the stochastic model around the endemic equilibrium of the corresponding deterministic system. When the noise is large, we find that a large noise intensity has the effect of suppressing the epidemic, so that it dies out. Finally, these results are illustrated by computer simulations.

Keywords SIRS epidemic model, nonlinear incidence rate, extinction, persistence, threshold.

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1. Introduction

Mathematical models describe the population dynamics of infectious diseases, which have played an important role in the better understanding of the epidemiological modes and disease control for a long time. Developing realistic mathematical models for the transmission dynamics of infectious diseases has been attempted.

At present, the study of epidemic models mainly concerns the basic reproduction number, the extinction or permanence of a disease and the stability of the solutions. The basic reproduction number is an omnipresent concept of epidemiology. Its success is born of its distinct biological explanation, as well as from its important virtues. Many authors try to find the threshold conditions that determine whether the diseases will spread or go extinct (see, for example, [3, 7, 12, 16, 22, 23, 29, 33, 34] and references therein).

Incidence rate plays a crucial role in the modelling of epidemic dynamics. In many epidemic models, the bilinear incidence rate βSI and the standard incidence rate $\frac{\beta SI}{N}$ are frequently used. Many authors observed the influence of the nonlinearity of incidence rates for some disease transmissions, so they have suggested

[†]the corresponding author. Email address:zhaoyn111@163.com(Y. Zhao)
[1]School of Mathematics, Jilin University, Changchun 130012, China
[2]School of Science, Changchun University, Changchun 130021, China
[3]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
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the bilinear incidence rate should be modified into nonlinear infection rate. For example, Capasso and Serio [4] investigated the cholera epidemic spread in Bari in 1973 and described an incidence rate which adopts a form $\frac{\beta SI}{1+\alpha I}$, which has been construed as saturated incidence rate. It seems more reasonable than the bilinear incidence rate βSI because it includes the change in behavior and crowding effect of the infective individuals and prevents the contact rate unbounded.

Xiao and Ruan [24] first proposed a non-monotonic incident rate $\frac{\beta SI}{1+\alpha I^2}$, where βI measures the infection force of the disease and $\frac{1}{1+\alpha I^2}$ describes the psychological or inhibitory effect from the behavioral change of the susceptible individuals when the number of infective individuals is very large. They proved that the endemic equilibrium of an SIRS model with this non-monotonic incidence rate and no delays, is globally asymptotically stable by applying Dulac function.

More recently, Yang and Xiao [28] extended the result of Xiao and Ruan [24] to the incident rate of a specific form, namely, $\frac{\beta I(t-\tau)S(t)}{1+\alpha I^h(t-\tau)}$ $(h \geq 1)$. For the following SIRS epidemic model with a nonlinear incidence rate and time delay,

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \dfrac{\beta S(t) I(t-\tau)}{1+\alpha I^h(t-\tau)} + \delta R(t), \\ \dot{I}(t) = \dfrac{\beta S(t) I(t-\tau)}{1+\alpha I^h(t-\tau)} - (\mu+\gamma+\varepsilon) I(t), \\ \dot{R}(t) = \gamma I(t) - (\mu+\delta) R(t), \end{cases} \tag{1.1}$$

where Λ is the birth rate, μ is the natural death rate, γ is the recovery rate of the infective individuals, δ is the rate at which recovered individuals lose immunity and return to the susceptible class, β is the transmission rate, ε is the death rate due to disease and the parameters are all positive. Yang and Xiao show there exists the basic reproduction number

$$R_0 = \frac{\Lambda\beta}{\mu(\mu+\gamma+\varepsilon)}, \tag{1.2}$$

which is independent of the form of the nonlinear incidence rate. If $R_0 \leq 1$, then the disease-free equilibrium is globally asymptotic stable, whereas if $R_0 > 1$, then the unique endemic equilibrium is globally asymptotically stable in the interior of the feasible region for the model for no latency, and periodic solutions can arise at a critical latency. The obtained results improve those in Xu and Ma [25] for $h = 1$(saturation effect) and Xiao and Ruan [24] for $h = 2$. Furthermore, a variety of nonlinear incidence rates have been used in the literature (see, for example, [6, 11, 20, 21, 26] and the references cited therein).

In fact, epidemic models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in comparison to their deterministic counterparts. Recently, several authors studied stochastic biological systems, see [1, 2, 5, 8, 13, 15, 18, 30, 32]. Liu and Wang [14] discussed the logistic equation with infinite delay and corresponding stochastic system. They established the sufficient conditions for extinction, non-persistence in the mean, and weak persistence of the solution. Yang et al. [27] include stochastic perturbations in the SIR epidemic model with saturated incidence and they investigated the dynamics. Under appropraite conditions the solution has the ergodic property if $R_0 > 1$, and it is exponential stable if $R_0 \leq 1$. However, to the best of our knowledge, some more desired and important properties of the stochastic epidemic model with nonlinear incidence rate $\frac{\beta SI}{1+\alpha I^h}$ $(h > 0)$ are not studied yet, for example, extinction, persistence and threshold of the stochastic models.

In this paper we assume fluctuations in the parameter β, as in [8], so that $\beta dt \to \beta dt + \sigma dB(t)$, where $B(t)$ is a standard Brownian motion with intensity $\sigma^2 > 0$. The stochastic version corresponding to (1.1) for no latency is

$$\begin{cases} dS(t) = \left(\Lambda - \mu S(t) - \dfrac{\beta S(t)I(t)}{1+\alpha I^h(t)} + \delta R(t)\right)dt - \dfrac{\sigma S(t)I(t)}{1+\alpha I^h(t)}dB(t), \\ dI(t) = \left(\dfrac{\beta S(t)I(t)}{1+\alpha I^h(t)} - (\mu+\gamma+\varepsilon)I(t)\right)dt + \dfrac{\sigma S(t)I(t)}{1+\alpha I^h(t)}dB(t), \\ dR(t) = (\gamma I(t) - (\mu+\delta)R(t))dt, \end{cases} \tag{1.3}$$

where $h > 0$. In this paper, we investigate the dynamics of system (1.3) and we give a threshold which can easily determine the extinction and persistence of the disease.

This paper is organized as follows. In Section 2 we show there is a unique positive solution of system (1.3) and in Section 3 we present the condition so that the disease will die out. The condition for the disease being persistent is given in Sections 4 and in Section 5, when $R_0 > 1$ we discuss when the solution of (1.3) oscillates around the endemic equilibrium $P^*(S^*, I^*, R^*)$. In Sections 6 outcomes of numerical simulations are reported to support the analytical results.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \text{Prob})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all Prob-null sets), and let B(t) be a scalar Brownian motion defined on the probability space.

2. Global positive solutions

In this section we show that the solution of system (1.3) is positive and global.

Theorem 2.1. *There is a unique solution $Y(t) = (S(t), I(t), R(t))$ of system (1.3) on $t \geqslant 0$ for any initial value $Y(0) = (S(0), I(0), R(0)) \in \mathbb{R}^3_+$, and the solution will remain in $\mathbb{R}^3_+$ with probability 1, namely, $Y(t) \in \mathbb{R}^3_+$ for all $t \geqslant 0$ almost surely.*

Proof. Consider

$$\begin{cases} dS(t) = (\Lambda - \mu S(t) - \dfrac{\beta S(t)e^{v(t)}}{1+\alpha e^{hv(t)}} + \delta R(t))dt - \dfrac{\sigma S(t)e^{v(t)}}{1+\alpha e^{hv(t)}}dB(t), \\ dv(t) = [\dfrac{\beta S(t)}{1+\alpha e^{hv(t)}} - (\mu+\gamma+\varepsilon) - \dfrac{\sigma^2 S^2(t)}{2(1+\alpha e^{hv(t)})^2}]dt + \dfrac{\sigma S(t)}{1+\alpha e^{hv(t)}}dB(t), \\ dR(t) = [\gamma e^{v(t)} - (\mu+\delta)R(t)]dt. \end{cases} \tag{2.1}$$

Since the coefficients of system (2.1) are locally Lipschitz continuous, there is a unique local solution of system (2.1). Let $I = e^v$, and Itô's formula implies that system (1.3) has a unique local solution. Hence it suffices to prove that the unique local solution of system (1.3) is global and positive.

Since the following argument is similar to that in [9] (Theorem 2.1), we only sketch the proof here. Let $k_0 \geq 0$ be sufficiently large so that $S(0), I(0)$ and $R(0)$ all lie within the interval $[1/k_0, k_0]$. F or each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : \min\{S(t), I(t), R(t)\} \le \frac{1}{k} \quad \text{or} \quad \max\{S(t), I(t), R(t)\} \ge k\},$$

where τ_e is the explore time. We show $\tau_\infty := \lim_{k\to\infty} \tau_k = \infty$ a.s. For $t \le \tau_k$, we see, for each k,

$$d(S + I + R) = [\Lambda - \mu(S + I + R) - \varepsilon I]dt \le [\Lambda - \mu(S + I + R)]dt$$

and so

$$S(t) + I(t) + R(t) \le \begin{cases} \frac{\Lambda}{\mu}, & \text{if } S(0) + I(0) + R(0) \le \frac{\Lambda}{\mu}, \\ S(0) + I(0) + R(0), & \text{if } S(0) + I(0) + R(0) > \frac{\Lambda}{\mu}, \end{cases} := M.$$

Let $T > 0$ be arbitrary. For any $0 \le t \le \tau_k \wedge T$, applying Itô's formula, we obtain

$$\begin{aligned} LV &= \Lambda - \mu S - (\mu + \varepsilon)I - \mu R - \frac{\Lambda}{S} + \mu - \frac{\beta S}{1 + \alpha I^h} - \frac{\delta R}{S} + \frac{\beta I}{1 + \alpha I^h} + (\mu + \gamma + \varepsilon) \\ &\quad - \frac{\gamma I}{R} + (\mu + \delta) + \frac{\sigma^2 (S^2 + I^2)}{2(1 + \alpha I^h)^2} \\ &\le \Lambda + \mu + \beta M + (\mu + \gamma + \varepsilon) + (\mu + \delta) + \sigma^2 M^2 \\ &:= K. \end{aligned}$$

The remainder part is a standard process for the proof of the existence and unique solution. So we omit it. It can found in in Gray et al. [8] and Ji et al. [9]. □

Remark 2.1. From Theorem 2.1 for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$, there is a unique global solution $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ almost surely of system (1.3). Hence

$$d(S + I + R) \le [\Lambda - \mu(S + I + R))]dt,$$

and

$$S(t) + I(t) + R(t) \le \frac{\Lambda}{\mu} + e^{-\mu t}\Big(S(0) + I(0) + R(0) - \frac{\Lambda}{\mu}\Big).$$

If $S(0) + I(0) + R(0) \le \frac{\Lambda}{\mu}$, then $S(t) + I(t) + R(t) \le \frac{\Lambda}{\mu}$ a.s. so the region

$$\Gamma = \{(S, I, R) \in \mathbb{R}^3_+ : S > 0, I > 0, R > 0, S + I + R \le \frac{\Lambda}{\mu}\} \tag{2.2}$$

is a positively invariant set of system (1.3).

3. Extinction

Let

$$\tilde{R}_0 = \frac{\frac{\beta\Lambda}{\mu} - \frac{\sigma^2\Lambda^2}{2\mu^2}}{\mu + \gamma + \varepsilon} = R_0 - \frac{\sigma^2\Lambda^2}{2\mu^2(\mu + \gamma + \varepsilon)}, \tag{3.1}$$

and R_0 is defined as in (1.2).

For later applications, let us cite a Strong Law of Large Numbers (see, Mao [19]) as the following lemma.

Lemma 3.1 (Strong Law of Large Numbers). *Let $M = \{M_t\}_{t\geq 0}$ be a real-value continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0. \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0. \quad a.s.$$

Theorem 3.1. *Let $Y(t) = (S(t), I(t), R(t))$ be the solution of system* (1.3) *with initial value $Y(0) \in \Gamma$. Assume that (a) $\sigma^2 > \beta^2/2(\mu+\gamma+\varepsilon)$, or (b) $\tilde{R}_0 < 1$ and $\sigma^2 \leq \beta\mu/\Lambda$. Then*

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leqslant -\lambda < 0 \qquad a.s., \tag{3.2}$$

where $\lambda = (\mu+\gamma+\varepsilon) - \frac{\beta^2}{2\sigma^2}$, if (a) holds; $\lambda = (\mu+\gamma+\varepsilon)(1-\tilde{R}_0)$, if (b) holds (namely, $I(t)$ tends to zero exponentially a.s. In other words, the disease will die out with probability 1). Moreover,

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}, \qquad \lim_{t\to\infty} R(t) = 0 \qquad a.s. \tag{3.3}$$

Proof. By Itô's formula,

$$\begin{aligned} d(\log I) &= [\frac{\beta S}{1+\alpha I^h} - (\mu+\gamma+\varepsilon) - \frac{\sigma^2 S^2}{2(1+\alpha I^h)^2}]dt + \frac{\sigma S}{1+\alpha I^h} dB(t) \\ &= f(x)dt + \frac{\sigma S}{1+\alpha I^h} dB(t), \end{aligned}$$

where $f : (0, \frac{\Lambda}{\mu}] \to R$ is defined by

$$\begin{aligned} f(x) &= \beta x - (\mu+\gamma+\varepsilon) - \frac{1}{2}\sigma^2 x^2 \\ &= -\frac{\sigma^2}{2}(x - \frac{\beta}{\sigma^2})^2 - (\mu+\gamma+\varepsilon) + \frac{\beta^2}{2\sigma^2}, \qquad x = \frac{S}{1+\alpha I^h} \in (0, \frac{\Lambda}{\mu}]. \end{aligned} \tag{3.4}$$

An integration can be found such that

$$\frac{\log I(t)}{t} \leqslant \frac{\log I(0)}{t} + \frac{1}{t}\int_0^t f(x)dr + \frac{1}{t}\int_0^t \frac{\sigma S(r)}{1+\alpha I^h(r)} dB(r). \tag{3.5}$$

First we consider case (a). Note

$$\begin{aligned} f(x) &= -\frac{\sigma^2}{2}(x - \frac{\beta}{\sigma^2})^2 - (\mu+\gamma+\varepsilon) + \frac{\beta^2}{2\sigma^2} \\ &\leqslant -(\mu+\gamma+\varepsilon) + \frac{\beta^2}{2\sigma^2}, \end{aligned} \tag{3.6}$$

which is negative by the condition in case (a). Substituting (3.6) into (3.5) yields

$$\frac{\log I(t)}{t} \leqslant [-(\mu+\gamma+\varepsilon) + \frac{\beta^2}{2\sigma^2}] + \frac{M(t)}{t} + \frac{\log I(0)}{t}, \tag{3.7}$$

where $M(t) := \int_0^t \frac{\sigma S(r)}{1+\alpha I^h(r)} dB(r)$. Compute by the boundedness of S that

$$\limsup_{t\to\infty} \frac{< M, M >_t}{t} \leqslant \frac{\sigma^2\Lambda^2}{\mu^2} < \infty \quad a.s.$$

Then the strong law of large numbers, Lemma 3.1, yields $\lim_{t\to\infty} \frac{M(t)}{t} = 0$ $a.s.$ It follows from (3.7), we obtain

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leqslant -(\mu+\gamma+\varepsilon) + \frac{\beta^2}{2\sigma^2} < 0 \quad a.s.. \tag{3.8}$$

Next we consider case (b). We consider the quadratic function f defined by (3.4). By the condition in case (b), it is easy to see that

$$\bar{x} = \frac{\beta}{\sigma^2} \geqslant \frac{\Lambda}{\mu}.$$

Then $f(x)$ takes its maximum value

$$f(\hat{x}) = f(\frac{\Lambda}{\mu}) = \beta\frac{\Lambda}{\mu} - (\mu+\gamma+\varepsilon) - \frac{1}{2}\sigma^2(\frac{\Lambda}{\mu})^2 = (\tilde{R}_0 - 1)(\mu+\gamma+\varepsilon). \tag{3.9}$$

Substituting (3.9) into (3.5) yields

$$\frac{\log I(t)}{t} \leqslant (\tilde{R}_0 - 1)(\mu+\gamma+\varepsilon) + \frac{M(t)}{t} + \frac{\log I(0)}{t}.$$

If $\tilde{R}_0 < 1$, then

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leqslant (\tilde{R}_0 - 1)(\mu+\gamma+\varepsilon) < 0 \quad a.s. \tag{3.10}$$

From (3.8) and (3.10), we have

$$\lim_{t\to\infty} I(t) = 0 \ a.s. \tag{3.11}$$

Next we prove (3.3). From (1.3), we have

$$d(S+I+R) = [\Lambda - \mu(S+I+R) - \varepsilon I]dt. \tag{3.12}$$

We solve to obtain

$$S(t) + I(t) + R(t) = e^{-\mu t}\{[S(0)+I(0)+R(0)] + \int_0^t [\Lambda - \varepsilon I(s)]e^{\mu s} ds\}.$$

Applying L'Hospital's rule, we get

$$\lim_{t\to\infty} (S(t)+I(t)+R(t)) = \frac{\Lambda}{\mu} \quad a.s.$$

Applying a similar method to the last equation of (1.3), we have

$$\lim_{t\to\infty} R(t) = 0 \quad a.s.$$

Together with (3.11), we obtain

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu} \quad a.s.$$

The proof is complete. □

Remark 3.1. Theorem 3.1 tells us the disease will die out if $\tilde{R}_0 < 1$ and the white noise is not large. If the white noise is large enough such that the condition in case(a) is satisfied then the disease will also die out.

4. Persistence

Definition 4.1 ([10]). System (1.3) is said to be persistent in the mean, if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(r)dr > 0, \quad a.s.$$

We further need the following lemma(see, Zhao [31]).

Lemma 4.1. *Let $f \in C[[0,\infty)\times\Omega,(0,\infty)]$ and $F(t) \in C([0,\infty)\times\Omega,R)$. If there exist positive constants λ_0, λ and T such that*

$$\log f(t) \le \lambda t - \lambda_0 \int_0^t f(s)ds + F(t) \quad a.s.$$

for all $t \ge T$, and $\lim_{t\to\infty}\frac{F(t)}{t} = 0$ a.s., then

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds \le \frac{\lambda}{\lambda_0} \quad a.s.$$

For convenience, we introduce some notations, defining $\langle x(t)\rangle$ as

$$\langle x(t)\rangle = \frac{1}{t}\int_0^t x(r)dr,$$

$$\tilde{I}_* = \begin{cases} \dfrac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)}{\beta(\dfrac{\mu+\varepsilon}{\mu}+\dfrac{\gamma}{\mu+\delta}) + [\alpha(\mu+\gamma+\varepsilon)+\dfrac{\alpha\sigma^2\Lambda^2}{2\mu^2}](\dfrac{\Lambda}{\mu})^{h-1}} := \tilde{I}_{1*}, \ h \ge 1, \\ \left(\dfrac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)}{\beta(\dfrac{\mu+\varepsilon}{\mu}+\dfrac{\gamma}{\mu+\delta})(\dfrac{\Lambda}{\mu})^{1-h} + \alpha(\mu+\gamma+\varepsilon)+\dfrac{\alpha\sigma^2\Lambda^2}{2\mu^2}}\right)^{\dfrac{1}{h}} := \tilde{I}_{2*}, \ 0<h<1, \end{cases} \tag{4.1}$$

and

$$\tilde{I}^* = \frac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1) + \dfrac{\sigma^2\Lambda^2}{2\mu^2}[1+\alpha h(\dfrac{\Lambda}{\mu})^h)]}{\beta(\dfrac{\mu+\varepsilon}{\mu}+\dfrac{\gamma}{\mu+\delta})}. \tag{4.2}$$

Theorem 4.1. *If $\tilde{R}_0 > 1$, then the solution $(S(t), I(t), R(t))$ of system (1.3) with any initial value $(S(0), I(0), R(0)) \in \Gamma$ has the following property:*

$$\tilde{I}_* \le \liminf_{t\to\infty}\langle I(t)\rangle \le \limsup_{t\to\infty}\langle I(t)\rangle \le \tilde{I}^* \qquad a.s. \tag{4.3}$$

where $\tilde{I}_*$ and $\tilde{I}^*$ are defined in (4.1) and (4.2).

Proof. Integration yields

$$\begin{cases} \dfrac{S(t)-S(0)}{t} = \Lambda - \mu\langle S(t)\rangle - \beta\langle \dfrac{S(t)I(t)}{1+\alpha I^h(t)}\rangle + \delta\langle R(t)\rangle - \dfrac{\sigma}{t}\displaystyle\int_0^t \dfrac{S(r)I(r)}{1+\alpha I^h(r)}dB(r), \\ \dfrac{I(t)-I(0)}{t} = \beta\langle \dfrac{S(t)I(t)}{1+\alpha I^h(t)}\rangle - (\mu+\gamma+\varepsilon)\langle I(t)\rangle + \dfrac{\sigma}{t}\displaystyle\int_0^t \dfrac{S(r)I(r)}{1+\alpha I^h(r)}dB(r), \\ \dfrac{R(t)-R(0)}{t} = \gamma\langle I(t)\rangle - (\mu+\delta)\langle R(t)\rangle. \end{cases}$$

Then

$$\langle S(t)\rangle = \frac{\Lambda}{\mu} - \left(\frac{\mu+\varepsilon}{\mu} + \frac{\gamma}{\mu+\delta}\right)\langle I(t)\rangle + \varphi(t), \tag{4.4}$$

where $\varphi(t)$ is defined by

$$\varphi(t) = -\frac{1}{\mu}\left(\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} + \frac{\delta}{\mu+\delta}\frac{R(t)-R(0)}{t}\right).$$

Note (2.2), so $\lim_{t\to\infty}\varphi(t) = 0$ $a.s.$ By Itô's formula, we have

$$\begin{aligned} d(\log I + \frac{\alpha}{h}I^h) &= [\beta S - (\mu+\gamma+\varepsilon) - \alpha(\mu+\gamma+\varepsilon)I^h - \frac{\sigma^2 S^2}{2(1+\alpha I^h)^2} \\ &\quad + \frac{\alpha(h-1)\sigma^2 S^2 I^h}{2(1+\alpha I^h)^2}]dt + \sigma S dB(t) \\ &\geq \{\beta S - (\mu+\gamma+\varepsilon) - \frac{\sigma^2}{2}(\frac{\Lambda}{\mu})^2 - [\alpha(\mu+\gamma+\varepsilon) + \frac{\alpha\sigma^2\Lambda^2}{2\mu^2}]I^h\}dt \\ &\quad + \sigma S dB(t). \end{aligned} \tag{4.5}$$

Then

$$\begin{aligned} &\frac{\log I(t) - \log I(0) + \alpha(I^h(t) - I^h(0))}{t} \\ &\geq \beta\langle S(t)\rangle - (\mu+\gamma+\varepsilon) - \frac{\sigma^2\Lambda^2}{2\mu^2} - [\alpha(\mu+\gamma+\varepsilon) \\ &\quad + \frac{\alpha\sigma^2\Lambda^2}{2\mu^2}]\langle I^h(t)\rangle + \frac{\sigma}{t}\int_0^t S(r)dB(r). \end{aligned} \tag{4.6}$$

We now break the proof into two cases.

Case 1. Let $h \geq 1$. Using (4.4) and (4.6) we get

$$\begin{aligned} &\frac{\log I(t) - \log I(0) + \alpha(I^h(t) - I^h(0))}{t} \\ &\geq \beta\frac{\Lambda}{\mu} - (\mu+\gamma+\varepsilon) - \frac{\sigma^2\Lambda^2}{2\mu^2} - \beta(\frac{\mu+\varepsilon}{\mu} + \frac{\gamma}{\mu+\delta})\langle I(t)\rangle - [\alpha(\mu+\gamma+\varepsilon) \\ &\quad + \frac{\alpha\sigma^2\Lambda^2}{2\mu^2}]\langle I^h(t)\rangle + \beta\varphi(t) + \frac{M(t)}{t} \\ &\geq (\tilde{R}_0 - 1)(\mu+\gamma+\varepsilon) - \{\beta(\frac{\mu+\varepsilon}{\mu} + \frac{\gamma}{\mu+\delta}) + [\alpha(\mu+\gamma+\varepsilon) \\ &\quad + \frac{\alpha\sigma^2\Lambda^2}{2\mu^2}](\frac{\Lambda}{\mu})^{h-1}\}\langle I(t)\rangle + \beta\varphi(t) + \frac{M(t)}{t}, \end{aligned}$$

where $M(t) := \sigma\int_0^t S(r)dB(r)$. Compute by the boundedness of S that

$$\limsup_{t\to\infty}\frac{< M, M >_t}{t} \leqslant \frac{\sigma^2\Lambda^2}{\mu^2} < \infty \quad a.s.$$

Then the strong law of large numbers, Lemma 3.1, yields $\lim_{t\to\infty}\frac{M(t)}{t}=0$ *a.s.* Then

$$\langle I(t)\rangle \geq \frac{1}{l}[(\tilde{R}_0-1)(\mu+\gamma+\varepsilon)+\beta\varphi(t)+\frac{M(t)}{t}-\frac{\log I(t)+\alpha I^h(t)}{t}]. \tag{4.7}$$

where $l=\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})+[\alpha(\mu+\gamma+\varepsilon)+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}](\frac{\Lambda}{\mu})^{h-1}$, Note (2.2), we have $-\infty<\log I(t)\leq \log\frac{\Lambda}{\mu}$. Then

$$\liminf_{t\to\infty}\langle I(t)\rangle \geq \frac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)}{\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})+[\alpha(\mu+\gamma+\varepsilon)+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}](\frac{\Lambda}{\mu})^{h-1}}=\tilde{I}_{1*}. \tag{4.8}$$

Case 2. Let $0<h<1$. Using (4.4) and (4.6) we get

$$\begin{aligned}
&\frac{\log I(t)-\log I(0)+\alpha(I^h(t)-I^h(0))}{t}\\
&\geq \beta\frac{\Lambda}{\mu}-(\mu+\gamma+\varepsilon)-\frac{\sigma^2\Lambda^2}{2\mu^2}-\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})\langle I(t)\rangle-[\alpha(\mu+\gamma+\varepsilon)\\
&\quad+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}]\langle I^h(t)\rangle+\beta\varphi(t)+\frac{M(t)}{t}\\
&\geq (\tilde{R}_0-1)(\mu+\gamma+\varepsilon)-[\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})(\frac{\Lambda}{\mu})^{1-h}+\alpha(\mu+\gamma+\varepsilon)+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}]\langle I^h(t)\rangle\\
&\quad+\beta\varphi(t)+\frac{M(t)}{t}.
\end{aligned}$$

Then

$$\langle I^h(t)\rangle \geq \frac{1}{l'}[(\tilde{R}_0-1)(\mu+\gamma+\varepsilon)+\beta\varphi(t)+\frac{M(t)}{t}-\frac{\log I(t)+\alpha I^h(t)}{t}].$$

where $l'=\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})(\frac{\Lambda}{\mu})^{1-h}+\alpha(\mu+\gamma+\varepsilon)+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}$. By Hölder inequality, then

$$\langle I(t)\rangle^h \geq \langle I^h(t)\rangle \geq \frac{1}{l'}[(\tilde{R}_0-1)(\mu+\gamma+\varepsilon)+\beta\varphi(t)+\frac{M(t)}{t}-\frac{\log I(t)+\alpha I^h(t)}{t}]. \tag{4.9}$$

Taking the limit inferior of both sides (4.9) leads to

$$\liminf_{t\to\infty}\langle I(t)\rangle \geq \left(\frac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)}{\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})(\frac{\Lambda}{\mu})^{1-h}+\alpha(\mu+\gamma+\varepsilon)+\frac{\alpha\sigma^2\Lambda^2}{2\mu^2}}\right)^{\frac{1}{h}}:=\tilde{I}_{2*}. \tag{4.10}$$

Also by the first equality of (4.5), we have

$$d(\log I++\frac{\alpha}{h}I^h)\leq[\beta S-(\mu+\gamma+\varepsilon)+\frac{1}{2}\alpha h\sigma^2(\frac{\Lambda}{\mu})^{h+2}]dt+\sigma SdB(t).$$

Then

$$\begin{aligned}
&\frac{\log I(t)-\log I(0)+\alpha(I^h(t)-I^h(0))}{t}\\
&\leq \beta\langle S(t)\rangle-(\mu+\gamma+\varepsilon)+\frac{1}{2}\alpha h\sigma^2(\frac{\Lambda}{\mu})^{h+2}+\frac{\sigma}{t}\int_0^t S(r)dB(r)\\
&\leq \beta\frac{\Lambda}{\mu}-(\mu+\gamma+\varepsilon)-\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})\langle I(t)\rangle+\frac{1}{2}\alpha h\sigma^2(\frac{\Lambda}{\mu})^{h+2}+\frac{M(t)}{t}+\beta\varphi(t)\\
&\leq (\mu+\gamma+\varepsilon)(\tilde{R}_0-1)+\frac{\sigma^2\Lambda^2}{2\mu^2}[1+\alpha h(\frac{\Lambda}{\mu})^h)]-\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})\langle I(t)\rangle+\frac{M(t)}{t}+\beta\varphi(t),
\end{aligned}$$

then

$$\begin{aligned}&\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})\langle I(t)\rangle\\ \le\ &(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)+\frac{\sigma^2\Lambda^2}{2\mu^2}[1+\alpha h(\frac{\Lambda}{\mu})^h)]+\frac{M(t)}{t}+\beta\varphi(t)-\frac{\log I(t)-\log I(0)}{t}\\ &+\alpha\frac{I^h(0)}{t}.\end{aligned}$$

By Lemma 4.1, then

$$\limsup_{t\to\infty}\langle I(t)\rangle\leqslant\frac{(\mu+\gamma+\varepsilon)(\tilde{R}_0-1)+\frac{\sigma^2\Lambda^2}{2\mu^2}[1+\alpha h(\frac{\Lambda}{\mu})^h)]}{\beta(\frac{\mu+\varepsilon}{\mu}+\frac{\gamma}{\mu+\delta})}:=\tilde{I}^*. \tag{4.11}$$

Therefore, by (4.8), (4.10) and (4.11), we have the assertion (4.3). The proof is complete. □

Remark 4.1. From Theorem 3.1 and Theorem 4.1 if $\sigma^2\le\frac{\beta\mu}{\Lambda}$ then the value of $\tilde{R}_0$ being below 1 or above 1 will lead to the disease dying out or prevailing. As a result we consider $\tilde{R}_0$ as the threshold of system (1.3).

5. Asymptotic behavior around the endemic equilibrium P^*

Theorem 5.1. *Let* $(S(t),I(t),R(t))$ *be the solution of system* (1.3) *with initial value* $(S(0),I(0),R(0))\in\Gamma$*. If* $R_0>1$*, then*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\int_0^t[(S(r)-S^*)^2+(I(r)-I^*)^2+(R(r)-R^*)^2]dr\\ \leqslant&\frac{\sigma^2\Lambda^2(2\mu+\varepsilon)(1+\alpha I^{*h})I^*}{2k\beta\mu^2}a.s.\end{aligned} \tag{5.1}$$

where $P^*=(S^*,I^*,R^*)$ *is the endemic equilibrium of system* (1.1) *and* k *is positive constant defined as* (5.2).

Proof. Define a C^2-function $V:\mathbb{R}^3_+\to\mathbb{R}_+$ by

$$\begin{aligned}V(S,I,R)\ =\ &\frac{(2\mu+\varepsilon)(1+\alpha I^{*h})}{\beta}(I-I^*-I^*\log\frac{I}{I^*})\\ &+\frac{1}{2}[(S-S^*)+(I-I^*)+(R-R^*)]^2+\frac{2\mu+\varepsilon}{2\gamma}(R-R^*)^2\\ :=\ &\frac{(2\mu+\varepsilon)(1+\alpha I^{*h})}{\beta}V_1+V_2+\frac{2\mu+\varepsilon}{\gamma}V_3,\end{aligned}$$

where $V_1=I-I^*-I^*\log\frac{I}{I^*}$, $V_2=[(S-S^*)+(I-I^*)+(R-R^*)]^2$, $V_3=(R-R^*)^2$.

Then

$$\begin{aligned} LV_1 &= (I-I^*)(\frac{\beta S}{1+\alpha I^h} - \frac{\beta S^*}{1+\alpha {I^*}^h}) + \frac{\sigma^2 S^2 I^*}{2(1+\alpha I^h)^2} \\ &= -\frac{\alpha\beta S(I-I^*)^2}{(1+\alpha I^h)(1+\alpha {I^*}^h)} + \frac{\beta(S-S^*)(I-I^*)}{1+\alpha {I^*}^h} + \frac{\sigma^2 S^2 I^*}{2(1+\alpha I^h)^2} \\ &\leq \frac{\beta(S-S^*)(I-I^*)}{1+\alpha {I^*}^h} + \frac{1}{2}\sigma^2 I^* (\frac{\Lambda}{\mu})^2, \end{aligned}$$

$$\begin{aligned} LV_2 = & -\mu(S-S^*)^2 - (\mu+\varepsilon)(I-I^*)^2 - \mu(R-R^*)^2 - (2\mu+\varepsilon)(S-S^*)(I-I^*) \\ & -2\mu(S-S^*)(R-R^*) - (2\mu+\varepsilon)(I-I^*)(R-R^*) \end{aligned}$$

and

$$LV_3 = \gamma(I-I^*)(R-R^*) - (\mu+\delta)(R-R^*)^2.$$

Hence

$$\begin{aligned} LV \ \leq\ & -\mu(S-S^*)^2 - (\mu+\varepsilon)(I-I^*)^2 - [\mu + \frac{(2\mu+\varepsilon)(\mu+\delta)}{\gamma}](R-R^*)^2 \\ & -2\mu(S-S^*)(R-R^*) + \frac{\sigma^2 I^*(2\mu+\varepsilon)(1+\alpha {I^*}^h)}{2\beta}(\frac{\Lambda}{\mu})^2 \\ \leq\ & -\frac{\mu(2\mu+\varepsilon)(\mu+\delta)}{2\mu\gamma + (2\mu+\varepsilon)(\mu+\delta)}(S-S^*)^2 - (\mu+\varepsilon)(I-I^*)^2 \\ & -\frac{(2\mu+\varepsilon)(\mu+\delta)}{2\gamma}(R-R^*)^2 + \frac{\sigma^2 I^*(2\mu+\varepsilon)(1+\alpha {I^*}^h)}{2\beta}(\frac{\Lambda}{\mu})^2 \\ \leq\ & -k(S-S^*)^2 - k(I-I^*)^2 - k(R-R^*)^2 + \frac{\sigma^2 I^*(2\mu+\varepsilon)(1+\alpha {I^*}^h)}{2\beta}(\frac{\Lambda}{\mu})^2 \\ :=\ & F(t), \end{aligned}$$

where

$$k = \min\{\frac{\mu(2\mu+\varepsilon)(\mu+\delta)}{2\mu\gamma + (2\mu+\varepsilon)(\mu+\delta)}, (\mu+\varepsilon), \frac{(2\mu+\varepsilon)(\mu+\delta)}{2\gamma}\}. \tag{5.2}$$

Therefore,

$$dV \leqslant F(t)dt + \frac{\sigma S(I-I^*)}{1+\alpha I^h}dB(t).$$

Integrating both side of it from 0 to t, yields

$$V_2(t) - V_2(0) \leqslant \int_0^t F(s)ds + \int_0^t \frac{\sigma S(I-I^*)}{1+\alpha I^h}dB(s). \tag{5.3}$$

Let $M(t) := \int_0^t \frac{\sigma S(I-I^*)}{1+\alpha I^h}dB(s)$, Compute by the boundedness of $S,\ I$ that

$$\limsup_{t\to\infty} \frac{< M, M >_t}{t} \leqslant 4\sigma^2(\frac{\Lambda}{\mu})^4 < \infty.$$

Then the strong law of large numbers, Lemma 3.1, yields $\lim_{t\to\infty}\frac{M(t)}{t} = 0$ $a.s.$, which together with (5.3) implies

$$\limsup_{t\to\infty} \frac{\int_0^t F(s)ds}{t} \geqslant 0 \quad a.s.$$

Consequently, (5.1) is proved. The proof is complete. □

Remark 5.1. From Theorem 5.1 the distance between the solution $Y(t)=(S(t),I(t),R(t))$ and the endemic equilibrium $P^* = (S^*, I^*, R^*)$ of system (1.1) has the form

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \parallel Y(s) - P^* \parallel^2 ds \leqslant K \parallel \sigma \parallel^2, \quad a.s.$$

where K is a positive constant. We can think of it as approximate stability provided $\parallel \sigma \parallel^2$ is sufficiently small.

6. Numerical simulations

We choose the parameters in system (1.3) with initial value (0.8, 0,4, 0.5) as follows:

$$h = 0.5,\ \Lambda = 0.2,\ \beta = 0.4,\ \mu = 0.1,\ \alpha = 0.2,\ \gamma = 0.3,\ \varepsilon = 0.2,\ \delta = 0.1,\ \sigma = 0.4. \tag{6.1}$$

Note that $\tilde{R}_0 = 0.8 < 1$ and $\sigma^2 = 0.16 \leq \beta\mu/\Lambda = 0.2$, and then by Theorem 3.1 (b), the solution $Y(t)$ of system (1.3) obeys

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leqslant (\mu+\gamma+\varepsilon)(\tilde{R}_0 - 1) = -0.12 < 0, \qquad a.s.$$

That is $I(t)$ will tend to zero exponentially with probability one. Also for the corresponding deterministic model (1.1), $R_0 = 1.3333 > 1$. Fig. 1 supports our results.

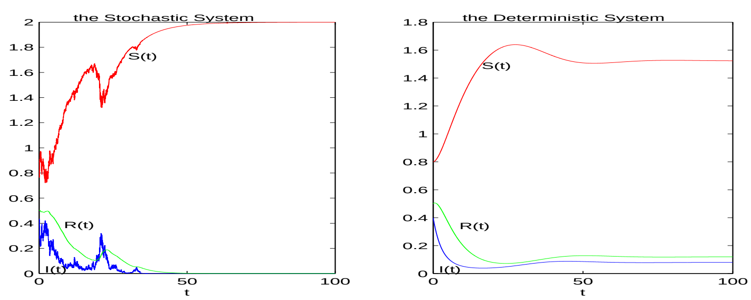

Figure 1. Computer simulation of the path $S(t)$, $I(t)$, $R(t)$ for model (1.3) and its corresponding deterministic model (1.1) for parameter values $h = 0.5$, $\Lambda = 0.2$, $\beta = 0.4$, $\mu = 0.1$, $\alpha = 0.2$, $\gamma = 0.3$, $\varepsilon = 0.2$, $\delta = 0.1$ and $\sigma = 0.4$, such that $\tilde{R}_0 < 1$, but $R_0 > 1$.

We choose the parameters in system (1.3) as follows:

$$h = 0.5,\ \Lambda = 0.2,\ \beta = 0.6,\ \mu = 0.1,\ \alpha = 0.2,\ \gamma = 0.3,\ \varepsilon = 0.2,\ \delta = 0.1,\ \sigma = 0.1. \tag{6.2}$$

Note that $\tilde{R}_0 \doteq 1.9667 > 1$ and $R_0 = 2 > 1$, then by Theorem 4.1, we conclude that the solution $Y(t)$ of system (1.3) obeys

$$0.0216 \leq \liminf_{t\to\infty}\langle I(t)\rangle \leq \limsup_{t\to\infty}\langle I(t)\rangle \leq 0.5969 \qquad a.s.$$

Then the disease will prevail. Fig. 2 (the left plot) supports our results.

By Theorem 5.1, we conclude that the difference between the solution of system (1.3) and P^* is related with white noise. As expected, the solution is oscillating

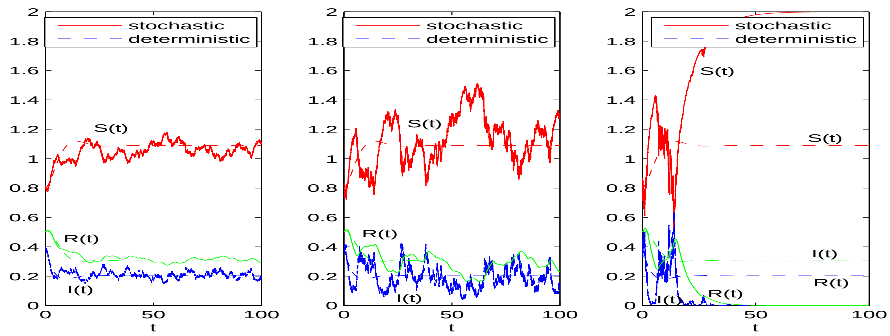

Figure 2. Computer simulation of the path $S(t)$, $I(t)$, $R(t)$ for the SDE SIRS model (1.3) and its corresponding deterministic model (1.1) for parameter values $\Lambda = 0.2$, $\beta = 0.6$, $\mu = 0.1$, $\alpha = 0.2$, $\gamma = 0.3$, $\varepsilon = 0.2$, $\delta = 0.1$, $h = 0.5$ and $\sigma = 0.1$ (left plot), and differing values of $\sigma = 0.3$ (middle plot) and $\sigma = 0.7$ (right plot).

around the endemic equilibrium P^* for a long time. To further illustrate the effect of the noise intensity σ on the model (1.3), we keep all the parameters of (6.2) unchanged but increase σ to 0.3 (see Fig.2 (the left plot and middle plot)). Comparing the middle plot with the left plot, the fluctuation of the solution of system (1.3) is getting weaker with the noise getting smaller.

We keep the system parameters the same as (6.2), but let $\sigma = 0.7$. It is easy to verify that the system parameters obey the condition (a) of Theorem 3.1, as $\sigma^2 > \max\{\beta^2/2(\mu+\gamma+\varepsilon), \beta\mu/\Lambda\} = 0.3$. We can therefore conclude that the solution of (1.3) obeys

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leqslant \frac{\beta^2}{2\sigma^2} - (\mu+\gamma+\varepsilon) = -0.2326 < 0, \quad \lim_{t\to\infty} S(t) = \Lambda/\mu = 2.$$

That is, $I(t)$ will tend to zero exponentially with probability one. The computer simulation shown in Fig. 2 (the right plot) clearly supports this result, showing extinction of the disease by the large noise.

7. Conclusions

In this paper we consider stochastic SIRS epidemic models with nonlinear incidence rates of the form $\frac{\beta SI}{1+\alpha I^h}$ $(h > 0)$ and we establish conditions for extinction and persistence of disease. We examine the threshold $\tilde{R}_0$ which is less than the corresponding deterministic version R_0 when the noise is small. We also show that the solution of the model oscillates around the endemic equilibrium of the corresponding deterministic system, and the intensity of fluctuation is proportional to white noise.

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