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OSCILLATION AND ASYMPTOTIC BEHAVIOR OF THIRD-ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DEVIATING ARGUMENTS*

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Abstract This paper concerns the oscillation and asymptotic behavior of a class of third-order nonlinear neutral delay differential equations with distributed deviating arguments. By employing a generalized Riccati transformation and integral averaging technique, we establish some sufficient conditions to ensure that all solutions of the considered equations are either oscillatory or converge to zero, which extend and improve some known results in the literature.

Keywords Oscillation, asymptotic property, third-order neutral delay differential equations, Riccati transformation.

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1. Introduction

In recent years, the oscillation theory of various classes of third-order functional differential equations and functional dynamic equations on time scales has received much attention from researchers in different fields, since it has wide applications in natural sciences and engineering [14] and [26], and we refer the reader to the papers [1, 2, 4, 6–8, 10, 11, 18, 21]. Specially, there has been much research activity concerning the oscillatory and asymptotic properties of third-order neutral differential equations and dynamic equations on time scales for the sake of the development of qualitative theory and practical reasons, in which some results can be found in [5, 9, 15, 17, 22, 25, 27] and the references are cited therein. However, there are few results dealing with the oscillation and asymptotic behavior of third-order neutral differential equations and dynamic equations on time scales with distributed deviating arguments in the literature, and we refer the reader to the papers [12, 13, 19, 20, 23, 24].

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In what follows, the third-order neutral differential equation

$$\left[r(t)([x(t)+p(t)x(\tau(t))]'')^{\alpha}\right]'+q(t)f(x(\sigma(t)))=0 \tag{1.1}$$

and its special cases have been studied by Baculíková and Džurina [5] and Li et al. [17] under the assumption $0 \le p(t) \le P < 1$, while Jiang et al. [15] and Thandapani and Li [22] investigated the oscillation of (1.1) complementing the results in the previous mentioned papers by assuming that $0 \le p(t) \le P < \infty$. Candan [9] and Zhang et al. [27] also established some oscillation criteria for the considered third-order neutral dynamic equations on time scales in the case $0 \le p(t) \le P < 1$.

Fu et al. [13], Şenel and Utku [19], and Tian et al. [23] studied the following equations

$$\left[r(t)\Big(\Big[x(t)+\int_a^b p(t,\xi)x(\tau(t,\xi))d\xi\Big]''\Big)^{\alpha}\right]'+\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi=0,$$

$$\left[r(t)\Big(\Big[x(t)+\int_a^b p(t,\xi)x(\tau(t,\xi))\Delta\xi\Big]^{\Delta\Delta}\Big)^{\alpha}\right]^{\Delta}+\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))\Delta\xi=0,$$

where $0 \le \int_a^b p(t,\xi)d\xi \le P < 1$ or $0 \le \int_a^b p(t,\xi)\Delta\xi \le P < 1$. Furthermore, Elabbasy and Moaaz [12] considered

$$\Big(r_2(t)\Big((r_1(t)(z'(t))^{\alpha_1})'\Big)^{\alpha_2}\Big)'+\int_a^b q(t,\xi)f(x(g(t,\xi)))d\sigma(\xi)=0,$$

where $z(t)=x(t)+p(t)x(\tau(t))$, $0 \le p(t) \le P < 1$. Unfortunately, the above mentioned results are not suitable to apply for the case $p(t) \ge 1$. Hence, Tunç [24] established some criteria for

$$\Big(r(t)\Big(\Big(x(t)+p(t)x(\tau(t))\Big)''\Big)^{\alpha}\Big)'+\int_a^b q(t,\xi)x^{\alpha}(\phi(t,\xi))d\xi=0$$

in the case $p(t) \ge 1$.

In this paper, we are interested in the oscillatory and asymptotic properties of the following third-order nonlinear neutral delay differential equation with distributed deviating arguments

$$\left[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\right]'+\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi=0, \tag{1.2}$$

where $z(t)=x(t)+p(t)x(\tau(t))$, $t \ge t_0 > 0$, $0 \le c < d$. We also suppose that the following hypotheses are satisfied:

(A1) $a(t), b(t) \in C([t_0,\infty),(0,\infty))$, $\int_{t_0}^{\infty} a^{-\frac{1}{\alpha_1}}(t)dt=\infty$, $\int_{t_0}^{\infty} b^{-\frac{1}{\alpha_2}}(t)dt=\infty$;

(A2) $p(t) \in C([t_0,\infty),[1,\infty))$ with $p(t) \not\equiv 1$ and $q(t,\xi) \in C([t_0,\infty)\times[c,d],[0,\infty))$;

(A3) $\tau(t) \in C^1([t_0,\infty),\mathbb{R})$, $\tau(t) \le t$, $\tau'(t) > 0$ and $\lim_{t\to\infty}\tau(t)=\infty$;

(A4) $\sigma(t,\xi) \in C([t_0,\infty)\times[c,d],\mathbb{R})$ is a nonincreasing function for ξ satisfying $\liminf_{t\to\infty}\sigma(t,\xi)=\infty$ for $\xi \in [c,d]$;

(A5) $f(x) \in C(\mathbb{R},\mathbb{R})$ is assumed to satisfy $xf(x) > 0$ and there exists a positive constant K such that $f(x)/x^{\alpha_3} \ge K$ for all $x \ne 0$;

(A6) α_i are ratios of positive odd integers, $i = 1,\ 2,\ 3$ and $\alpha_1\alpha_2 \geq 1$.

On the basis of the ideas exploited by Tian et al. [23] and Tunç [24], the main objective of this paper is to establish some new criteria for the oscillation and asymptotic behavior of (1.2) under the assumption $p(t) \geq 1$. Furthermore, the obtained sufficient conditions generalize and improve some known results in the literature.

2. Some Lemmas

In this section, we present some useful lemmas, which are essential to prove our main results. For simplicity throughout this paper, we use the following notations for sufficiently large t_1 with $t_1 \geq t_0$:

$$\sigma_1(t) = \sigma(t,c),\ \sigma_2(t) = \sigma(t,d),\ \rho_+(t) = \max\{0, \rho(t)\},$$
$$\delta_1(t,t_1) = \int_{t_1}^{t} b^{-\frac{1}{\alpha_2}}(s)ds,\ \delta_2(t,t_1) = \Big(\frac{\delta_1(t,t_1)}{a(t)}\Big)^{\frac{1}{\alpha_1}},$$
$$\delta_3(t,t_1) = \int_{t_1}^{t} \delta_2(s,t_1)ds,\ t \geq t_1.$$

Furthermore, assume that for $t \geq t_1$

$$\hat{p}(t) = \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\Big) > 0, \tag{2.1}$$
$$\tilde{p}(t) = \frac{1}{p(\tau^{-1}(t))}\Big(1 - \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)), t_1)}{p(\tau^{-1}(\tau^{-1}(t)))\delta_3(\tau^{-1}(t), t_1)}\Big) > 0, \tag{2.2}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. Then let

$$q_1(t) = K\int_c^d q(t,\xi)\hat{p}^{\alpha_3}(\sigma(t,\xi))d\xi,$$
$$q_2(t) = K\int_c^d q(t,\xi)\tilde{p}^{\alpha_3}(\sigma(t,\xi))d\xi.$$

Lemma 2.1 ([16]). *Let $\gamma \geq 1$ be a ratio of two odd integers. Then*

$$A^{1+\frac{1}{\gamma}} - (A-B)^{1+\frac{1}{\gamma}} \leq \frac{1}{\gamma}B^{\frac{1}{\gamma}}[(\gamma+1)A - B], \tag{2.3}$$

for all $AB \geq 0$.

Lemma 2.2. *Assume that conditions (A1)-(A6) hold, and $x(t)$ is an eventually positive solution of* (1.2). *Then there are only the following two possible cases of $z(t)$ for large t:*

(I) $z(t) > 0,\ z'(t) > 0,\ (a(t)(z'(t))^{\alpha_1})' > 0$ *and* $\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]' \leq 0$;

(II) $z(t) > 0,\ z'(t) < 0,\ (a(t)(z'(t))^{\alpha_1})' > 0$ *and* $\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]' \leq 0$.

The proof of Lemma 2.2 is similar to that of some existing papers, such as Sun et al. [21, Lemma 2.2], Elabbasy and Moaaz [12, Lemma 2.1]. So it is omitted.

Lemma 2.3. *Assume that conditions (A1)-(A6) and* (2.1) *hold. Furthermore, let* $x(t)$ *be an eventually positive solution of* (1.2) *and suppose that the corresponding* $z(t)$ *has the property (II) in Lemma 2.2. If*

$$\int_{t_0}^{\infty}\Big[\frac{1}{a(u)}\int_u^{\infty}\Big(\frac{1}{b(v)}\int_v^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_2}}dv\Big]^{\frac{1}{\alpha_1}}du=\infty, \tag{2.4}$$

then $\lim_{t\to\infty}x(t)=0$.

Proof. From the assumption, there exists $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t,\xi))>0$, $\xi\in[c,d]$ for $t\geq t_1$. From the definition of $z(t)$, we have

$$\begin{aligned} z(\tau^{-1}(t))&=x(\tau^{-1}(t))+p(\tau^{-1}(t))x(t),\\ x(t)&\geq\frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))}, \end{aligned} \tag{2.5}$$

which can be seen from [19, (2.4)] or [3, (8.6)]. Since $z'(t)<0$ for $t\geq t_1$ and $\tau(t)\leq t$, we get

$$x(t)\geq\hat{p}(t)z(\tau^{-1}(t)),$$

which yields that

$$x(\sigma(t,\xi))\geq\hat{p}(\sigma(t,\xi))z(\tau^{-1}(\sigma(t,\xi))) \tag{2.6}$$

for $t\geq t_2\geq t_1$. Using (1.2), (A5) and (2.6), we obtain

$$\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]'+K\int_c^d q(t,\xi)\hat{p}^{\alpha_3}(\sigma(t,\xi))z^{\alpha_3}(\tau^{-1}(\sigma(t,\xi)))d\xi\leq 0, \tag{2.7}$$

which leads to

$$\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]'\leq -q_1(t)z^{\alpha_3}(\tau^{-1}(\sigma_1(t))), \tag{2.8}$$

for $t\geq t_2$. Integrating (2.8) from t to ∞, we have

$$b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\geq\int_t^{\infty}q_1(s)z^{\alpha_3}(\tau^{-1}(\sigma_1(s)))ds. \tag{2.9}$$

Since $z(t)$ has the property (II), there exists a constant $l\geq 0$ such that $\lim_{t\to\infty}z(t)=l$. We claim that $l=0$. Otherwise, assume that $l>0$. Then there exists $t_3\geq t_2$ such that $\tau^{-1}(\sigma_1(t))\geq t_3$ and $z(\tau^{-1}(\sigma_1(t)))\geq l$, $t\geq t_3$. It can be seen from (2.9) that

$$(a(t)(z'(t))^{\alpha_1})'\geq\Big(\frac{l^{\alpha_3}}{b(t)}\int_t^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_2}}.$$

Integrating again from t to ∞, we get

$$-a(t)(z'(t))^{\alpha_1}\geq l^{\frac{\alpha_3}{\alpha_2}}\int_t^{\infty}\Big(\frac{1}{b(v)}\int_v^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_2}}dv,$$

due to $z'(t)<0$, which yields that

$$-z'(t)\geq\Big[\frac{l^{\frac{\alpha_3}{\alpha_2}}}{a(t)}\int_t^{\infty}\Big(\frac{1}{b(v)}\int_v^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_2}}dv\Big]^{\frac{1}{\alpha_1}}.$$

Integrating the above inequality from t_3 to ∞, we obtain

$$z(t_3) > l^{\frac{\alpha_3}{\alpha_1\alpha_2}} \int_{t_3}^{\infty} \Big[\frac{1}{a(u)} \int_u^{\infty} \Big(\frac{1}{b(v)} \int_v^{\infty} q_1(s)ds\Big)^{\frac{1}{\alpha_2}} dv\Big]^{\frac{1}{\alpha_1}} du,$$

which contradicts (2.4). Thus, we have $\lim_{t\to\infty} z(t) = l = 0$. We can also get $\lim_{t\to\infty} x(t) = 0$ from the fact that $0 < x(t) \le z(t)$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that conditions (A1)–(A6) and* (2.2) *hold. Furthermore, let $x(t)$ be an eventually positive solution of* (1.2) *and suppose that the corresponding $z(t)$ has the property (I) in Lemma 2.2. Then for sufficiently large t, we have*

$$\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]' + q_2(t) z^{\alpha_3}(\tau^{-1}(\sigma_2(t))) \le 0. \tag{2.10}$$

Proof. From the assumption, there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \ge t_1$. $b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}$ is nonincreasing for $t \ge t_1$, so we deduce that

$$\begin{aligned} a(t)(z'(t))^{\alpha_1} &= a(t_1)(z'(t_1))^{\alpha_1} + \int_{t_1}^{t} \frac{b^{\frac{1}{\alpha_2}}(s)(a(s)(z'(s))^{\alpha_1})'}{b^{\frac{1}{\alpha_2}}(s)} ds \\ &\ge \delta_1(t,t_1) b^{\frac{1}{\alpha_2}}(t)(a(t)(z'(t))^{\alpha_1})'. \end{aligned} \tag{2.11}$$

From (2.11), it is easy to see that

$$\Big(\frac{a(t)(z'(t))^{\alpha_1}}{\delta_1(t,t_1)}\Big)' = \frac{b^{-\frac{1}{\alpha_2}}(t)\big[\delta_1(t,t_1) b^{\frac{1}{\alpha_2}}(t)(a(t)(z'(t))^{\alpha_1})' - a(t)(z'(t))^{\alpha_1}\big]}{(\delta_1(t,t_1))^2} \le 0,$$

for $t \ge t_1$, which yields that $z'(t)/\delta_2(t,t_1)$ is nonincreasing for $t \ge t_1$. Then we get

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} \frac{z'(s)}{\delta_2(s,t_1)} \delta_2(s,t_1) ds \\ &\ge \frac{\delta_3(t,t_1)}{\delta_2(t,t_1)} z'(t). \end{aligned} \tag{2.12}$$

Based on (2.12), there exists $t_2 \ge t_1$ such that

$$\Big(\frac{z(t)}{\delta_3(t,t_1)}\Big)' = \frac{\delta_3(t,t_1)z'(t) - \delta_2(t,t_1)z(t)}{(\delta_3(t,t_1))^2} \le 0, \tag{2.13}$$

which yields that

$$z(\tau^{-1}(\tau^{-1}(t))) \le \frac{\delta_3(\tau^{-1}(\tau^{-1}(t)),t_1) z(\tau^{-1}(t))}{\delta_3(\tau^{-1}(t),t_1)}, \ t \ge t_2, \tag{2.14}$$

due to $\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t))$. Substituting (2.14) into (2.5), we obtain

$$x(t) \ge \tilde{p}(t) z(\tau^{-1}(t)),$$

which leads to

$$x(\sigma(t,\xi)) \ge \tilde{p}(\sigma(t,\xi)) z(\tau^{-1}(\sigma(t,\xi))), \ t \ge t_2. \tag{2.15}$$

Combining (1.2), (A5) and (2.15), we have (2.10). This completes the proof of Lemma 2.4. □

3. Main results

In this section, we respectively consider two cases $\tau(t) \geq \sigma(t,\xi)$ and $\tau(t) \leq \sigma(t,\xi)$, $\xi \in [c,d]$. We now establish the oscillation criteria for (1.2) with the first case.

Theorem 3.1. *Assume that conditions (A1)–(A6),* (2.1), (2.2), (2.4) *and* $\tau(t) \geq \sigma(t,\xi)$, $\xi \in [c,d]$ *are satisfied. Furthermore, assume that there exist functions* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *and* $\eta(t) \in C([t_0,\infty),[0,\infty))$ *such that* $b(t)\eta(t) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \int_{t_*}^{t} (\varphi(s) - G(s))\, ds = \infty, \tag{3.1}$$

for sufficiently large t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$*, where*

$$\begin{aligned}
\varphi(t) =&\rho(t)q_2(t)\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_3(t,t_1)}\Big)^{\alpha_3} \\
&+ \frac{\alpha_3}{\alpha_1\alpha_2}\rho(t)\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{1+\frac{1}{\alpha_1\alpha_2}} - \rho(t)(b(t)\eta(t))', \\
G(t) =&\frac{\lambda\Big[\rho'_+(t) + \Big(1+\frac{1}{\alpha_1\alpha_2}\Big)\alpha_3\rho(t)\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{\frac{1}{\alpha_1\alpha_2}}\Big]^{\alpha_1\alpha_2+1}}{(\rho(t)\gamma(t)\delta_2(t,t_1))^{\alpha_1\alpha_2}}, \\
\gamma(t) =&\begin{cases} m_1, & m_1 \text{ is any positive constant, } if\ \alpha_1\alpha_2 \leq \alpha_3, \\ m_2(\delta_3(t,t_1))^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}, & m_2 \text{ is any positive constant, } if\ \alpha_1\alpha_2 > \alpha_3, \end{cases} \\
\lambda =&\Big(\frac{\alpha_1\alpha_2}{\alpha_3}\Big)^{\alpha_1\alpha_2}\Big(\frac{1}{\alpha_1\alpha_2+1}\Big)^{\alpha_1\alpha_2+1}.
\end{aligned}$$

Then every solution of (1.2) *is either oscillatory or converges to zero.*

Proof. Suppose to the contrary that $x(t)$ is a eventually positive solution of (1.2). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$, for $t \geq t_1$. Lemma 2.2 verifies that $z(t)$ satisfies either property (I) or (II) for $t \geq t_1$.

Assume first that $z(t)$ has the property (I). Define a Riccati transformation $\omega(t)$ by

$$\omega(t) = \rho(t)\Big[\frac{b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)} + b(t)\eta(t)\Big],\ t \geq t_1. \tag{3.2}$$

Then $\omega(t) > 0$ for $t \geq t_1$, and

$$\begin{aligned}
\omega'(t) =&\rho'(t)\Big[\frac{b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)} + b(t)\eta(t)\Big] \\
&+ \rho(t)\Big[\frac{b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)} + b(t)\eta(t)\Big]', \\
=&\frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)(b(t)\eta(t))' + \rho(t)\frac{\big[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\big]'}{z^{\alpha_3}(t)} \\
&- \frac{\alpha_3\rho(t)b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}z'(t)}{z^{\alpha_3+1}(t)}.
\end{aligned} \tag{3.3}$$

From (2.11), we get

$$z'(t) \geq \delta_2(t,t_1)\Big(b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\Big)^{\frac{1}{\alpha_1\alpha_2}}. \tag{3.4}$$

From (2.10), we have

$$\frac{\left[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\right]'}{z^{\alpha_3}(t)} \leq -\frac{q_2(t)z^{\alpha_3}(\tau^{-1}(\sigma_2(t)))}{z^{\alpha_3}(t)}. \tag{3.5}$$

(2.13) yields that $z(t)/\delta_3(t,t_1)$ is nonincreasing for $t \geq t_1$, and since $\sigma(t,\xi) \leq \tau(t)$, we obtain $\tau^{-1}(\sigma_2(t)) \leq t$, which lead to

$$\frac{z(\tau^{-1}(\sigma_2(t)))}{z(t)} \geq \frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_3(t,t_1)}. \tag{3.6}$$

The definition of $\omega(t)$ verifies that

$$\frac{\omega(t)}{\rho(t)} - b(t)\eta(t) = \frac{b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}}{z^{\alpha_3}(t)}. \tag{3.7}$$

Combining (3.3), (3.4), (3.5), (3.6) and (3.7), we have

$$\begin{aligned}\omega'(t) \leq& \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)(b(t)\eta(t))' - \rho(t)q_2(t)\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_3(t,t_1)}\Big)^{\alpha_3} \\ &- \alpha_3\rho(t)\delta_2(t,t_1)z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t)\Big(\frac{\omega(t)}{\rho(t)} - b(t)\eta(t)\Big)^{1+\frac{1}{\alpha_1\alpha_2}}.\end{aligned} \tag{3.8}$$

Next, we will compute $z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t)$ and consider the following two cases:

Case (i) Let $\alpha_1\alpha_2 \leq \alpha_3$. From $z'(t) > 0$, there exists a constant $h_1 > 0$ such that

$$z(t) \geq z(t_1) = h_1,\ t \geq t_1,$$

which implies that

$$z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t) \geq m_1,\ t \geq t_1, \tag{3.9}$$

where $m_1 = h_1^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}$.

Case (ii) Let $\alpha_1\alpha_2 > \alpha_3$. Based on (2.13), there exist constants $h_2 > 0$ and $t_2 \geq t_1$ such that

$$\frac{z(t)}{\delta_3(t,t_1)} \leq \frac{z(t_2)}{\delta_3(t_2,t_1)} = h_2,\ t \geq t_2.$$

Hence,

$$z^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}(t) \geq m_2(\delta_3(t,t_1))^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}, \tag{3.10}$$

where $m_2 = h_2^{\frac{\alpha_3}{\alpha_1\alpha_2}-1}$. Combining (3.8) with (3.9) and (3.10), we get

$$\begin{aligned}\omega'(t) \leq& \frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)(b(t)\eta(t))' - \rho(t)q_2(t)\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_3(t,t_1)}\Big)^{\alpha_3} \\ &- \alpha_3\rho(t)\gamma(t)\delta_2(t,t_1)\Big(\frac{\omega(t)}{\rho(t)} - b(t)\eta(t)\Big)^{1+\frac{1}{\alpha_1\alpha_2}}.\end{aligned} \tag{3.11}$$

Applying Lemma 2.1, we have

$$\Big(\frac{\omega(t)}{\rho(t)} - b(t)\eta(t)\Big)^{1+\frac{1}{\alpha_1\alpha_2}} \geq \Big(\frac{\omega(t)}{\rho(t)}\Big)^{1+\frac{1}{\alpha_1\alpha_2}} + \frac{1}{\alpha_1\alpha_2}(b(t)\eta(t))^{1+\frac{1}{\alpha_1\alpha_2}}$$

$$-\Big(1+\frac{1}{\alpha_1\alpha_2}\Big)\frac{(b(t)\eta(t))^{\frac{1}{\alpha_1\alpha_2}}}{\rho(t)}\omega(t). \tag{3.12}$$

Substituting (3.12) into (3.11), we see that

$$\begin{aligned}\omega'(t) \leq& \rho(t)(b(t)\eta(t))' - \rho(t)q_2(t)\Big(\frac{\delta_3(\tau^{-1}(\sigma_2(t)),t_1)}{\delta_3(t,t_1)}\Big)^{\alpha_3}\\ &-\frac{\alpha_3}{\alpha_1\alpha_2}\rho(t)\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{1+\frac{1}{\alpha_1\alpha_2}}\\ &+\Big[\frac{\rho'_+(t)}{\rho(t)}+\Big(1+\frac{1}{\alpha_1\alpha_2}\Big)\alpha_3\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{\frac{1}{\alpha_1\alpha_2}}\Big]\omega(t)\\ &-\frac{\alpha_3\gamma(t)\delta_2(t,t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}\omega^{1+\frac{1}{\alpha_1\alpha_2}}(t)\\ =&-\varphi(t)+C(t)\omega(t)-D(t)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(t),\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}C(t)&=\frac{\rho'_+(t)}{\rho(t)}+\Big(1+\frac{1}{\alpha_1\alpha_2}\Big)\alpha_3\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{\frac{1}{\alpha_1\alpha_2}},\\ D(t)&=\frac{\alpha_3\gamma(t)\delta_2(t,t_1)}{\rho^{\frac{1}{\alpha_1\alpha_2}}(t)}.\end{aligned}$$

Using the inequality (see [5])

$$C\omega - D\omega^{1+\frac{1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha},\ D>0, \tag{3.14}$$

we get

$$\begin{aligned}\omega'(t) &\leq -\varphi(t)+\frac{(\alpha_1\alpha_2)^{\alpha_1\alpha_2}}{(\alpha_1\alpha_2+1)^{\alpha_1\alpha_2+1}}\frac{C^{\alpha_1\alpha_2+1}(t)}{D^{\alpha_1\alpha_2}(t)}\\ &=-\varphi(t)+G(t).\end{aligned} \tag{3.15}$$

Integrating (3.15) from t_2 to t, we get

$$\int_{t_2}^t (\varphi(s)-G(s))\,ds \leq \omega(t_2)-\omega(t) < \omega(t_2),$$

for all sufficiently large t, which contradicts (3.1).

Secondly, assume that $z(t)$ has the property (II). Based on Lemma 2.3, we obtain $\lim_{t\to\infty} x(t)=0$. The proof is complete. □

Let

$$\mathbb{D}=\{(t,s)\in\mathbb{R}^2 : t_0\leq s\leq t<\infty\} \text{ and } \mathbb{D}_0=\{(t,s)\in\mathbb{R}^2 : t_0\leq s<t<\infty\}.$$

The function $H(t,s)\in C(\mathbb{D},\mathbb{R})$ is said to belong to the class X, denoted by $H(t,s)\in X$, if it satisfies

(i) $H(t,t)=0$ for $t\geq t_0$ and $H(t,s)>0$ on $\mathbb{D}_0$;

(ii) $\partial H(t,s)/\partial s\leq 0$ on $\mathbb{D}_0$.

Theorem 3.2. *Assume that conditions (A1)–(A6),* (2.1)*,* (2.2)*,* (2.4) *and* $\tau(t) \geq \sigma(t,\xi)$*,* $\xi \in [c,d]$ *are satisfied. Furthermore, assume that there exist functions* $H(t,s) \in X$*,* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$*,* $\eta(t) \in C([t_0,\infty),[0,\infty))$ *and* $h(t,s) \in C(\mathbb{D}_0,\mathbb{R})$ *such that* $b(t)\eta(t) \in C^1([t_0,\infty),\mathbb{R})$*,*

$$-h(t,s) = \frac{\partial H(t,s)}{\partial s} + H(t,s)\Big[\frac{\rho'_+(s)}{\rho(s)} + \Big(1+\frac{1}{\alpha_1\alpha_2}\Big)\alpha_3\gamma(s)\delta_2(s,t_1)(b(s)\eta(s))^{\frac{1}{\alpha_1\alpha_2}}\Big],\ (t,s)\in\mathbb{D}_0, \tag{3.16}$$

and for sufficiently large t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$*,*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_*)}\int_{t_*}^{t}\Big[H(t,s)\varphi(s) - \frac{\lambda\rho(s)|h(t,s)|^{\alpha_1\alpha_2+1}}{(H(t,s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\Big]ds = \infty, \tag{3.17}$$

where λ*,* $\varphi(t)$ *and* $\gamma(t)$ *are defined as in Theorem 3.1. Then every solution of* (1.2) *is either oscillatory or converges to zero.*

Proof. Suppose that (1.2) has a eventually positive solution $x(t)$. Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \geq t_1$. From Lemma 2.2, it can be seen that $z(t)$ satisfies either property (I) or (II).

Assume first that $z(t)$ has the property (I). We define the Riccati transformation $\omega(t)$ by (3.2) and proceed as in the proof of Theorem 3.1 to obtain (3.13). Multiplying (3.13) by $H(t,s)$ and integrating the resulting inequality from t_2 ($t_2 \geq t_1$) to t, we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\varphi(s)ds \leq& -\int_{t_2}^{t} H(t,s)\omega'(s)ds + \int_{t_2}^{t} H(t,s)C(s)\omega(s)ds\\ &-\int_{t_2}^{t} H(t,s)D(s)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(s)ds\\ =&H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\Big(\frac{\partial H(t,s)}{\partial s} + H(t,s)C(s)\Big)\omega(s)ds\\ &-\int_{t_2}^{t} H(t,s)D(s)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(s)ds\\ =&H(t,t_2)\omega(t_2) - \int_{t_2}^{t} h(t,s)\omega(s)ds\\ &-\int_{t_2}^{t} H(t,s)D(s)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(s)ds\\ \leq&H(t,t_2)\omega(t_2)\\ &+\int_{t_2}^{t}\Big[|h(t,s)|\omega(s) - H(t,s)D(s)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(s)\Big]ds.\end{aligned}$$

Letting $C = |h(t,s)|$ and $D = H(t,s)D(s)$, and using the inequality (3.14), we get

$$\int_{t_2}^{t} H(t,s)\varphi(s)ds \leq H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\frac{\lambda\rho(s)|h(t,s)|^{\alpha_1\alpha_2+1}}{(H(t,s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}ds,$$

which yields that

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\Big[H(t,s)\varphi(s) - \frac{\lambda\rho(s)|h(t,s)|^{\alpha_1\alpha_2+1}}{(H(t,s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\Big]ds \leq \omega(t_2)$$

for all sufficiently large t, which contradicts (3.17).

On the other hand, assume that $z(t)$ has the property (II). By Lemma 2.3, we obtain $x(t) \to 0$ as $t \to \infty$. The proof is complete. $\square$

Next, we establish the oscillation criteria for (1.2) with the second case, i.e., $\tau(t) \le \sigma(t,\xi)$, $\xi \in [c,d]$.

Theorem 3.3. *Assume that conditions (A1)-(A6),* (2.1), (2.2), (2.4) *and* $\tau(t) \le \sigma(t,\xi)$, $\xi \in [c,d]$ *are satisfied. Furthermore, assume that there exist functions* $\rho(t) \in C^1([t_0,\infty),(0,\infty))$ *and* $\eta(t) \in C([t_0,\infty),[0,\infty))$ *such that* $b(t)\eta(t) \in C^1([t_0,\infty),\mathbb{R})$ *and*

$$\limsup_{t\to\infty} \int_{t_*}^{t} (\varphi_1(s) - G(s))\, ds = \infty, \tag{3.18}$$

for sufficiently large t_1 *and* t_* *with* $t_* \ge t_1 \ge t_0$, *where*

$$\begin{aligned}\varphi_1(t) =&\rho(t)q_2(t) - \rho(t)(b(t)\eta(t))' \\ &+ \frac{\alpha_3}{\alpha_1\alpha_2}\rho(t)\gamma(t)\delta_2(t,t_1)(b(t)\eta(t))^{1+\frac{1}{\alpha_1\alpha_2}},\end{aligned}$$

$G(t)$ *and* $\gamma(t)$ *are defined as in Theorem 3.1. Then every solution of* (1.2) *is either oscillatory or converges to zero.*

Proof. Suppose that (1.2) has a eventually positive solution $x(t)$. Then there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t,\xi)) > 0$, $\xi \in [c,d]$ for $t \ge t_1$. From Lemma 2.2, it can be seen that $z(t)$ satisfies either property (I) or (II).

Assume that $z(t)$ has the property (I). Proceeding as in the proof of Theorem 3.1, we have (3.3) and (3.5). In view of (A3) and $\tau(t) \le \sigma(t,\xi)$, $\xi \in [c,d]$, we get $\tau^{-1}(\sigma_2(t)) \ge t$. Since $z'(t) \ge 0$ for $t \ge t_1$, we obtain

$$z(\tau^{-1}(\sigma_2(t))) \ge z(t). \tag{3.19}$$

Combining (3.5) and (3.19), we have

$$\frac{\left[b(t)((a(t)(z'(t))^{\alpha_1})')^{\alpha_2}\right]'}{z^{\alpha_3}(t)} \le -q_2(t).$$

Then similar to the proof of Theorem 3.1, we get

$$\begin{aligned}\omega'(t) \le&\frac{\rho'_+(t)}{\rho(t)}\omega(t) + \rho(t)(b(t)\eta(t))' - \rho(t)q_2(t) \\ &- \alpha_3\rho(t)\gamma(t)\delta_2(t,t_1)\Big(\frac{\omega(t)}{\rho(t)} - b(t)\eta(t)\Big)^{1+\frac{1}{\alpha_1\alpha_2}} \\ =& - \varphi_1(t) + C(t)\omega(t) - D(t)\omega^{1+\frac{1}{\alpha_1\alpha_2}}(t),\end{aligned} \tag{3.20}$$

where $C(t)$ and $D(t)$ are defined as in Theorem 3.1. Using the inequality (3.14) again, we obtain

$$\omega'(t) \le -\varphi_1(t) + G(t),$$

where $G(t)$ is also defined as in Theorem 3.1. Integrating the above inequality from t_2 ($t_2 \ge t_1$) to t, we have a contradiction to (3.18).

On the other hand, assume that $z(t)$ has the property (II). By Lemma 2.3, we obtain $x(t) \to 0$ as $t \to \infty$. The proof is complete. $\square$

Theorem 3.4. *Assume that conditions (A1)-(A6),* (2.1), (2.2), (2.4) *and* $\tau(t) \leq \sigma(t,\xi)$, $\xi \in [c,d]$ *are satisfied. Furthermore, assume that there exist functions* $H(t,s) \in X$, $\rho(t) \in C^1([t_0,\infty),(0,\infty))$, $\eta(t) \in C([t_0,\infty),[0,\infty))$ *and* $h(t,s) \in C(\mathbb{D}_0,\mathbb{R})$ *such that* $b(t)\eta(t) \in C^1([t_0,\infty),\mathbb{R})$, (3.16) *holds and for sufficiently large* t_1 *and* t_* *with* $t_* \geq t_1 \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_*)} \int_{t_*}^{t} \left[H(t,s)\varphi_1(s) - \frac{\lambda\rho(s)|h(t,s)|^{\alpha_1\alpha_2+1}}{(H(t,s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}} \right] ds = \infty, \tag{3.21}$$

where λ *and* $\gamma(t)$ *are defined as in Theorem 3.1, and* $\varphi_1(t)$ *is defined as in Theorem 3.3. Then every solution of* (1.2) *is either oscillatory or converges to zero.*

Following from (3.20), the proof of Theorem 3.4 is similar to that of Theorem 3.2, so we omit it here.

Remark 3.1. With different choices of the functions $\rho(t)$ and $\eta(t)$, one can derive a number of oscillation criteria for (1.2) from Theorems 3.1 and 3.3.

Remark 3.2. One can see that the third-order neutral differential equation considered in [24] is a special case of (1.2). It would be interesting to find a different method to investigate (1.2) in the case $\alpha_1\alpha_2 > 0$, which can also extend and improve the results in [24].

Remark 3.3. The main results of this paper complement and improve the results obtained by Elabbasy and Moaaz [12] due to their application in the case $p(t) \geq 1$.

4. Examples

Firstly, we will examine an example to illustrate Theorem 3.1.

Example 4.1. For $t > 1$, consider a third-order neutral delay differential equation

$$\left[\left(\left((t-1)\left(\left(x(t)+5x\left(\frac{t}{3}\right)\right)'\right)^5\right)'\right)^{\frac{1}{3}}\right]' + \int_1^2 25^3 t\xi x^3\left(\frac{t}{3}-\xi\right)d\xi = 0, \tag{4.1}$$

where $\alpha_1 = 5$, $\alpha_2 = 1/3$, $\alpha_3 = 3$, $c = 1$, $d = 2$, $f(x) = x^3$, $a(t) = t-1$, $b(t) = 1$, $p(t) = 5$, $q(t,\xi) = 25^3 t\xi$, $\tau(t) = t/3$, $\sigma(t,\xi) = t/3 - \xi$. Let $t_0 = t_1 = 1$. Then we have

$$\begin{aligned}
&\sigma_2(t) = \sigma(t,2) = \frac{t}{3} - 2,\\
&\delta_1(t,t_1) = \delta_1(t,1) = t-1,\\
&\delta_2(t,t_1) = \left(\frac{\delta_1(t,1)}{t-1}\right)^{\frac{1}{5}} = 1,\\
&\delta_3(t,t_1) = \delta_3(t,1) = t-1,\\
&\delta_3(\tau^{-1}(t),t_1) = \delta_3(3t,1) = 3t-1,\\
&\delta_3(\tau^{-1}(\tau^{-1}(t)),t_1) = \delta_3(9t,1) = 9t-1,\\
&\delta_3(\tau^{-1}(\sigma_2(t)),t_1) = \delta_3(t-6,1) = t-7.
\end{aligned}$$

Furthermore, we get

$$\hat{p}(t) = \frac{4}{25} > 0,$$

$$\tilde{p}(t) = \frac{1}{5}\Big(1 - \frac{1}{5}\cdot\frac{9t-1}{3t-1}\Big) = \frac{1}{25}\Big(2 - \frac{2}{3t-1}\Big) \geq \frac{1}{25} > 0,$$
$$q_1(t) = \int_1^2 \Big(\frac{4}{25}\Big)^3 \cdot 25^3 t\xi d\xi = 96t,$$
$$q_2(t) \geq \int_1^2 \Big(\frac{1}{25}\Big)^3 \cdot 25^3 t\xi d\xi = \frac{3}{2}t.$$

It is easy to verify that

$$\begin{aligned}&\int_{t_0}^{\infty}\Big[\frac{1}{a(u)}\int_u^{\infty}\Big(\frac{1}{b(v)}\int_v^{\infty} q_1(s)ds\Big)^{\frac{1}{\alpha_2}}dv\Big]^{\frac{1}{\alpha_1}}du\\ =&\int_1^{\infty}\Big[\frac{1}{u-1}\int_u^{\infty}\Big(\int_v^{\infty} 96sds\Big)^3 dv\Big]^{\frac{1}{5}}du\\ =&\infty.\end{aligned}$$

Therefore, conditions (A1)-(A6), (2.1), (2.2) and (2.4) are satisfied. We choose $\rho(t) = 1$, $\eta(t) = 1/t$ and $t_2 = 7$. Applying Theorem 3.1, it remains to check (3.1), where

$$\begin{aligned}\lambda &= \Big(\frac{5}{9}\Big)^{\frac{5}{3}}\Big(\frac{3}{8}\Big)^{\frac{8}{3}},\\ \varphi(t) &\geq \frac{3t}{2}\Big(\frac{t-7}{t-1}\Big)^3 + \frac{9}{5}m_1\Big(\frac{1}{t}\Big)^{\frac{8}{5}} - \Big(\frac{1}{t}\Big)'\\ &= \frac{3t}{2}\Big(\frac{t-7}{t-1}\Big)^3 + \frac{9m_1}{5}t^{-\frac{8}{5}} + t^{-2},\\ G(t) &= \frac{\lambda\Big(\frac{24}{5}m_1\Big(\frac{1}{t}\Big)^{\frac{3}{5}}\Big)^{\frac{8}{3}}}{m_1^{\frac{5}{3}}}\\ &= \Big(\frac{24}{5}\Big)^{\frac{8}{3}}\lambda m_1 t^{-\frac{8}{5}}.\end{aligned}$$

We can see that

$$\int_7^t (\varphi(s) - G(s))\,ds \to \infty,$$

as $t \to \infty$. Hence, by Theorem 3.1, every solution of (4.1) is either oscillatory or converges to zero.

Secondly, the following example illustrates Theorem 3.4.

Example 4.2. For $t > k_1$, consider a third-order neutral delay differential equation

$$\Big((t-k_1)\Big(x(t) + \frac{3t+4}{t+1}x(t-k_1)\Big)'\Big)'' + \int_0^1 \frac{24q_0\xi}{t^2}x^{\alpha_3}\Big(t - k_1 + \frac{1}{\xi}\Big)d\xi = 0, \quad (4.2)$$

where k_1 and q_0 are positive constants. Let $\alpha_1 = \alpha_2 = 1$, $\alpha_3 \geq 1$, $c = 0$, $d = 1$, $f(x) = x^{\alpha_3}$, $a(t) = t - k_1$, $b(t) = 1$, $\tau(t) = t - k_1$, $\sigma(t,\xi) = t - k_1 + 1/\xi$,

$$p(t) = \frac{3t+4}{t+1},\ q(t,\xi) = \frac{24q_0\xi}{t^2}.$$

Choose $t_0 = t_1 = t_2 = k_1$. Then we get

$$3 \leq p(t) < 4,$$

$$\begin{aligned}
&\delta_1(t,t_1) = t - k_1,\\
&\delta_2(t,t_1) = 1,\\
&\delta_3(t,t_1) = t - k_1,\\
&\delta_3(\tau^{-1}(t),t_1) = \delta_3(t+k_1,k_1) = t,\\
&\delta_3(\tau^{-1}(\tau^{-1}(t)),t_1) = \delta_3(t+2k_1,k_1) = t+k_1.
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
\hat{p}(t) &> \frac{1}{4}\Big(1-\frac{1}{3}\Big) = \frac{1}{6} > 0,\\
\tilde{p}(t) &> \frac{1}{4}\Big(1-\frac{1}{3}\cdot\frac{t+k_1}{t}\Big) > \frac{1}{12} > 0,\\
q_1(t) &> \int_0^1 \frac{1}{6}\cdot\frac{24q_0\xi}{t^2}d\xi = \frac{2q_0}{t^2},\\
q_2(t) &> \int_0^1 \frac{1}{12}\cdot\frac{24q_0\xi}{t^2}d\xi = \frac{q_0}{t^2},
\end{aligned}$$

for $t \geq t_2$. It is easy to verify that

$$\begin{aligned}
&\int_{t_0}^{\infty}\Big[\frac{1}{a(u)}\int_u^{\infty}\Big(\frac{1}{b(v)}\int_v^{\infty}q_1(s)ds\Big)^{\frac{1}{\alpha_2}}dv\Big]^{\frac{1}{\alpha_1}}du\\
>&\int_{k_1}^{\infty}\frac{1}{u-k_1}\int_u^{\infty}\int_v^{\infty}\frac{2q_0}{s^2}dsdvdu\\
=&\infty.
\end{aligned}$$

Therefore, conditions (A1)-(A6), (2.1), (2.2) and (2.4) are satisfied. We choose $\rho(t) = t^2$, $\eta(t) = 0$ and $H(t,s) = (t-s)^2$. Then $h(t,s) = (t-s)(4-2ts^{-1})$. Applying Theorem 3.4, it remains to check (3.21), where

$$\varphi_1(t) = \rho(t)q_2(t) > t^2\cdot\frac{q_0}{t^2} = q_0.$$

Then we have

$$\begin{aligned}
&\frac{1}{H(t,t_2)}\int_{t_2}^{t}\Big[H(t,s)\varphi_1(s) - \frac{\lambda\rho(s)|h(t,s)|^{\alpha_1\alpha_2+1}}{(H(t,s)\gamma(s)\delta_2(s,t_1))^{\alpha_1\alpha_2}}\Big]ds\\
>&\frac{1}{(t-k_1)^2}\int_{k_1}^{t}\Big[(t-s)^2\cdot q_0 - \frac{\frac{1}{4\alpha_3}s^2|(t-s)(4-2ts^{-1})|^2}{(t-s)^2m_1}\Big]ds\\
=&\frac{1}{(t-k_1)^2}\int_{k_1}^{t}\Big[q_0(t^2-2ts+s^2) - \frac{1}{4\alpha_3m_1}(4t^2-16ts+16s^2)\Big]ds\\
\to&\infty,
\end{aligned}$$

as $t\to\infty$, if $q_0 \geq 4/(\alpha_3m_1)$. Hence, by Theorem 3.4, every solution of (4.2) is either oscillatory or converges to zero.

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