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APPROXIMATE CONTROLLABILITY OF SECOND-ORDER IMPULSIVE STOCHASTIC DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT DELAY*

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Abstract In this paper we study a kind of second-order impulsive stochastic differential equations with state-dependent delay in a real separable Hilbert space. Some sufficient conditions for the approximate controllability of this system are formulated and proved under the assumption that the corresponding deterministic linear system is approximately controllable. The results concerning the existence and approximate controllability of mild solutions have been addressed by using strongly continuous cosine families of operators and the contraction mapping principle. At last, an example is given to illustrate the theory.

Keywords Approximate controllability, state-dependent delay, impulsive stochastic differential equations, resolvent operator.

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1. Introduction

The stochastic differential equations (SDEs) in both finite dimensional and infinite dimensional spaces have been extensively studied. It has played an important role in many ways such as option pricing, forecast of the growth of population and so on. For first-order SDEs, some qualitative properties such as existence, controllability and stability have been investigated in several papers [5, 12, 19, 25]. In setting of second-order systems, it is advantages to treat second-order abstract differential equations directly rather than convert them to first-order systems. The second-order SDEs are the precise model in continuous time to account for integrated processes that can be made stationary. For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation through second-order SDEs. Recently, the problem of controllability for second-order SDEs has received considerable attention.

Mahmudov and Mckibben [14] focused on the approximate controllability problem for the class of abstract neutral semi-linear stochastic evolution equations in a real separable Hilbert space. Based on the theory of strongly continuous cosine families and Sadovskii fixed point theorem, Parthasarathy and Arjunan [17] obtained the controllability results of second-order impulsive stochastic differential

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and neutral differential systems with state-dependent delay. By the means of the Leray-Schauder Alternative fixed point theorem, Arthi et al. [2] proved the existence and controllability results for second-order impulsive stochastic evolution systems with state-dependent delay. The approximate controllability of a class of second-order neutral stochastic differential equations with infinite delay and Poisson jumps was considered by Muthukumar and Rajivganthi [16]. Das et al. [6] studied the existence and uniqueness of mild solution and approximate controllability for the second-order stochastic neutral partial differential equation with state-dependent delay.

However, in above mentioned papers, the authors didn't consider the damped term $x'(\cdot)$ in defining the exact and approximate controllability of the systems. It is not coincide with the definition of the controllability, because apart from $x(t)$, $x'(t)$ is also a state variable of a second-order system. Kang et al. [11] studied the exact controllability for the second-order differential inclusion in Banach spaces. With the help of a fixed point theorem for condensing maps due to Martelli [15], the authors found a control $u(\cdot)$ in $L_2(J,U)$ such that the solution satisfies $x(b) = x_1$ and $x'(b) = y_1$. Afterwards Balachandran and Kim [3] made some remarks on the paper [11] and indicated that the result of [11] is true only for finite dimensional Banach spaces. Very recently, after taking into account the damped term $x'(t)$ in defining the approximate controllability of the second-order abstract system, Li and Ma [13] established a new set of sufficient conditions for the approximate controllability of second-order impulsive functional differential system with infinite delay in Banach spaces. Inspired by the above mentioned works [2, 13, 17], the main purpose of this paper is to investigate the approximate controllability for the following second-order impulsive stochastic differential equations with state-dependent delay

$$
\begin{aligned}
& d[x'(t)] = [Ax(t) + Bu(t)]dt + f(t, x_{\rho(t,x_t)}, x'(t))d\omega(t), t \in J = [0,b], t \neq t_k, \\
& x_0 = \phi \in \mathcal{B},\ x'(0) = \zeta \in H, \\
& \Delta x|_{t=t_k} = I_k^1(x(t_k)),\ k = 1, 2, ..., m, \\
& \Delta x'|_{t=t_k} = I_k^2(x(t_k)),\ k = 1, 2, ..., m,
\end{aligned}
\tag{1.1}
$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operator $\{C(t)\}_{t\in R}$ on a Hilbert space H. The state variable $x(\cdot)$ takes the values in H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. The control function $u(\cdot)$ takes values in $L_2^{\mathcal{F}}(J,U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $\{\omega(t)\}_{t\geq 0}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. Moreover, $L(K,H)$ denotes the space of all bounded linear operators from K into H endowed with the same norm $\|\cdot\|$, simply $L(H)$ if $K = H$. For $t \in J$, x_t represents the function $x_t : (-\infty, 0] \to H$ defined by $x_t(\theta) = x(t+\theta), -\infty < \theta \leq 0$ which belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. Assume that $f : J\times\mathcal{B}\times H \to L_Q(K,H), \rho : J\times\mathcal{B} \to (-\infty, b]$, $I_k^i : H \to H, i = 1, 2$ are appropriate functions and will be specified later. Moreover, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k. Similarly $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of $x'(t)$ at $t = t_k$.

The rest of this paper is organized as follows. In Section 2, we recall some essential facts. In Section 3, we derive the existence of mild solution to problem (1.1). In section 4, we present the approximate controllability result. In Section 5, an example is provided to illustrate our results. We end this article with conclusion in Section 6.

2. Preliminaries

In this section, we review some concepts, notations and properties necessary to establish our results. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space furnished with a complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to H$ and a collection of random variables $S = \{x(t, \omega) : \Omega \to H \mid t \in J\}$ is called a stochastic process. Usually we write $x(t)$ instead of $x(t, \omega)$ and $x(t) : J \to H$ in the space of S. Let $\alpha_n(t)(n = 1, 2, \cdots)$ be a sequence of real-valued independent one-dimensional standard Brownian motions over $(\Omega, \mathcal{F}, P)$. Set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\alpha_n(t)\eta_n, \qquad t \geq 0,$$

where $\{\eta_n\}(n = 1, 2, \cdots)$ is a complete orthonormal basis in K and $\lambda_n \geq 0(n = 1, 2, \cdots)$ are nonnegative real numbers.

Let $Q \in L(K, K)$ be an operator defined by $Q\eta_n = \lambda_n\eta_n$ with $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\mathrm{Tr}(Q)$ denotes trace of Q. The K-valued stochastic process $\{\omega(t), t \geq 0\}$ is called a Q-Wiener process. It is assumed that $\mathcal{F}_t = \sigma(\omega(s) : 0 \leq s \leq t)$ is the σ-algebra generated by ω and $\mathcal{F}_b = \mathcal{F}$. Let $\xi \in L(K, H)$ and define

$$\|\xi\|_Q^2 = \mathrm{Tr}(\xi Q \xi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\xi\eta_n\|^2.$$

If $\|\xi\|_Q^2 < \infty$, then ξ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators from K into H. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\|\cdot\|_Q$, where $\|\xi\|_Q^2 = \langle \xi, \xi \rangle$ is a Hilbert space with the above norm topology. The collection of all strongly-measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega, H)$, which is a Banach space when endowed with the norm $\|x\|_{L_2} = (E\|x\|^2)^{\frac{1}{2}}$, where the expectation E is defined by $Ex = \int_\Omega x(\omega)dP$. Let $C(J, L_2(\Omega, H))$ be the Banach space of all continuous maps from J into $L_2(\Omega, H)$ satisfying the conditions $\sup_{t \in J} E\|x(t)\|^2 < \infty$. An important subspace of $L_2(\Omega, H)$ is given by $L_2^0(\Omega, H) = \{x \in L_2^0 : x \text{ is } \mathcal{F}_0\text{-adapted}\}$. Further, we denote $\mathcal{C} = \{x \in C(J, L_2(\Omega, H)) \mid x \text{ is } \mathcal{F}_t\text{-adapted }\}$, which is also a Banach space equipped with the norm $\|x\|_{\mathcal{C}} = \sup_{t \in J}(E\|x(t)\|^2)^{\frac{1}{2}}$. For more details reader may refer the reference [18].

The theory of cosine functions of operator plays an essential role in investigating the existence and controllability of mild solutions. Next we introduce the following definition.

Definition 2.1 (see [20,21])**.** A one parameter family $\{C(t)\}_{t \in R}$, of bounded linear operators defined on a Banach space H is called a strongly continuous cosine family

if
(i) $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s,t\in R$;
(ii) $C(0)=I$, I is the identity operators in H;
(iii) $C(t)x$ is strongly continuous in t on R for each fixed $x\in H$.

The strongly continuous sine family $\{S(t)\}_{t\in R}$, associated to the given strongly continuous cosine family $\{C(t)\}_{t\in R}$, is defined by

$$S(t)x=\int_0^t C(s)xds,\quad x\in H,\quad t\in R.$$

Moreover, M and N are positive constants such that $\|C(t)\|\le M$ and $\|S(t)\|\le N$ for every $t\in J$.

The infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\in R}$ is the operator $A:H\to H$ defined by

$$Ax=\frac{d^2}{dt^2}C(t)x\mid_{t=0},\quad x\in D(A),$$

where $D(A)=\{x\in H: C(t)x$ is twice continuously differentiable in $t\}$, endowed with the norm $\|x\|_A=\|x\|+\|Ax\|, x\in D(A)$.

Define $\mathbb{E}=\{x\in H: C(t)x$ is once continuously differentiable in $t\}$, endowed with the norm $\|x\|_{\mathbb{E}}=\|x\|+\sup\limits_{0\le t\le 1}\|AS(t)x\|, x\in\mathbb{E}$, then $\mathbb{E}$ is a Banach space. It follows that $AS(t):\mathbb{E}\to H$ is a bounded linear operator and $AS(t)x\to 0$ as $t\to 0$ for each $x\in\mathbb{E}$. The following properties are well known [22]:

$$S(t+s)=C(t)S(s)+C(s)S(t), \tag{2.1}$$
$$C(t+s)=C(t)C(s)+AS(s)S(t), \tag{2.2}$$
$$AS(s)S(t)=\frac{1}{2}[C(t+s)-C(t-s)]. \tag{2.3}$$

The existence of solutions of the second order abstract Cauchy problem

$$\begin{aligned} x''(t)&=Ax(t)+h(t),\ t\in J,\\ x(0)&=\epsilon_0,\\ x'(0)&=\epsilon_1, \end{aligned} \tag{2.4}$$

where $h:J\to H$ is an integral function, has been discussed in [20]. While, the existence of solutions for semilinear second order abstract Cauchy problem has been studied in [21]. In addition, the solution of (2.4) is introduced in [21] as follows. When $t\in J$, the function $x(\cdot)$ given by

$$x(t)=C(t)\epsilon_0+S(t)\epsilon_1+\textstyle\int_0^t S(t-s)h(s)ds,\ t\in J \tag{2.5}$$

is called a mild solution of (2.4), and that when $\epsilon_1\in\mathbb{E}$ the function $x(\cdot)$ is continuously differentiable and

$$x'(t)=AS(t)\epsilon_0+C(t)\epsilon_1+\int_0^t C(t-s)h(s)ds,\ t\in J.$$

In what follows, we put $t_0=0, t_{m+1}=b$ and a function $x:[\sigma,\tau]\to H$ is said to be a normalized piecewise continuous function on $[\sigma,\tau]$ if x is piecewise continuous and left continuous on $(\sigma,\tau]$. We denote by $\mathcal{PC}([\sigma,\tau],H)$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable process from $[\sigma,\tau]$ into H. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, H-valued stochastic process $x: J\to H$ such that $x(\cdot)$ is continuous at $t\neq t_k, x(t_k^-)=x(t_k)$ and $x(t_k^+)$ exists, for $k=1,2,\cdots,m$. It is clear that $\mathcal{PC}$ endowed with the norm $\|x\|_{\mathcal{PC}}=\sup\limits_{s\in J}(E\|x(s)\|^2)^{\frac{1}{2}}$ is a Banach space, where $\|\cdot\|$ is any norm of H.

For $x\in\mathcal{PC}$,we denote the function $\widetilde{x}_k\in C([t_k,t_{k+1}];L_2(\Omega,H))$ for $k=0,1,2,\cdots,m$, by

$$\widetilde{x}_k(t)=\begin{cases}x(t) & \text{for} \quad t\in(t_k,t_{k+1}],\\ x(t_k^+) & \text{for} \quad t=t_k.\end{cases}$$

A normalized piecewise continuous function $x:[\sigma,\tau]\to H$ is said to be normalized piecewise smooth on $[\sigma,\tau]$ if x is continuously differentiable except on a finite set S, the left derivative exists on $(\sigma,\tau]$ and the right derivative exists on $[\sigma,\tau)$. In this case, we present by $x'(t)$ the left derivative at $t\in(\sigma,\tau]$ and by $x'(\sigma)$ the right derivative at σ. We denote by $\mathcal{PC}^1([\sigma,\tau],H)$ the space of normalized piecewise smooth functions from $[\sigma,\tau]$ into H and by $\mathcal{PC}^1$ the space of $\mathcal{F}_t$-adapted measurable, H-valued stochastic process $x:J\to H$ such that $x(\cdot)$ is piecewise smooth. Obviously, $\mathcal{PC}^1$ is also a Banach space with the norm $\|x\|_{\mathcal{PC}^1}=\max\{\|x\|_{\mathcal{PC}},\|x'\|_{\mathcal{PC}}\}$.

In this paper, the phase space $(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ denotes a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping from $(-\infty,0]$ into H and such that the following axioms hold (Hale and Kato [9]).

(A) if $x:(-\infty,\sigma+b]\to H, b>0$, is such that $x_\sigma\in\mathcal{B}$ and $x\mid_{[\sigma,\sigma+b]}\in\mathcal{PC}([\sigma,\sigma+b],H)$, then for every $t\in[\sigma,\sigma+b)$ the following conditions hold:
(i) x_t is in $\mathcal{B}$;
(ii) $\|x(t)\|\leq\widetilde{H}\|x_t\|_{\mathcal{B}}$;
(iii) $\|x_t\|_{\mathcal{B}}\leq K(t-\sigma)\sup\{\|x(s)\|:\sigma\leq s\leq t\}+M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,
where $K,M:[0,\infty)\to[1,\infty)$, K is continuous, M is locally bounded and $\widetilde{H}>0$ is a constant, $\widetilde{H},K,M$ are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axioms.

Lemma 2.1 ([25]). *Let $x:(-\infty,b]\to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0=\phi\in L_2^0(\Omega,\mathcal{B})$ and $x(\cdot)|_J\in\mathcal{PC}$, then,*

$$\|x_s\|_{\mathcal{B}}\leq M_bE\|\phi\|_{\mathcal{B}}+K_b\sup_{0\leq s\leq b}E\|x(s)\|,$$

where $K_b=\sup\{K(t):0\leq t\leq b\}$ and $M_b=\sup\{M(t):0\leq t\leq b\}$.

In order to define the solution of the system(1.1), we consider the space

$$\mathcal{B}'_{h_1}=\{x:(-\infty,b]\to H \text{ such that } x(\cdot)|_J\in\mathcal{PC}, x_0\in\mathcal{B}\}$$

and

$$\mathcal{B}'_{h_2}=\{x\in\mathcal{B}'_{h_1}, x'(\cdot)\mid_J\in\mathcal{PC}\}.$$

Let $\|\cdot\|_{\mathcal{B}'_{h_1}},\|\cdot\|_{\mathcal{B}'_{h_2}}$ be the seminorm in $\mathcal{B}'_{h_1}$ and $\mathcal{B}'_{h_2}$, and they are defined by

$$\|x\|_{\mathcal{B}'_{h_1}}=\|\phi\|_{\mathcal{B}}+\sup_{s\in J}\|x(s)\|, \quad x\in\mathcal{B}'_{h_1}$$

and

$$\|x\|_{\mathcal{B}'_{h_2}} = \max\{\|x\|_{\mathcal{B}'_{h_1}}, \sup_{s\in J}\|x'(s)\|\}, \quad x \in \mathcal{B}'_{h_2}.$$

3. Existence results

In this section, we study the existence of mild solutions for the impulsive stochastic differential system (1.1). We present the definition of mild solutions for the system firstly.

Definition 3.1. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b) \to H$ is called a mild solution of the abstract Cauchy problem (1.1) if
(i) $x_0 = \phi$, $x_{\rho(s,x_s)} \in \mathcal{B}$, satisfying $x_0 \in L_2^0(\Omega, H)$, $x(\cdot)|_J \in \mathcal{PC}$;
(ii) the impulsive conditions $\triangle x|_{t=t_k} = I_k^1(x(t_k))$, $\triangle x'|_{t=t_k} = I_k^2(x(t_k))$, $k = 1, 2, \cdots, m$;
(iii) $x(t)$ satisfies the following integral equation:

$$x(t) = C(t)\phi(0) + S(t)\zeta + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)$$
$$+ \sum_{0<t_k<t} C(t-t_k)I_k^1(x(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x(t_k)),\ t \in J.$$

In this paper, we assume that $\rho : J\times\mathcal{B} \to (-\infty, b]$ is continuous. In the following, we give the following hypotheses firstly.
(H_1) For each $0 \le t < b$, the operator $\alpha(\alpha I + \Gamma_t^b)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$, where the controllability operator Γ_t^b, associated with (1.1) is defined as

$$\Gamma_t^b = \int_t^b S(b-s)BB^*S^*(b-s)ds.$$

(H_2) $f : J \times \mathcal{B} \times H \to L_Q(K, H)$ is a continuous function and there exist positive constants k_1 and k_2 such that

$$\|f(t, \varpi_1, \nu_1) - f(t, \varpi_2, \nu_2)\|_Q \le k_1\|\varpi_1 - \varpi_2\|_{\mathcal{B}} + k_2\|\nu_1 - \nu_2\|$$

for every $\varpi_1, \varpi_2 \in \mathcal{B}$ and $\nu_1, \nu_2 \in H$.
(H_3) The functions $I_k^i : H \to H$ are continuous and there exist positive constants $L(I_k^i),\ i = 1, 2,\ k = 1, 2, \cdots, m$ such that

$$\|I_k^i(\nu_1) - I_k^i(\nu_2)\|^2 \le L(I_k^i)\|\nu_1 - \nu_2\|^2,$$

for each $\nu_1, \nu_2 \in H$.
(H_4) $\max\{\phi_1, \phi_2\} < 1$, where

$$\begin{aligned}\phi_1 =& 32N^2\mathrm{Tr}(Q)b(1 + 6(\frac{N^2K^2b}{\alpha})^2)\eta + 8(1 + 6(\frac{N^2K^2b}{\alpha})^2)[M^2\sum_{k=1}^m L(I_k^1)\\ &+ N^2\sum_{k=1}^m L(I_k^2)],\\ \phi_2 =& 32\mathrm{Tr}(Q)b[M^2 + 6N^2(\frac{MNK^2b}{\alpha})^2]\eta + 8[\widetilde{N}^2 + 6M^2(\frac{MNK^2b}{\alpha})^2]\sum_{k=1}^m L(I_k^1)\\ &+ 8[M^2 + 6N^2(\frac{MNK^2b}{\alpha})^2]\sum_{k=1}^m L(I_k^2),\end{aligned}$$

and $\widetilde{N} = \sup_{s\in J} \|AS(t)\|_{\mathcal{L}(\mathbb{E},H)}$, $\eta = 2k_1^2K_b^2 + k_2^2$.

Theorem 3.1. *If* (H_1)-(H_4) *are satisfied, then system* (1.1) *has a mild solution on* J *for all* $u \in L_2^{\mathcal{F}}(J,U)$.

Proof. Let $l_f = \max_{t\in J} \|f(t,0,0)\|_Q$, $\|B\| < K$. Define the feedback control function

$$\begin{aligned} u(t) = & B^*S^*(b-t)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & - \textstyle\int_0^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) \\ & - \sum_{k=1}^m S(b-t_k)I_k^2(x(t_k))]. \end{aligned}$$

For $\phi \in \mathcal{B}$, we define $\widetilde{\phi}$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ C(t)\phi(0) + S(t)\zeta, & t \in J, \end{cases}$$

and then $\widetilde{\phi} \in \mathcal{B}'_{h_1}$.

We define $y(t) = AS(t)\phi(0) + C(t)\zeta,\ t \in J$. Let $x(t) = \widetilde{x}(t) + \widetilde{\phi}(t),\ x'(t) = \widetilde{x}'(t) + y(t),\ -\infty < t \le b$. It is straightforward that x satisfies

$$\begin{aligned} x(t) = & C(t)\phi(0) + S(t)\zeta + \textstyle\int_0^t S(t-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) \\ & + \textstyle\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & - \textstyle\int_0^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) \\ & - \sum_{k=1}^m S(b-t_k)I_k^2(x(t_k))]d\eta + \sum_{0<t_k<t} C(t-t_k)I_k^1(x(t_k)) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k^2(x(t_k)), \quad t \in J, \end{aligned}$$

if and only if $\widetilde{x}$ satisfies $\widetilde{x}_0 = 0$, and

$$\begin{aligned} \widetilde{x}(t) = & \textstyle\int_0^t S(t-s)f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s))d\omega(s) \\ & + \textstyle\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & - \textstyle\int_0^b S(b-s)f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s))d\omega(s) \\ & - \sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)) - \sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k) + \widetilde{\phi}(t_k))]d\eta \\ & + \sum_{0<t_k<t} C(t-t_k)I_k^1(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)),\ t \in J. \end{aligned}$$

It is also easy to verify that x' satisfies

$$\begin{aligned} x'(t) = & AS(t)\phi(0) + C(t)\zeta + \textstyle\int_0^t C(t-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) \\ & + \textstyle\int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & - \textstyle\int_0^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) - \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) \end{aligned}$$

$$-\sum_{k=1}^{m} S(b-t_k)I_k^2(x(t_k))]d\eta + \sum_{0<t_k<t} AS(t-t_k)I_k^1(x(t_k))$$
$$+\sum_{0<t_k<t} C(t-t_k)I_k^2(x(t_k)), \quad t\in J,$$

if and only if $\widetilde{x}'$ satisfies

$$\begin{aligned}\widetilde{x}'(t) = &\int_0^t C(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&+\int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\
&-\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&-\sum_{k=1}^{m} C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^{m} S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\\
&+\sum_{0<t_k<t} AS(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))+\sum_{0<t_k<t} C(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k)),\\
&t\in J.\end{aligned}$$

Let $\mathcal{B}''_{h_1}=\{\widetilde{x}\in\mathcal{B}'_{h_1}:\widetilde{x}_0=0\in\mathcal{B}\}$. For any $\widetilde{x}\in\mathcal{B}''_{h_1}$, $\|\widetilde{x}\|_{\mathcal{B}''_{h_1}}=\|\widetilde{x}_0\|_{\mathcal{B}}+\sup_{s\in J}\|\widetilde{x}(s)\|=\sup_{s\in J}\|\widetilde{x}(s)\|$, and thus $(\mathcal{B}''_{h_1},\|\cdot\|_{\mathcal{B}''_{h_1}})$ is a Banach space.

Let $Z=\mathcal{B}''_{h_1}\times\mathcal{PC}^1$ be the space

$$Z=\{(\widetilde{x},\widetilde{z}):\widetilde{x}\in\mathcal{B}''_{h_1},\widetilde{z}\in\mathcal{PC}^1 \text{ and } \widetilde{x}'(t)=\widetilde{z}(t) \text{ for } t\in J, t\neq t_k\}$$

provided with the norm

$$\|(\widetilde{x},\widetilde{z})\|_Z=\max\{\|\widetilde{x}\|_{\mathcal{B}''_{h_1}},\|\widetilde{z}\|_{\mathcal{PC}^1}\}.$$

It is now shown, $(\widetilde{x},\widetilde{z})\in Z$ implies $\widetilde{x}\in\mathcal{B}'_{h_2}$.

On the space Z, we define the nonlinear operator $\Phi(\widetilde{x},\widetilde{z})=(\Phi_1(\widetilde{x},\widetilde{z}),\Phi_2(\widetilde{x},\widetilde{z}))$, where

$$\begin{aligned}&\Phi_1(\widetilde{x},\widetilde{z})(t)\\
=&\int_0^t S(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&+\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\
&-\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&-\sum_{k=1}^{m} C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^{m} S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\\
&+\sum_{0<t_k<t} C(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))+\sum_{0<t_k<t} S(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k)),\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\Phi_2(\widetilde{x},\widetilde{z})(t)\\
=&\int_0^t C(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&+\int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\
&-\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&-\sum_{k=1}^{m} C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^{m} S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\end{aligned}\tag{3.2}$$

$$+\sum_{0<t_k<t} AS(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))+\sum_{0<t_k<t} C(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k)).$$

The continuity and well definition of Φ follow directly from the assumptions. Next, we will show that the operator Φ has a fixed point.

Let $\mathcal{Q}=\{(\widetilde{x},\widetilde{z})\in Z:\|(\widetilde{x},\widetilde{z})\|_Z\le r\}$, where r is a positive constant. For $(\widetilde{x},\widetilde{z})\in\mathcal{Q}$, by Lemma 2.1 and $\widetilde{x}_0=0$, we can obtain the following estimates:

$$\|\widetilde{x}_t\|_{\mathcal{B}}\le M_bE\|\widetilde{x}_0\|_{\mathcal{B}}+K_b\sup_{0\le s\le b}E\|\widetilde{x}(s)\|\le K_br,$$

and

$$\begin{aligned}\|\widetilde{x}_t+\widetilde{\phi}_t\|_{\mathcal{B}}&\le\|\widetilde{x}_t\|_{\mathcal{B}}+\|\widetilde{\phi}_t\|_{\mathcal{B}}\\&\le M_bE\|\widetilde{x}_0\|_{\mathcal{B}}+K_b\sup_{0\le s\le b}E\|\widetilde{x}(s)\|+M_bE\|\widetilde{\phi}_0\|_{\mathcal{B}}+K_b\sup_{0\le s\le b}E\|\widetilde{\phi}(s)\|\\&\le K_b(r+M\|\phi(0)\|+M\|\zeta\|)+M_b\|\phi\|_{\mathcal{B}}.\end{aligned}$$

So, we can obtain

$$E\|\widetilde{x}_t+\widetilde{\phi}_t\|_{\mathcal{B}}^2\le 4K_b^2(r^2+M^2\|\phi(0)\|^2+N^2\|\zeta\|^2))+4M_b^2\|\phi\|_{\mathcal{B}}^2=:r_1 \tag{3.3}$$

and

$$\begin{aligned}E\|\widetilde{x'}(t)+y(t)\|^2&\le 2r^2+2E\|AS(t)\phi(0)+C(t)\zeta\|^2\\&\le 2(r^2+2\widetilde{N}^2\|\phi(0)\|^2+2M^2\|\zeta\|^2)\\&=:r_2.\end{aligned} \tag{3.4}$$

For $(\widetilde{x},\widetilde{z})\in\mathcal{Q}$, by taking expectation on (3.1), then from (H_2)-(H_3), we have

$$\begin{aligned}&E\|\Phi_1(\widetilde{x},\widetilde{z})(t)\|^2\\&=E\|\textstyle\int_0^t S(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\&\quad+\textstyle\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\&\quad-\textstyle\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\&\quad-\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\\&\quad+\sum_{0<t_k<t}C(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\\&\quad+\sum_{0<t_k<t}S(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\&\le 4E\|\textstyle\int_0^t S(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\|^2\\&\quad+4E\|\textstyle\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\&\quad-\textstyle\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\&\quad-\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\|^2\\&\quad+4E\|\sum_{0<t_k<t}C(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\&\quad+4E\|\sum_{0<t_k<t}S(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\&:=I_1+I_2+I_3+I_4.\end{aligned} \tag{3.5}$$

From condition (H_2) and above mentioned estimates, we can find that

$$\begin{aligned}
I_1 &= 4E\|\int_0^t S(t-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\|^2\\
&\le 4Tr(Q)N^2\int_0^t E\|f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))\|_Q^2ds\\
&= 4Tr(Q)N^2\int_0^t E\|f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))-f(s,0,0)\\
&\quad+f(s,0,0)\|_Q^2ds\\
&\le 4Tr(Q)N^2\int_0^t[2E\|f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))-f(s,0,0)\|_Q^2\\
&\quad+2E\|f(s,0,0)\|_Q^2]ds\\
&\le 8Tr(Q)N^2\int_0^t[2k_1^2E\|\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}\|_{\mathcal{B}}^2+2k_2^2E\|\widetilde{x}'(s)+y(s)\|^2+l_f^2]ds\\
&\le 8Tr(Q)N^2b[2k_1^2r_1+2k_2^2r_2+l_f^2].
\end{aligned}$$

Then by using (H_3) and above mentioned estimates, we get

$$\begin{aligned}
I_3 &= 4E\|\sum_{0<t_k<t}C(t-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\
&= 4E\|\sum_{0<t_k<t}C(t-t_k)[I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^1(\widetilde{\phi}(t_k))+I_k^1(\widetilde{\phi}(t_k))]\|^2\\
&\le 4M^2\sum_{0<t_k<t}[2L(I_k^1)E\|\widetilde{x}(t_k)\|^2+2E\|I_k^1(\widetilde{\phi}(t_k))\|^2]\\
&\le 8M^2\sum_{k=1}^m[L(I_k^1)r^2+\|I_k^1(\widetilde{\phi}(t_k))\|^2]
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= 4E\|\sum_{0<t_k<t}S(t-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\
&= 4E\|\sum_{0<t_k<t}S(t-t_k)[I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^2(\widetilde{\phi}(t_k))+I_k^2(\widetilde{\phi}(t_k))]\|^2\\
&\le 8N^2\sum_{k=1}^m[L(I_k^2)E\|\widetilde{x}(t_k)\|^2+E\|I_k^2(\widetilde{\phi}(t_k))\|^2]\\
&\le 8N^2\sum_{k=1}^m[L(I_k^2)r^2+\|I_k^2(\widetilde{\phi}(t_k))\|^2].
\end{aligned}$$

Similarly, from the expressions of (I_1), (I_3) and (I_4), we have

$$\begin{aligned}
I_2 &= 4E\|\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}[x_1-C(b)\phi(0)-S(b)\zeta\\
&\quad-\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&\quad-\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))]d\eta\|^2\\
&\le 4(\tfrac{N^2K^2b}{\alpha})^2E\|x_1-C(b)\phi(0)-S(b)\zeta\\
&\quad-\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\
&\quad-\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))\|^2\\
&\le 24(\tfrac{N^2K^2b}{\alpha})^2\{\|x_1\|^2+M^2\|\phi(0)\|^2+N^2\|\zeta\|^2+2N^2\mathrm{Tr}(Q)b[2k_1^2r_1+2k_2^2r_2+l_f^2]\\
&\quad+2M^2\sum_{k=1}^m[L(I_k^1)r^2+\|I_k^1(\widetilde{\phi}(t_k))\|^2]+2N^2\sum_{k=1}^m[L(I_k^2)r^2+\|I_k^2(\widetilde{\phi}(t_k))\|^2]\}.
\end{aligned}$$

Thus, for $(\widetilde{x},\widetilde{z})\in\mathcal{Q}$, we have

$$
\begin{aligned}
&E\|\Phi_1(\widetilde{x},\widetilde{z})(t)\|^2\\
\le\,& I_1+I_2+I_3+I_4\\
\le\,& 8\mathrm{Tr}(Q)N^2b[2k_1^2r_1+2k_2^2r_2+l_f^2]+24(\tfrac{N^2K^2b}{\alpha})^2\{\|x_1\|^2+M^2\|\phi(0)\|_{\mathcal{B}}^2+N^2\|\zeta\|^2\\
&+2N^2\mathrm{Tr}(Q)b[2k_1^2r_1+2k_2^2r_2+l_f^2]+2M^2\sum_{k=1}^m[L(I_k^1)r^2+\|I_k^1(\widetilde{\phi}(t_k))\|^2]\\
&+2N^2\sum_{k=1}^m[L(I_k^2)r^2+\|I_k^2(\widetilde{\phi}(t_k))\|^2]\}+8M^2\sum_{k=1}^m[L(I_k^1)r^2+\|I_k^1(\widetilde{\phi}(t_k))\|^2]\\
&+8N^2\sum_{k=1}^m[L(I_k^2)r^2+\|I_k^2(\widetilde{\phi}(t_k))\|^2]\\
:=\,& 8\mathrm{Tr}(Q)N^2bN_1+24(\tfrac{N^2K^2b}{\alpha})^2N_2+8N_3,
\end{aligned}
$$

where

$$
\begin{aligned}
N_1&=2k_1^2r_1+2k_2^2r_2+l_f^2,\\
N_2&=\|x_1\|^2+M^2\|\phi(0)\|^2+N^2\|\zeta\|^2+2N^2Tr(Q)bN_1+2N_3,\\
N_3&=M^2\sum_{k=1}^m[L(I_k^1)r^2+\|I_k^1(\widetilde{\phi}(t_k))\|^2]+N^2\sum_{k=1}^m[L(I_k^2)r^2+\|I_k^2(\widetilde{\phi}(t_k))\|^2].
\end{aligned}
$$

Now let $8\mathrm{Tr}(Q)N^2bN_1+24(\frac{N^2K^2b}{\alpha})^2N_2+8N_3<r^2$, and substitute r_1 and r_2 into this inequality, which is equivalent to

$$
\begin{aligned}
&8\mathrm{Tr}(Q)N^2b\{2k_1^2[4K_b^2(M^2\|\phi(0)\|^2+N^2\|\zeta\|^2)+4M_b^2\|\phi\|_{\mathcal{B}}^2]\\
&+2k_2^2[2(2\widetilde{N}^2\|\phi(0)\|^2+2M^2\|\zeta\|^2)]+l_f^2\}+24(\tfrac{N^2K^2b}{\alpha})^2\{\|x_1\|^2+M^2\|\phi(0)\|^2\\
&+N^2\|\zeta\|^2+2N^2\mathrm{Tr}(Q)b\{2k_1^2[4K_b^2(M^2\|\phi(0)\|^2+N^2\|\zeta\|^2)+4M_b^2\|\phi\|_{\mathcal{B}}^2]\\
&+2k_2^2[2(2\widetilde{N}^2\|\phi(0)\|^2+2M^2\|\zeta\|^2)]+l_f^2\}+2M^2\sum_{k=1}^m\|I_k^1(\widetilde{\phi}(t_k))\|^2\\
&+2N^2\sum_{k=1}^m\|I_k^2(\widetilde{\phi}(t_k))\|^2\}+8M^2\sum_{k=1}^m\|I_k^1(\widetilde{\phi}(t_k))\|^2+8N^2\sum_{k=1}^m\|I_k^2(\widetilde{\phi}(t_k))\|^2\\
<\,& r^2\{1-8\mathrm{Tr}(Q)N^2b(8k_1^2K_b^2+4k_2^2)\\
&-24(\tfrac{N^2K^2b}{\alpha})^2[2N^2\mathrm{Tr}(Q)b(8k_1^2K_b^2+4k_2^2)+2M^2\sum_{k=1}^mL(I_k^1)\\
&+2N^2\sum_{k=1}^mL(I_k^2)]-8M^2\sum_{k=1}^mL(I_k^1)-8N^2\sum_{k=1}^mL(I_k^2)\}.
\end{aligned}
\tag{3.6}
$$

Then there exists r^2 such that (3.6) holds if,

$$
\begin{aligned}
\phi_1&=32N^2\mathrm{Tr}(Q)b(1+6(\tfrac{N^2K^2b}{\alpha})^2)(2k_1^2K_b^2+k_2^2)\\
&\quad+8(1+6(\tfrac{N^2K^2b}{\alpha})^2)[M^2\sum_{k=1}^mL(I_k^1)+N^2\sum_{k=1}^mL(I_k^2)]\\
&<1.
\end{aligned}
\tag{3.7}
$$

Similarly, taking expectation on (3.2),

$$
E\|\Phi_2(\widetilde{x},\widetilde{z})(t)\|^2\le 8M^2\mathrm{Tr}(Q)bN_1+24(\tfrac{MNK^2b}{\alpha})^2N_2+8N_4,
$$

where

$$N_4 = \widetilde{N}^2 \sum_{k=1}^{m}[L(I_k^1)r^2 + \|I_k^1(\widetilde{\phi}(t_k))\|^2] + M^2 \sum_{k=1}^{m}[L(I_k^2)r^2 + \|I_k^2(\widetilde{\phi}(t_k))\|^2].$$

Now let $8M^2\mathrm{Tr}(Q)bN_1 + 24(\frac{MNK^2b}{\alpha})^2N_2 + 8N_4 < r^2$. Similarly, substitute r_1 and r_2 into this inequality, we abtain

$$\begin{aligned}\phi_2 = {} & 32\mathrm{Tr}(Q)b[M^2 + 6N^2(\tfrac{MNK^2b}{\alpha})^2](2k_1^2K_b^2 + k_2^2) + 8[\widetilde{N}^2 \\ & +6M^2(\tfrac{MNK^2b}{\alpha})^2]\sum_{k=1}^{m}L(I_k^1) + 8[M^2 + 6N^2(\tfrac{MNK^2b}{\alpha})^2]\sum_{k=1}^{m}L(I_k^2) \\ & < 1.\end{aligned} \tag{3.8}$$

Therefore, Φ maps $\mathcal{Q}$ into $\mathcal{Q}$, when $\max\{\phi_1, \phi_2\} < 1$.

Next, we show that Φ is a contraction mapping on $\mathcal{Q}$. Let $(\widetilde{x}, \widetilde{z}), (\widetilde{v}, \widetilde{w}) \in \mathcal{Q}$, then we get

$$\begin{aligned} & E\|\Phi_1(\widetilde{x}, \widetilde{z})(t) - \Phi_1(\widetilde{v}, \widetilde{w})(t)\|^2 \\ & \le 4E\|\int_0^t S(t-s)f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s))d\omega(s) \\ & \quad - \int_0^t S(t-s)f(s, \widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}, \widetilde{v}'(s) + y(s))d\omega(s)\|^2 \\ & \quad +4E\|\int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & \quad - \int_0^b S(b-s)f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s))d\omega(s) \\ & \quad - \sum_{k=1}^{m} C(b-t_k)I_k^1(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)) - \sum_{k=1}^{m} S(b-t_k)I_k^2(\widetilde{x}(t_k) + \widetilde{\phi}(t_k))]d\eta \\ & \quad - \int_0^t S(t-\eta)BB^*S^*(b-\eta)(\alpha I + \Gamma_0^b)^{-1}[x_1 - C(b)\phi(0) - S(b)\zeta \\ & \quad - \int_0^b S(b-s)f(s, \widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}, \widetilde{v}'(s) + y(s))d\omega(s) \\ & \quad - \sum_{k=1}^{m} C(b-t_k)I_k^1(\widetilde{v}(t_k) + \widetilde{\phi}(t_k)) - \sum_{k=1}^{m} S(b-t_k)I_k^2(\widetilde{v}(t_k) + \widetilde{\phi}(t_k))]d\eta\|^2 \\ & \quad +4E\|\sum_{0<t_k<t} C(t-t_k)I_k^1(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)) - \sum_{0<t_k<t} C(t-t_k)I_k^1(\widetilde{v}(t_k) + \widetilde{\phi}(t_k))\|^2 \\ & \quad +4E\|\sum_{0<t_k<t} S(t-t_k)I_k^2(\widetilde{x}(t_k) + \widetilde{\phi}(t_k)) - \sum_{0<t_k<t} S(t-t_k)I_k^2(\widetilde{v}(t_k) + \widetilde{\phi}(t_k))\|^2 \\ & := J_1 + J_2 + J_3 + J_4. \end{aligned}$$

From the condition (H_2), we get

$$\begin{aligned} J_1 & = 4E\|\int_0^t S(t-s)[f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s)) \\ & \quad -f(s, \widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}, \widetilde{v}'(s) + y(s))]d\omega(s)\|^2 \\ & \le 4N^2\mathrm{Tr}(Q)\int_0^t E\|f(s, \widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}, \widetilde{x}'(s) + y(s)) \\ & \quad -f(s, \widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)} + \widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}, \widetilde{v}'(s) + y(s))\|_Q^2 ds \\ & \le 4N^2\mathrm{Tr}(Q)\int_0^t[2k_1^2E\|\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)} - \widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}\|_{\mathcal{B}}^2 + 2k_2^2E\|\widetilde{x}'(s) - \widetilde{v}'(s)\|^2]ds. \end{aligned}$$

In view of

$$\|\widetilde{x}_s - \widetilde{v}_s\|_{\mathcal{B}} \le K_b \sup_{0\le\tau\le s}\|\widetilde{x}(\tau) - \widetilde{v}(\tau)\|,$$

we obtain

$$\begin{aligned}J_1 &\le 8N^2\mathrm{Tr}(Q)b(k_1^2K_b^2\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+k_2^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1})\\ &=8N^2\mathrm{Tr}(Q)bk_1^2K_b^2\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+8N^2Tr(Q)bk_2^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}.\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}J_2 \le\ & 4(\tfrac{N^2K^2b}{\alpha})^2E\|[\textstyle\int_0^b S(b-s)f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))d\omega(s)\\ &-\textstyle\int_0^b S(b-s)f(s,\widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)},\widetilde{v}'(s)+y(s))d\omega(s)]\\ &+[\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m C(b-t_k)I_k^1(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))]\\ &+[\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-\sum_{k=1}^m S(b-t_k)I_k^2(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))]\|^2\\ \le\ & 4(\tfrac{N^2K^2b}{\alpha})^2[3\mathrm{Tr}(Q)N^2bE\|f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s))\\ &-f(s,\widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)},\widetilde{v}'(s)+y(s))\|_Q^2\\ &+3M^2\sum_{k=1}^m E\|I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^1(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))\|^2\\ &+3N^2\sum_{k=1}^m E\|I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^2(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))\|^2]\\ \le\ & 12(\tfrac{N^2K^2b}{\alpha})^2\{\mathrm{Tr}(Q)N^2b[2k_1^2K_b^2\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+2k_2^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}]\\ &+M^2\sum_{k=1}^m L(I_k^1)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+N^2\sum_{k=1}^m L(I_k^2)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}\}\\ \le\ & 12(\tfrac{N^2K^2b}{\alpha})^2[2\mathrm{Tr}(Q)N^2bk_1^2K_b^2+M^2\sum_{k=1}^m L(I_k^1)+N^2\sum_{k=1}^m L(I_k^2)]\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}\\ &+24\mathrm{Tr}(Q)N^2bk_2^2(\tfrac{N^2K^2b}{\alpha})^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1},\end{aligned}$$

and

$$\begin{aligned}J_3 &\le 4M^2\sum_{k=1}^m L(I_k^1)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}},\\ J_4 &\le 4N^2\sum_{k=1}^m L(I_k^2)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}.\end{aligned}$$

Then, we have

$$\begin{aligned}&E\|\Phi_1(\widetilde{x},\widetilde{z})(t)-\Phi_1(\widetilde{v},\widetilde{w})(t)\|^2\\ \le\ & [8N^2\mathrm{Tr}(Q)bk_1^2K_b^2+24N^2\mathrm{Tr}(Q)bk_1^2K_b^2(\tfrac{N^2K^2b}{\alpha})^2\\ &+12M^2(\tfrac{N^2K^2b}{\alpha})^2\sum_{k=1}^m L(I_k^1)+12N^2(\tfrac{N^2K^2b}{\alpha})^2\sum_{k=1}^m L(I_k^2)\\ &+4M^2\sum_{k=1}^m L(I_k^1)+4N^2\sum_{k=1}^m L(I_k^2)]\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}\\ &+[8N^2\mathrm{Tr}(Q)bk_2^2+24N^2\mathrm{Tr}(Q)bk_2^2(\tfrac{N^2K^2b}{\alpha})^2]\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}\\ =\ & 4(1+3(\tfrac{N^2K^2b}{\alpha})^2)[2N^2\mathrm{Tr}(Q)bk_1^2K_b^2+M^2\sum_{k=1}^m L(I_k^1)\\ &+N^2\sum_{k=1}^m L(I_k^2)]\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+8N^2\mathrm{Tr}(Q)bk_2^2(1+3(\tfrac{N^2K^2b}{\alpha})^2)\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}.\end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned}
& E\|\Phi_2(\widetilde{x},\widetilde{z})(t)-\Phi_2(\widetilde{v},\widetilde{w})(t)\|^2 \\
\le\ & 4E\|\textstyle\int_0^t C(t-s)[f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s)) \\
& -f(s,\widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)},\widetilde{v}'(s)+y(s))]d\omega(s)\|^2 \\
& +4E\|\textstyle\int_0^t C(t-\eta)BB^*S^*(b-\eta)(\alpha I+\Gamma_0^b)^{-1}\{-\int_0^b S(b-s) \\
& \times[f(s,\widetilde{x}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{x}_s+\widetilde{\phi}_s)},\widetilde{x}'(s)+y(s)) \\
& -f(s,\widetilde{v}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)}+\widetilde{\phi}_{\rho(s,\widetilde{v}_s+\widetilde{\phi}_s)},\widetilde{v}'(s)+y(s))]d\omega(s) \\
& -\sum_{k=1}^m C(b-t_k)[I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^1(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))] \\
& -\sum_{k=1}^m S(b-t_k)[I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^2(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))]\}d\eta\|^2 \\
& +4E\|\sum_{k=1}^m AS(b-t_k)[I_k^1(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^1(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))]\|^2 \\
& +4E\|\sum_{k=1}^m C(b-t_k)[I_k^2(\widetilde{x}(t_k)+\widetilde{\phi}(t_k))-I_k^2(\widetilde{v}(t_k)+\widetilde{\phi}(t_k))]\|^2 \\
\le\ & 8M^2\mathrm{Tr}(Q)b(k_1^2K_b^2\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+k_2^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}) \\
& +12(\tfrac{MNK^2b}{\alpha})^2[N^2\mathrm{Tr}(Q)b(2k_1^2K_b^2\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+2k_2^2\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}) \\
& +M^2\sum_{k=1}^m L(I_k^1)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+N^2\sum_{k=1}^m L(I_k^2)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}] \\
& +4\widetilde{N}^2\sum_{k=1}^m L(I_k^1)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}}+4M^2\sum_{k=1}^m L(I_k^2)\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}} \\
=\ & [(M^2+3N^2(\tfrac{MNK^2b}{\alpha})^2)(8\mathrm{Tr}(Q)bk_1^2K_b^2+4\sum_{k=1}^m L(I_k^2)) \\
& +4(\widetilde{N}^2+3M^2(\tfrac{MNK^2b}{\alpha})^2)\sum_{k=1}^m L(I_k^1)]\|\widetilde{x}-\widetilde{v}\|^2_{\mathcal{B}''_{h_1}} \\
& +8\mathrm{Tr}(Q)bk_2^2(M^2+3N^2(\tfrac{MNK^2b}{\alpha})^2)\|\widetilde{z}-\widetilde{w}\|^2_{\mathcal{PC}^1}.
\end{aligned}\tag{3.10}$$

The above inequalities (3.9) and (3.10) and the assumption $\max\{\phi_1,\phi_2\}<1$ imply that Φ is a contraction mapping. Hence there exists a unique fixed point $(\widetilde{x},\widetilde{z})\in\mathcal{Q}$. Then the function $x(\cdot)=\widetilde{x}(\cdot)+\widetilde{\phi}(\cdot)\in\mathcal{B}'_{h_2}$ is a mild solution of (1.1). This completes the proof. □

4. Approximate controllability

In this section, we compare approximate controllability of the semilinear system (1.1) with approximate controllability of the associated linear system. For this reason, we consider the linear system

$$x''(t)=Ax(t)+Bu(t),\quad t\in J,\tag{4.1}$$

with initial condition

$$\begin{aligned} x(0)&=\phi(0),\\ x'(0)&=\zeta. \end{aligned}\tag{4.2}$$

First, we show the definition of the approximate controllability of systems (4.1)-(4.2).

Definition 4.1. Systems (4.1)-(4.2) are said to be approximately controllable on J if $\overline{D} = H \times H$, where $D = \{x(b,\phi(0),\zeta,u), y(b,\phi(0),\zeta,u) : u \in L_2^{\mathcal{F}}(J,U)\}$, $y(\cdot,\phi(0),\zeta,u) = x'(\cdot,\phi(0),\zeta,u)$ and $x(\cdot,\phi(0),\zeta,u)$ is a mild solution of (4.1)-(4.2).

The following result has been established by Fattorini [7] and Triggiani [23,24]. We introduce the sets

$$D_\infty(A) = \bigcap_{n=1}^{\infty} D(A^n),$$
$$U_\infty = \{u \in U : Bu \in D_\infty(A)\},$$
$$X_0 = \bigcup_{t>0} T(t)(X),$$
$$U_0 = \{u \in U : Bu \in X_0\},$$

where $T(t)$ is the analytic semigroup generated by A [1,8]. It is clear that $U_0 \subseteq U_\infty$.

We are on the position to give the approximate controllability of (4.1)-(4.2), that is

Theorem 4.1. (see [7,23,24])

(i) *Systems* (4.1)-(4.2) *are approximately controllable on J if, and only if, $x^*, y^* \in H^*$ are such that $B^*S(t)x^* + B^*C(t)^*y^* = 0$, for $t \in J$, then $x^* = y^* = 0$.*

(ii) *If $Sp\{A^nBU_\infty : n \geq 0\}$ is dense in H, then systems* (4.1)-(4.2) *are approximately controllable on J.*

(iii) *If BU_0 is dense in BU and system* (4.1)-(4.2) *are approximately controllable on J, then $Sp\{A^nBU_0 : n \geq 0\}$ is dense in H.*

Next, we discuss the approximate controllability of the semilinear system (1.1). Before stating and proving our main result, we give the definition of approximate controllability firstly.

Definition 4.2. System (1.1) is said to be approximately controllable on J if $\overline{\mathcal{R}(f,\phi,\zeta)} = H \times H$, where $\mathcal{R}(f,\phi,\zeta) = \{x(b,\phi,\zeta,u), y(b,\phi,\zeta,u) : u \in L_2^{\mathcal{F}}(J,U)\}$, $y(\cdot,\phi,\zeta,u) = x'(\cdot,\phi,\zeta,u)$ and $x(\cdot,\phi,\zeta,u)$ is a mild solution of (1.1).

Now, under the above conditions, we prove the following approximately controllable theorem.

Theorem 4.2. *Assume that BU_0 is dense in BU and the conditions (H_1)-(H_4) are satisfied. If systems* (4.1)-(4.2) *are approximately controllable on J, then system* (1.1) *is approximately controllable on J.*

Proof. It follows by the approximately controllability of (4.1)-(4.2) on J, we obtain (H_1) is satisfied. Because the hypotheses of Theorem 3.1 are fulfilled, for each $u \in L_2^{\mathcal{F}}(J,U)$, there is a unique mild solution of (1.1). Let $(\widetilde{x},\widetilde{z})$ be a fixed point of Φ in $\mathcal{Q}$. $x(\cdot) = \widetilde{x}(\cdot) + \widetilde{\phi}(\cdot)$ is the mild solution of (1.1) on J. By the conditions (H_2) and the proof of Theorem 3.1, we know

$$E\|f(t,x_{\rho(t,x_t)},x'(t))\|_Q^2 \leq 2k_1^2 r_1 + 2k_2^2 r_2 + l_f^2.$$

We fix $z = (z_1,z_2) \in H \times H$ and take $0 < b_n < b$ such that $b_n \to b$ as $n \to \infty$. Let $x_n = x(b_n,\phi,\zeta,0)$ and $y_n = y(b_n,\phi,\zeta,0)$. It follows from the properties established in Section 2 that $x_n \in \mathbb{E}$. In addition, it follows from Theorem 4.1 that system (4.1) with initial conditions $x(0) = x_n$ and $x'(0) = y_n$ is approximate controllable on $[0, b-b_n]$. Consequently, there exists a control function $w_n(\cdot) \in L_p^{\mathcal{F}}([0, b-b_n], U)$ such that

$$\begin{aligned}&\int_0^{b-b_n} S(b-b_n-s)Bw_n(s)ds + C(b-b_n)x_n + S(b-b_n)y_n - z_1\\ =&\int_{b_n}^{b} S(b-s)Bv_n(s)ds + C(b-b_n)x_n + S(b-b_n)y_n - z_1 \to 0, \quad n\to\infty\end{aligned}$$

and

$$\begin{aligned}&\int_0^{b-b_n} C(b-b_n-s)Bw_n(s)ds + AS(b-b_n)x_n + C(b-b_n)y_n - z_2\\ =&\int_{b_n}^{b} C(b-s)Bv_n(s)ds + AS(b-b_n)x_n + C(b-b_n)y_n - z_2 \to 0, \quad n\to\infty,\end{aligned}$$

where $v_n(s) = w_n(s-b_n)$. $L_p^{\mathcal{F}}([0, b-b_n], U)$ denotes the closed subspace of $L_p([0, b-b_n], U)$ consisting of $\mathcal{F}$-adapted processes. We define

$$u_n(s) = \begin{cases} 0, & 0 \le s \le b_n,\\ v_n(s), & b_n < s \le b.\end{cases}$$

Next, we denote the abbreviate notation with $x(\cdot) = x(\cdot, \phi, \zeta, u_n)$ and $y(\cdot) = y(\cdot, \phi, \zeta, u_n)$. By the uniqueness of solutions, we have

$$\begin{aligned}x_n =& C(b_n)\phi(0) + S(b_n)\zeta + \int_0^{b_n} S(b_n-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \sum_{0<t_k<b_n} C(b_n-t_k)I_k^1(x(t_k)) + \sum_{0<t_k<b_n} S(b_n-t_k)I_k^2(x(t_k)),\\ y_n =& AS(b_n)\phi(0) + C(b_n)\zeta + \int_0^{b_n} C(b_n-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \sum_{0<t_k<b_n} AS(b_n-t_k)I_k^1(x(t_k)) + \sum_{0<t_k<b_n} C(b_n-t_k)I_k^2(x(t_k)).\end{aligned}$$

Combing these expressions with (2.1) and (2.2), we obtain

$$\begin{aligned}&x(b, \phi, \zeta, u_n)\\ =& C(b)\phi(0) + S(b)\zeta + \int_0^b S(b-s)Bu_n(s)ds + \int_0^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x(t_k))\\ =& C(b)\phi(0) + S(b)\zeta + \int_{b_n}^b S(b-s)Bv_n(s)ds + \int_0^{b_n} S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \int_{b_n}^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s) + \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k))\\ &+ \sum_{k=1}^m S(b-t_k)I_k^2(x(t_k))\\ =& C(b)\phi(0) + S(b)\zeta + \int_{b_n}^b S(b-s)Bv_n(s)ds\\ &+S(b-b_n)\int_0^{b_n} C(b_n-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+C(b-b_n)\int_0^{b_n} S(b_n-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \int_{b_n}^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\\ &+ \sum_{k=1}^m C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^m S(b-t_k)I_k^2(x(t_k))\\ =& C(b)\phi(0) + S(b)\zeta + \int_{b_n}^b S(b-s)Bv_n(s)ds + \int_{b_n}^b S(b-s)f(s, x_{\rho(s,x_s)}, x'(s))d\omega(s)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{m} C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^{m} S(b-t_k)I_k^2(x(t_k)) + S(b-b_n)[y_n - AS(b_n)\phi(0)\\
&-C(b_n)\zeta - \sum_{0<t_k<b_n} AS(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} C(b_n-t_k)I_k^2(x(t_k))]\\
&+C(b-b_n)[x_n - C(b_n)\phi(0) - S(b_n)\zeta - \sum_{0<t_k<b_n} C(b_n-t_k)I_k^1(x(t_k))\\
&-\sum_{0<t_k<b_n} S(b_n-t_k)I_k^2(x(t_k))]\\
=&\int_{b_n}^{b} S(b-s)Bv_n(s)ds + \int_{b_n}^{b} S(b-s)f(s,x_{\rho(s,x_s)},x'(s))d\omega(s)\\
&+\sum_{k=1}^{m} C(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^{m} S(b-t_k)I_k^2(x(t_k))\\
&+S(b-b_n)[y_n - \sum_{0<t_k<b_n} AS(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} C(b_n-t_k)I_k^2(x(t_k))]\\
&+C(b-b_n)[x_n - \sum_{0<t_k<b_n} C(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} S(b_n-t_k)I_k^2(x(t_k))].
\end{aligned}
$$

Because the function f is bounded on $\mathcal{Q}$, we infer that $\int_{b_n}^{b} S(b-s)f(s,x_{\rho(s,x_s)}, x'(s))d\omega(s) \to 0$, as $n \to \infty$. Furthermore, from (2.1) and (2.2), all the summation terms could be canceled out as $n \to \infty$. Thus we obtain $x(b,\phi,\zeta,u_n) \to z_1$ as $n \to \infty$.

Similarly,

$$
\begin{aligned}
&y(b,\phi,\zeta,u_n)\\
=&AS(b)\phi(0) + C(b)\zeta + \int_0^b C(b-s)Bu_n(s)ds + \int_0^b C(b-s)f(s,x_{\rho(s,x_s)},x'(s))d\omega(s)\\
&+\sum_{k=1}^{m} AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^{m} C(b-t_k)I_k^2(x(t_k))\\
=&AS(b)\phi(0) + C(b)\zeta + \int_{b_n}^b C(b-s)Bv_n(s)ds + \int_{b_n}^b C(b-s)f(s,x_{\rho(s,x_s)},x'(s))d\omega(s)\\
&+\sum_{k=1}^{m} AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^{m} C(b-t_k)I_k^2(x(t_k))\\
&+C(b-b_n)[y_n - AS(b_n)\phi(0) - C(b_n)\zeta - \sum_{0<t_k<b_n} AS(b_n-t_k)I_k^1(x(t_k))\\
&-\sum_{0<t_k<b_n} C(b_n-t_k)I_k^2(x(t_k))] + AS(b-b_n)[x_n - C(b_n)\phi(0) - S(b_n)\zeta\\
&-\sum_{0<t_k<b_n} C(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} S(b_n-t_k)I_k^2(x(t_k))]\\
=&\int_{b_n}^b C(b-s)Bv_n(s)ds + \int_{b_n}^b C(b-s)f(s,x_{\rho(s,x_s)},x'(s))d\omega(s)\\
&+\sum_{k=1}^{m} AS(b-t_k)I_k^1(x(t_k)) + \sum_{k=1}^{m} C(b-t_k)I_k^2(x(t_k)) + C(b-b_n)[y_n\\
&-\sum_{0<t_k<b_n} AS(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} C(b_n-t_k)I_k^2(x(t_k))] + AS(b-b_n)[x_n\\
&-\sum_{0<t_k<b_n} C(b_n-t_k)I_k^1(x(t_k)) - \sum_{0<t_k<b_n} S(b_n-t_k)I_k^2(x(t_k))].
\end{aligned}
$$

Since f is bounded, we infer that $\int_{b_n}^b C(b-s)f(s,x_{\rho(s,x_s)},x'(s))d\omega(s) \to 0$, when $n \to \infty$. Again, as in $x(b,\phi,\zeta,u_n)$, from (2.1) and (2.2), all the summation terms could be canceled out as $n \to \infty$. Thus, $y(b,\phi,\zeta,u_n) \to z_2$ as $n \to \infty$.

This implies that $z \in \overline{\mathcal{R}(f,\phi,\zeta)}$. Because z was arbitrarily chosen, this completes the proof. $\square$

5. Example

In this section, we consider an application of the theory developed in the previous section. Let $H = L^2([0,\pi])$. We choose the space $\mathcal{B} = \mathcal{PC}_0 \times L^2(g,H)$ constructed in [10], which satisfies the axioms (A) and (B) with $\widetilde{H} = 1$, $M(t) = \varrho(-t)^{\frac{1}{2}}$ and $K(t) = 1 + (\int_{-t}^0 g(\theta)d\theta)^{\frac{1}{2}}$ for $t \geq 0$.

Let $A : H \to H$ be an operator defined by $Af = f''$ with domain $D(A) = \{f \in H : f$ and f' are absolutely continuous, $f'' \in H, f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine family of operators on H. The spectrum of A consists of the eigenvalues $-n^2$ for $n \in N$, which associated eigenvectors $z_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$. Furthermore, the set $\{z_n, n \in N\}$ is an orthonormal basis of H and the following properties hold:

(a) If $f \in D(A)$, then $Af = -\sum_{n=1}^{\infty} n^2\langle f, z_n\rangle z_n$.

(b) For $x \in H$, $C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, z_n\rangle z_n$ and $S(t)x = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x, z_n\rangle z_n$. Consequently, $\|C(t)\| = \|S(t)\| \leq 1$ for all $t \in R$.

(c) If Φ is the group of translations on H defined by $\Phi(t)x(\xi) = \widetilde{x}(\xi + t)$, where $\widetilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t))$ and $A = B^2$, where B is the generator of Φ and $\mathbb{E} = \{x \in H^1([0,\pi]) : x(0) = x(\pi) = 0\}$ (see [8] for details).

Consider the impulsive stochastic partial differential equation

$$\begin{aligned}
&d[\tfrac{\partial z(t,x)}{\partial t}] = [\tfrac{\partial^2 z(t,x)}{\partial x^2} + q(x)u(t)]dt + [\textstyle\int_{-\infty}^{t} c(s-t)z(s-\sigma(\|z(t,x)\|),x)ds \\
&\qquad\qquad + \tfrac{\partial z(t,x)}{\partial t}]d\alpha(t), \quad x \in [0,\pi],\ t \in J = [0,b],\ t \neq t_k, \\
&z(t,0) = z(t,\pi) = 0, \quad t \in J, \\
&\triangle z(t_k)(x) = \textstyle\int_0^{\pi} K_1(t_k,x,y)z(t_k,y)dy, \quad k = 1,2,\cdots,m, \\
&\triangle z'(t_k)(x) = \textstyle\int_0^{\pi} K_2(t_k,x,y)z(t_k,y)dy, \quad k = 1,2,\cdots,m, \\
&z(\tau,x) = \phi(\tau,x),\ \tau \in (-\infty,0], \\
&\tfrac{\partial z(0,x)}{\partial t} = \psi(x),
\end{aligned} \tag{5.1}$$

where we assume $\phi \in \mathcal{B}$, with the identifications $\phi(\tau)(x) = \phi(\tau,x), (\tau,x) \in (-\infty,0]\times[0,\pi]$. $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$. $\sigma : [0,\infty) \to [0,\infty)$, is continuously differentiable, and $\alpha(t)$ is a one-dimensional standard Brownian motion. We assume that the function q can be expressed in the form $q = \sum_{n=1}^{\infty} e^{-n^2}q_n z_n$, where $q_n \neq 0$ for all $n \in N$ and $\sum_{n=1}^{\infty} q_n^2 < \infty$. We define $B : R \to H$ by $Bu = qu$. Then $\|B\| \leq K = \sqrt{\sum_{n=1}^{\infty} e^{-2n^2}q_n^2}$. Let $z(t)(y) = z(t,y)$. Assume that the following conditions hold: (i) $c(t)$ is measurable and continuous with finite $\mathcal{K}_f = (\int_{-\infty}^0 \frac{c^2(\theta)}{g(\theta)}d\theta)^{\frac{1}{2}}$. (ii) $K_i(t,x,y) : J \to L^2(\triangle)$, $\triangle = [0,\pi]\times[0,\pi]$, $l_k^i := \int_0^{\pi}\int_0^{\pi}\|K_i(t_k,x,y)\|^2dxdy$, $i = 1,2,\ k = 1,2,\cdots,m$.

The problem (5.1) can be modeled as the abstract impulsive Cauchy problem

(1.1) by defining

$$\begin{aligned} f(t,\phi,\psi)(x) &= \textstyle\int_{-\infty}^{0} c(\theta)\phi(\theta,x)d\theta + \psi(x), \\ \rho(\theta,\phi) &= \theta - \sigma(\|\phi(0)\|), \\ I_k^1(w)(x) &= \textstyle\int_0^\pi K_1(t_k,x,y)w(y)dy,\ \ w \in H, \\ I_k^2(w)(x) &= \textstyle\int_0^\pi K_2(t_k,x,y)w(y)dy,\ \ w \in H. \end{aligned}$$

Define

$$\Gamma_t^b = \textstyle\int_t^b S(b-s)BB^*S^*(b-s)ds.$$

We claim that $B^*S^*(b-s)x^* + B^*C^*(b-s)y^* = 0,\ s \in J$ implies that $x^* = y^* = 0$. Indeed

$$\begin{aligned} &B^*S^*(b-s)x^* + B^*C^*(b-s)y^* = 0 \\ \Longrightarrow &\sum_{n=1}^{\infty} e^{-n^2} q_n \langle z_n, \sum_{k=1}^{\infty} \tfrac{\sin k(b-s)}{k} \langle x^*, z_k\rangle z_k \rangle \\ &\qquad + \sum_{n=1}^{\infty} e^{-n^2} q_n \langle z_n, \sum_{k=1}^{\infty} \cos k(b-s) \langle y^*, z_k\rangle z_k\rangle = 0 \\ \Longrightarrow &\sum_{n=1}^{\infty} e^{-n^2} q_n \tfrac{\sin n(b-s)}{n} \langle x^*, z_n\rangle + \sum_{n=1}^{\infty} e^{-n^2} q_n \cos n(b-s) \langle y^*, z_n\rangle = 0 \\ \Longrightarrow &\, x^* = y^* = 0. \end{aligned}$$

It follows from Theorem 4.1 that the linear systems (4.1)-(4.2) are approximately controllable on J. Then the operator $\alpha(\alpha I + \Gamma_t^b)^{-1} \to 0$ in the strong operator topology as $\alpha \to 0^+$ (see [4,23,24]). So assumption (H_1) is satisfied.

Under these conditions,

$$\begin{aligned} \|f(t,\phi,\psi)\|_{L^2}^2 &= \textstyle\int_0^\pi [\int_{-\infty}^0 c(\theta)\phi(\theta,x)d\theta + \psi(x)]^2 dx \\ &\le 2\textstyle\int_0^\pi [\int_{-\infty}^0 c(\theta)\phi(\theta,x)d\theta]^2 dx + 2\int_0^\pi \|\psi(x)\|^2 dx \\ &= 2\textstyle\int_0^\pi [\int_{-\infty}^0 \frac{c(\theta)}{g^{\frac{1}{2}}(\theta)} g^{\frac{1}{2}}(\theta)\phi(\theta,x)d\theta]^2 dx + 2\|\psi\|_{L^2}^2 \\ &\le 2\textstyle\int_0^\pi [\int_{-\infty}^0 \frac{c^2(\theta)}{g(\theta)} d\theta \cdot \int_{-\infty}^0 g(\theta)\|\phi(\theta,x)\|^2 d\theta] dx + 2\|\psi\|_{L^2}^2 \\ &= 2\textstyle\int_{-\infty}^0 \frac{c^2(\theta)}{g(\theta)} d\theta \cdot \int_{-\infty}^0 g(\theta) \int_0^\pi \|\phi(\theta,x)\|^2 dx d\theta + 2\|\psi\|_{L^2}^2 \\ &\le 2\mathcal{K}_f^2 \|\phi\|_{\mathcal{B}}^2 + 2\|\psi\|_{L^2}^2 \\ &\le 2\mathcal{K}^2(\|\phi\|_{\mathcal{B}}^2 + \|\psi\|_{L^2}^2), \end{aligned}$$

where $\mathcal{K}^2 = \max\{1, \mathcal{K}_f^2\}$ and which implies that the function f satisfies the following condition

$$\begin{aligned} &\|f(t,\phi_1,\psi_1) - f(t,\phi_2,\psi_2)\|_{L^2} \\ &= \{\textstyle\int_0^\pi [\int_{-\infty}^0 c(\theta)\phi_1(\theta,x)d\theta + \psi_1(x) - (\int_{-\infty}^0 c(\theta)\phi_2(\theta,x)d\theta + \psi_2(x))]^2 dx\}^{\frac{1}{2}} \\ &= \{\textstyle\int_0^\pi [\int_{-\infty}^0 c(\theta)(\phi_1(\theta,x) - \phi_2(\theta,x))d\theta + (\psi_1(x) - \psi_2(x))]^2 dx\}^{\frac{1}{2}} \\ &\le \sqrt{2\mathcal{K}_f^2 \|\phi_1 - \phi_2\|_{\mathcal{B}}^2 + 2\|\psi_1 - \psi_2\|_{L^2}^2} \\ &\le \sqrt{2}\mathcal{K}(\|\phi_1 - \phi_2\|_{\mathcal{B}} + \|\psi_1 - \psi_2\|_{L^2}). \end{aligned}$$

Hence, (H_2) is satisfied.

Similarly,

$$\begin{aligned}&\|I_k^i(w_1)-I_k^i(w_2)\|_{L^2}^2\\=&\int_0^\pi[\int_0^\pi K_i(t_k,x,y)w_1(y)dy-\int_0^\pi K_i(t_k,x,y)w_2(y)dy]^2dx\\=&\int_0^\pi[\int_0^\pi K_i(t_k,x,y)(w_1(y)-w_2(y))dy]^2dx\\=&\int_0^\pi[\int_0^\pi\|K_i(t_k,x,y)\|^2dy\cdot\int_0^\pi\|w_1(y)-w_2(y)\|^2dy]dx\\\le& l_k^i\|w_1-w_2\|_{L^2}^2.\end{aligned}$$

Hence (H_3) is satisfied.

Moreover, the function $t\to AS(t)$ is uniformly continuous into $\mathcal{L}(\mathbb{E},H)$ and $\|AS(t)\|_{\mathcal{L}(\mathbb{E},H)}\le 1$ for $t\in J$.

Let

$$\begin{aligned}\phi_1=\phi_2=&\,64\mathcal{K}^2\mathrm{Tr}(Q)b(1+6(\tfrac{K^2b}{\alpha})^2)\{2[1+(\int_{-b}^0 g(\theta)d\theta)^{\frac{1}{2}}]^2+1\}\\&+8(1+6(\tfrac{K^2b}{\alpha})^2)[\sum_{k=1}^m l_k^1+\sum_{k=1}^m l_k^2].\end{aligned}$$

The next proposition is a consequence of Theorem 4.2.

Proposition 5.1. *Assume $\phi_1<1$. Then the system* (5.1) *is approximate controllability.*

6. Conclusion

In this paper, we discussed approximate controllability results for the second-order impulsive stochastic differential equations with state-dependent delay by using phase space axioms. We have proved the result without compactness of family of cosine operators. Several explicit sufficient conditions of such systems have been established by utilizing the fixed point strategy. Finally, an example is illustrated for the effectiveness of the approximate controllability results.

For the future research, we will consider the control problem for second-order stochastic functional differential equations with infinite delay in L_p space by using fundamental solution theory. By this manner, we will show some interesting results for the system involving a linear term (non-uniformly bounded), which has a wide practical background such as some heat conduction models with fading memory.

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References

[1] W. Arendt, C. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Springer Basel, 2001.

[2] G. Arthi, H. P. Ju and H. Y. Jung, *Existence and controllability results for second-order impulsive stochastic evolution systems with state-dependent delay*, Appl. Math. Comput., 2014, 248, 328–341.

[3] K. Balachandran and J. H. Kim, *Remarks on the paper "Controllability of second order differential inclusion in Banach spaces"* [J. Math. Anal. Appl., 2003 285, 537–550], J. Math. Anal. Appl., 2006, 324(1), 746–749.

[4] A. E. Bashirov and N. I. Mahmudov, *On concepts of controllability for deterministic and stochastic systems*, SIAM J. Control Optim., 1999, 37, 1808–1821.

[5] H. Chen, C. Zhu and Y. Zhang, *A note on exponential stability for impulsive neutral stochastic partial functional differential equations*, Appl. Math. Comput., 2014, 227, 139–147.

[6] S. Das, D. Pandey and N. Sukavanam, *Existence of solution and approximate controllability of a second-order neutral stochastic differential equation with state-dependent delay*, Acta Mathematica Scientia, 2016, 36B(5), 1509–1523.

[7] H. O. Fattorini, *Controllability of higher order linear systems*, In Mathematical Theory of Control, Academic Press, New York, 1967, 301–312.

[8] H. O. Fattorini, *Second-order linear differential equations in Banach spaces*, 108, Elsevier Sience, North Holland, 1985.

[9] J. K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcialaj Ekvacioj, 1978, 21, 11–41.

[10] Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay*, In: Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.

[11] J. R. Kang, Y. C. Kwun and J. Y. Park, *Controllability of the second-order differential inclusion in Banach spaces*, J. Math. Anal. Appl., 2003, 285, 537–550.

[12] J. Klamka, *Stochastic controllability and minimum energy control of systems with multiple delays in control*, Appl. Math. Comput., 2008, 206, 704–715.

[13] M. L. Li and J. L. Ma, *Approximate controllability of second-order impulsive functional differential system with infinite delay in Banach spaces*, J. Appl. Anal. Comput., 2016, 6(2), 492–514.

[14] N. I. Mahmudov and M. A. McKibben, *Approximate controllability of second-order neutral stochastic evolution equations*, Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and Algorithms, 2006, 13, 619–634.

[15] M. Martelli, *A Rothe type theorem for noncompact acyclic-valued map*, Boll. Un. Mat. Ital., 1975, 4, 70–76.

[16] P. Muthukumar and C. Rajivganthi, *Approximate controllability of second-order neutral stochastic differential equations with infinite delay and poisson jumps*, J. Systems Science and Complexity, 2015, 28, 1033–1048.

[17] C. Parthasarathy and M. M. Arjunan, *Controllability results for second-order impulsive stochastic functional differential systems with state-dependent delay*, Electronic Journal of Mathematical Analysis and Applications, 2013, 1(1), 88–109.

[18] G. D. Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992.

[19] L. J. Shen and J. T. Sun, *Approximate controllability of stochastic impulsive functional systems with infinite delay*, Automatica, 2012, 48, 2705–2709.

[20] C. C. Travis and G. F. Webb, *Compactness, regularity, and uniform continuity properties of strongly continuous cosine families*, Houston Journal of Mathematics, 1977, 3, 555–567.

[21] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second-order differential equations*, Acta Mathematica Hungarica, 1978, 32, 76–96.

[22] C. C. Travis and G. F. Webb, *Second order differential equations in Banach space*, Proceedings International Symposium on Nonlinear Equations in Abatract Spaces, Academic Press, New York, 1987, 331–361.

[23] R. Triggiani, *On the relationship between first and second-order controllable systems in Banach spaces*, Springer-verlag, Berlin, Notes in Control and Inform, Sciences, 1978, 370–393.

[24] R. Triggiani, *On the relationship between first and second order controllable systems in Banach spaces*, SIAM J. Control Optim., 1978, 16, 847–859.

[25] Z. M. Yan and X. X. Yan, *Existence of solutions for impulsive partial stochastic neutral integrodifferential equations with state-dependent delay*, Collectanea Mathematica, 2013, 64, 235–250.